# On the exact symmetries of dynamical systems from their reduced system of equations

F.I. Arunaye

***Abstract***— In a recent paper we presented the computation of the exact symmetry transformations of dynamical systems from their reduced systems using the Kepler problem as vehicle. We also noted therein that this computational technique is applicable to systems that can be reduced to couple oscillator(s) and a conservation law both in two-and three-dimensions. In this paper we show in addition to the former that when the reduction variable for the radial component of the equation of motion is varied by the multiples of quadratic powers of the angular momentum, the exact symmetry transformations of the vector fields $\alpha v_1 \partial_1$ and $\alpha v_2 \partial_2$ are invariants in 2-dimension. We demonstrate this by using both the Kepler and the generalized Kepler problems in two dimensions. We also note that this is not necessarily the case for the dynamical system in 3-dimensions.

***Keywords***—Exact, symmetries, dynamical, systems, infinitesimal, generators, flow, Kepler, Lie, generalized-Kepler.

## I. Introduction

Symmetries in general and in particular Lie point symmetry analysis are formidable tools for finding solutions to differential equations ( be it ordinary or partial differential equations) [1],[2],[3],[4],[5]. Lie theory and more recently works/researches [2],[4],[6],[7],[8],[9],[10] in the subject emphasized the attainment of the infinitesimal generators of the Lie point symmetries for the differential equations. This approach may not be far from the consequences of the efforts required for the actual computational activities that are involved in obtaining the Lie point symmetry infinitesimal generators given the fact that some symmetry transformations of some dynamical systems are nonlocal types in their representations.[11],[10],[9],[8] More so, the determinations of the symmetries of most physical dynamical systems actually posed significant challenges in the literature [10],[11],[12] as in the understanding of their physical properties, visa vise constants of the motion, first integrals, linearization and orbit equations. Noether theory is significant in the aspect of variational symmetries in that it provided a straightforward link between symmetries and constants of the motion (first integrals).[1],[2],[3] It is also well known that in the case of the Kepler problem the Noether symmetries (five variational symmetries) which are subset of the Lie symmetries obtained in the literature.[11],[8],[10] The events that followed the analysis of complete symmetry groups of differential equations brought to the fore the reduction of order technique for obtaining the infinitesimal generators for dynamical systems.[11],[13] Krause [11], Nucci [13], Nucci et al[21] Leach et al [22] and a host of other papers revolutionized the entire symmetry analysis of differential equations although these did not amount to deviation from the original idea of Lie's but explicitly exposed the importance of nonlocal symmetries as well as opening the frontiers of the subject, specifically they amounted to the acknowledgement of nonlocal symmetries as the bases for the actual integrability of differential equations.[7],[11],[12] The reduction of order algorithm reduced dynamical systems to systems of oscillator(s) and conservation laws, which admits Lie algorithm for the determination of their infinitesimal symmetry generators. The applicability of the reduction of order algorithm is formidable for determining the Lie symmetry group of dynamical systems [6],[ 9],[10],[11],[13]. It was reported [14],[26], [27] that the literature refers to the vector fields of the infinitesimal generators as symmetries and nowhere in the literature were exact symmetries as presented therein was mention hitherto. In [14],[26] the exact symmetries of dynamical systems which are different in representations from vector fields of the infinitesimal generators of dynamical systems were computed from the Lie symmetry generators of their reduced systems obtained by the reduction of order algorithm. It was also shown [14],[29] that one could use analogous constants obtained from the Hamilton vector of dynamical systems instead of the Ermanno-Bernoulli constants to reduce dynamical systems to systems that admit Lie algorithm. In this paper we consider the cases where the respective natural variables for reducing dynamical systems into systems of oscillator(s) and conservation law(s) are constant multiples of the positive quadratic powers (or inverse quadratic powers) of the angular momentum of the dynamical systems in two dimensions. The computations of these exact symmetry transformations of the Kepler and generalized Kepler problems was undertaken and inferences was drawn. In section II, we review basic definitions [26],[28] that are crucial to the understanding of our discussion. Section III reviews the computation of the exact symmetry transformations of the Kepler problem in two-dimensions while in section IV we examined the exact symmetry transformations of both the Kepler problem and the generalized Kepler problem from the constant multiple of the reduction variables by the quadratic powers of the angular momentum point of view in two-dimensions. Section V is

Manuscript received February 13, 2008: Revised version received May 9, 2008. This work was supported in part by the Nigerian High Commission Kingston office, Jamaica. The author thanks the Acting High commissioner for excellent initiative in this direction. The author also thank the University of the West Indies, Mona, Kingston 7, Jamaica for Supports both financially and environment of research.

