Stability condition for a class of linear discrete systems

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***Abstract*—The paper deals with the proof of the stability condition for a certain class of linear time-discrete systems. From the mathematical point of view it shown that all roots of the speci£ed class of the n-th order linear equation are placed inside of the unit circle.**

***Index Terms*—Linear time-discrete systems, Eigenvalues, Polynomial roots.**

I. Introduction

FOR the linear stability of a general time-discrete dynamical system all the roots of the characteristic equation

$$A(\lambda) = a_n\lambda^n + a_{n-1}\lambda^{n-1} + ... + a_1\lambda + a_0 = 0 \qquad (1)$$

must be contained in the unit circle, $|\lambda| < 1$.
The Jury criterion ([2], [4]) gives a suf£cient and necessary condition that all roots of a given polynomial (1) where $a_i \in \Re$ are of modulus smaller than unity. To check this condition one applies the iterative scheme of the Jury table:

$$\begin{array}{rr} \forall_{0\le i\le n} & b_i := a_{n-i} \\ & \alpha_n := b_n/a_n \\ \forall_{1\le i\le n} & a_{i-1}^{new} := a_i - \alpha_n b_i \end{array} \qquad (2)$$

giving α_n and coef£cients $a_{n-1}...a_0$ for the next iteration.
The Jury criterion states that the eigenvalues are of the modulus smaller than unity if and only if $\forall_{1\le i\le n} |\alpha_i| < 1$. The criterion gives $2n$ (usually partly redundant) inequalities that de£ne hypersurfaces in coef£cient space. These hypersurfaces are given by algebraic equations; it is not necessary to compute the roots of the polynomial.
Whereas the Jury criterion is extremely helpful for small n and for numeric purposes, the hypersurface equations become very complex for large n, and one has to select the relevant hypersurface equations.

II. Problem description

The following theorem ensures the stability of a certain class of linear time-discrete systems without computing the polynomial roots or neither solution of the hypersurface equation set.

Theorem 1: Consider a n-th order equation

$$x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + ... + a_1x + a_0 = 0 \qquad (3)$$

with coef£cients $a_i \in \Re$ whereby

$$1 > a_{n-1} > a_{n-2} > ... > a_1 > a_0 > 0 \qquad (4)$$

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Manuscript received February 17, 2008; revised April 2, 2008.

Then, for all solutions of the equation (3) it holds

$$|x| < 1 \qquad (5)$$

It is necessary to say the presented theorem was already proved (see [5]) and discussed e.g. in [3], [6] and [7]. The novelty of here demonstrated approach of the proof consists in its simplicity that can be easily understood by everybody that has only a basic secondary education. We return to this topic since it helps to introduce a very important property of linear time-discrete systems (see previous section) in praxis.

Proof: The proof will be done by contradiction. Let's suppose that $\exists\, x : |x| \ge 1$. Since $x \ne 0$, the equation (3) can be divided by x^n and x^{n-1}:

$$1 + a_{n-1}\frac{1}{x} + a_{n-2}\frac{1}{x^2} + ... + a_1\frac{1}{x^{n-1}} + a_0\frac{1}{x^n} = 0 \qquad (6)$$

$$x + a_{n-1} + a_{n-2}\frac{1}{x} + ... + a_1\frac{1}{x^{n-2}} + a_0\frac{1}{x^{n-1}} = 0 \qquad (7)$$

After the substraction $(6) - (7)$ one receives

$$\begin{aligned} 1 - a_{n-1} \quad &+ \quad (a_{n-1} - a_{n-2})\frac{1}{x} + (a_{n-2} - a_{n-3})\frac{1}{x^2} + \\ &+ \quad .. + (a_1 - a_0)\frac{1}{x^{n-1}} + a_0\frac{1}{x^n} = x \end{aligned} \qquad (8)$$

and wihout the loss of generality it can be written

$$\begin{aligned} &|1 - a_{n-1} + (a_{n-1} - a_{n-2})\frac{1}{x} + \\ &+ .. + (a_1 - a_0)\frac{1}{x^{n-1}} + a_0\frac{1}{x^n}| = |x| \end{aligned} \qquad (9)$$

Following properties of the absolute value, the left side of the equation (9) can be estimated by the expression

$$\begin{aligned} &\left|1 - a_{n-1} + \frac{a_{n-1} - a_{n-2}}{x} + .. + \frac{a_1 - a_0}{x^{n-1}} + \frac{a_0}{x^n}\right| \\ \le \quad &|1 - a_{n-1}| + \left|\frac{a_{n-1} - a_{n-2}}{x}\right| + .. + \left|\frac{a_1 - a_0}{x^{n-1}}\right| + \left|\frac{a_0}{x^n}\right| \end{aligned} \qquad (10)$$

that, under consideration of (4), can be simpli£ed to the form

$$\begin{aligned} &\left|1 - a_{n-1} + \frac{a_{n-1} - a_{n-2}}{x} + .. + \frac{a_1 - a_0}{x^{n-1}} + \frac{a_0}{x^n}\right| \\ \le \quad &1 - a_{n-1} + \frac{a_{n-1} - a_{n-2}}{|x|} + .. + \frac{a_1 - a_0}{|x^{n-1}|} + \frac{a_0}{|x^n|} \end{aligned} \qquad (11)$$