F. I. Arunaye is with the Department of Mathematics and Computer Science U.W.I. Mona presently, but he is permanently with the Department of Mathematics and Computer Science, Delta State University, Abraka, Nigeria. Cell Phone 01 876 412 0835; e-mail festus.arunaye@uwimona.edu.jm; ifarunaye@yahoo.com

devoted to the treatment of the three-dimensional cases. In section VI we present concluding remarks.

## II. Basic Definitions and Concepts

Let $T: Y \to Y$ be a one-to-one, and onto mapping (transformation) defined on a sub-manifold $Y \subset M$. The totality of such transformations $\tau(M)$ form a group where the composition of mappings plays the part of a group operation and the identity transformation is designated $I_M$. The point $(y,t)$ to point $(\bar{y},\bar{t})$ transformation defined symmetry transformation in general conceptualization. If the point transformation depends on a group parameters $\alpha_i$ such that the point $(y,t;\boldsymbol{\alpha})$ is transformed to the point $(y,\bar{t};\boldsymbol{\alpha})$ where $\boldsymbol{\alpha} = (\alpha_1, \alpha_2 ..., \alpha_n)$, we have a parameter-dependent symmetry transformation. By this we mean the following symbolic transformations

$$\bar{t} = \bar{t}(y,t;\boldsymbol{\alpha}),\ \bar{y} = \bar{y}(y,t;\boldsymbol{\alpha});$$
$$\bar{\bar{y}} = \bar{\bar{y}}(\bar{y},\bar{t};\bar{\boldsymbol{\alpha}}) = \bar{\bar{y}}(y,t;\bar{\bar{\boldsymbol{\alpha}}}), \tag{1}$$

and for some $\bar{\bar{\boldsymbol{\alpha}}} = \bar{\bar{\boldsymbol{\alpha}}}(\bar{\boldsymbol{\alpha}},\boldsymbol{\alpha})$, the identity $\boldsymbol{\alpha} = \mathbf{0}$ ensured that $\bar{t}(y,t;0) = t$ and $\bar{y}(y,t;0) = y$ hold for the continuous group parameter $\boldsymbol{\alpha}$. Lie theory is centered on one parameter transformations which are flows $\bar{\bar{\lambda}}(\bar{\lambda},\lambda) = \bar{\lambda} + \lambda$.

### *A. Flows (Lie group of symmetry transformations)*

A flow or one parameter group of symmetry transformations of a space $Y \subset M$ onto itself is a set of functions $f_\lambda : Y \to Y$ such that the following composition and identity maps are respectively defined on the space $Y$,

(i) $f_{\lambda+\mu} = f_\lambda \circ f_\mu$ ;

(ii) $f_0 = id$ on $Y$.

*Theorem1*. The map $f_\lambda : Y \to Y$ is a flow if and only if there is a vector function $V$ on $Y$ such that $\bar{\mathbf{y}} = f_\lambda(\mathbf{y})$ is a solution of the equation

$$\frac{d\bar{\mathbf{y}}}{d\lambda} = V(\bar{\mathbf{y}}),\ \bar{\mathbf{y}} = \mathbf{y} \text{ when } \lambda = 0.$$

*Proof:* let $f_\lambda : Y \to Y$ be a flow, then $f_{\lambda+\mu}(\mathbf{y}) = f_\lambda[f_\mu(\mathbf{y})]$. On differentiating this with respect to $\mu$ we have the following relation

$$\frac{d}{d\mu} f_{\lambda+\mu}(\mathbf{y}) \equiv \frac{d}{d\lambda} f_{\lambda+\mu}(\mathbf{y}) = \frac{df_\mu}{d\mu}[f_\lambda(\mathbf{y})]. \tag{4}$$

By setting $\mu = 0$ in (4) we have that $\frac{d\bar{\mathbf{y}}}{d\lambda} = V(\bar{\mathbf{y}})$ where $\bar{\mathbf{y}} = f_\lambda(\mathbf{y})$, $V(\mathbf{y}) = \frac{d}{d\mu} f_\mu(\mathbf{y})\Big|_{\mu=0}$.

Conversely, if we let

$$\frac{d\bar{\mathbf{y}}}{d\lambda} = V(\bar{\mathbf{y}}), \tag{5}$$

such that $\bar{\mathbf{y}} = \mathbf{y}$ when $\lambda = 0$.