Since, it is supposed that $|x| \geq 1$, it holds

$$\frac{1}{|x|} \leq 1, \quad ..., \quad \frac{1}{|x^{n-1}|} \leq 1, \quad \frac{1}{|x^n|} \leq 1$$

and there can be found an upper estimation of the relation (11)

$$\begin{aligned} & 1 - a_{n-1} + \frac{a_{n-1} - a_{n-2}}{|x|} + .. + \frac{a_1 - a_0}{|x^{n-1}|} + \frac{a_0}{|x^n|} \\ \leq \; & 1 - a_{n-1} + a_{n-1} - a_{n-2} + .. + a_1 - a_0 + a_0 = 1 \end{aligned} \tag{12}$$

Going back to (9), it can be written

$$\left| 1 - a_{n-1} + \frac{a_{n-1} - a_{n-2}}{x} + .. + \frac{a_1 - a_0}{x^{n-1}} + \frac{a_0}{x^n} \right| = |x| \leq 1 \tag{13}$$

Following the assumption of the proof ($|x| \geq 1$) there exists the only one solution that satis£es the condition (13) and it is

$$|x| = 1 \tag{14}$$

After denoting $A_1 = 1 - a_{n-1}$, $A_2 = \frac{a_{n-1} - a_{n-2}}{x}$, ..., $A_n = \frac{a_1 - a_0}{x^{n-1}}$ and $A_{n+1} = \frac{a_0}{x^n}$, the expression (10) can be rewritten in the form

$$\begin{aligned} & |A_1 + A_2 + ... + A_n + A_{n+1}| \leq \\ \leq \; & |A_1| + |A_2| + ... + |A_n| + |A_{n+1}| \end{aligned} \tag{15}$$

and for (14) even as

$$\begin{aligned} & |A_1 + A_2 + ... + A_n + A_{n+1}| = \\ = \; & |A_1| + |A_2| + ... + |A_n| + |A_{n+1}| \end{aligned} \tag{16}$$

The equation (16) holds only in the case when all non-zero numbers A_i have the same value of $arg(A_i)$ (see Fig.1). Since $A_1 = 1 - a_{n-1} \in \Re^+$, $A_1 \neq 0$, $A_2 = \frac{a_{n-1} - a_{n-2}}{x} \neq 0$ and $arg(A_1)$ should equal $arg(A_2)$, one has to conclude that $A_2 \in \Re^+$. Moreover since the numerator of A_2 is the real positive number, the denominator $den(A_2) = x$ has to be a real positive number too. Therefore following (14) one gets $x = 1$.

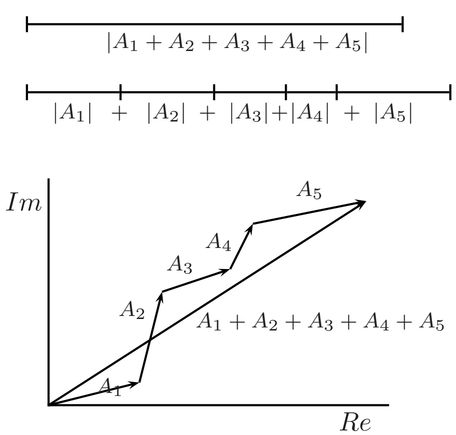

Fig. 1. Geometrical interpretation of the sum of vectors (complex numbers) A_1, A_2, A_3, A_4 and A_5

The value $x = 1$ remains the only possible solution of the equation (3). After its substitution there, one receives

$$1 + a_{n-1} + a_{n-2} + .. + a_1 + a_0 = 0 \tag{17}$$

Since (4) holds, it is the contradiction to the assumption ($x \geq 1$) and the statement speci£ed in the theorem is true. ■

Remark 1: To approve the Theorem 1 it is suf£cient to demonstrate that

A.) $a_{n-1} < 1$

B.) $a_0 > 0$

C.) $a_{n-i} < a_{n-i+1}$

III. Conclusion

It can be said that Jury criterion gives the suf£cient and necessary condition of stability for all linear discrete systems that satisfy the Jury stability scheme. The proposed criterion gives the suf£cient and necessary condition of stability for the class of linear discrete systems whose characteristic equation coef£cients (3) satisfy the condition (4).

Acknowledgment

The work has been supported by the Slovak Grant Agency, Grant No. VEGA 1/3089/06. This support is very gratefully acknowledged.

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