Then we have the integral equation

$$\bar{\mathbf{y}} = f_\lambda(\mathbf{y}) = \mathbf{y} + \int_0^\lambda V[f_{\lambda'}(\mathbf{y})]d\lambda' \tag{6}$$

and $\mathbf{y} = f_0(\mathbf{y})$.

But the function $g_\lambda(\mathbf{y}) = f_{\lambda+\mu}(\mathbf{y})$ is also a solution of (5) which satisfies $g_0(\mathbf{y}) = f_\mu(\mathbf{y})$.

Thus $g_\lambda(\mathbf{y}) = f_\lambda(g_0(\mathbf{y}))$

i.e. $f_{\lambda+\mu}(\mathbf{y}) = f_\lambda[f_\mu(\mathbf{y})]$. ■

If $F: Y \to \Re$ is a function then by Taylor's Theorem we have that

$$F(\bar{\mathbf{x}}) = \sum_{n=0}^{k} \frac{\lambda^n}{n!} \frac{d^n \bar{F}}{d\lambda^n}\Bigg|_{\lambda=0} \tag{7}$$

$$\text{But} \quad \frac{d\bar{F}}{d\lambda} = \sum_i \frac{d\bar{y}_i}{d\lambda}\frac{\partial \bar{F}}{\partial \bar{y}_i} = \sum_i \bar{v}_i \frac{\partial \bar{F}}{\partial \dot{y}_i}\Bigg|_{\lambda=0},$$
$$= \sum_i v_i \frac{\partial F}{\partial x_i} = WF \tag{8}$$

So (7) implies

$$F(\bar{\mathbf{x}}) = \sum_{n=0}^{k} \frac{\lambda^n}{n!} W^n F = e^{\lambda W} F(x), \tag{9}$$

and $W = \sum v_i \partial_i$ where $V = (v_1, v_2, ..., v_n)$, is called the vector field generating the flow $f_\lambda$ (commonly referred to as the symmetry generator).

### *B. Illustrative Examples*

*(i).* The flow generated by the vector field $V = t\partial_y$ on $R \times R = \{(y,t) \mid y,t \in R\}$ is given by the solution to the equations [26],[27],[28]

$$\frac{d\bar{\mathbf{y}}}{d\lambda} = \bar{t},\ \frac{d\bar{t}}{d\lambda} = 0$$

where $(\bar{\mathbf{y}},\bar{t}) = (\mathbf{y},t)$ when $\lambda = 0$.

i.e. $\bar{t} = t$, $\bar{\mathbf{y}} = \mathbf{y} + \lambda t$.

So the flow is given by

$$(\bar{\mathbf{y}},\bar{t}) = f_\lambda(\mathbf{y},t) = (\mathbf{y} + \lambda t, t). \tag{10}$$

Conversely given the flow $f_\lambda$, the vector field generating it is given by

$$\frac{d}{d\mu} f_\mu(\mathbf{y},t)\Bigg|_{\mu=0} \cdot \partial = t\partial_y + 0\partial_t = t\partial_y.$$

*(ii).* The vector field $V = yt\partial_y + t^2\partial_t$ generates the flow $f_\lambda$ given by the solution of the equations

$$\frac{d\bar{y}}{d\lambda} = \bar{y}\bar{t}\ ,\ \frac{d\bar{t}}{d\lambda} = \bar{t}^{\,2}.$$

The equations respectively give the solutions

$$\bar{y} = \frac{y}{1-\lambda t} \text{ and } \bar{t} = \frac{t}{1-\lambda t}.$$

Thus

$$(\bar{y},\bar{t}) = f_\lambda(y,t) = (1-\lambda t)^{-1}(y,t).$$

Conversely, calculating

$$\left.\frac{d}{d\mu} f_\mu(y,t)\right|_{\mu=0} \cdot \partial = yt\partial_y + t^2\partial_t\,, \tag{11}$$

which is the vector field generating the flow.

*C. Lie point symmetry and nonlocal symmetry*

The Lie theory of symmetry analysis of differential equations is anchored on the shore of extended (prolongation) vector fields. [4],[1],[2],[3] For a vector field given by the relation

$$V = \xi(y,t)\partial_t + \eta(y,t)\partial_x\,, \qquad (12\ 19)$$

the prolongation of $V$ to the nth order is defined by the relation

$$V^{(n)} = \xi\partial_t + \eta\partial_y + \eta'\partial_{y'} + \ldots + \eta^{(n)}\partial_{y(n)}\,, \tag{13}$$

where $\eta^{(n)} = \frac{d^n}{dx^n}(\eta - y'\xi) + y^{(n+1)}\xi$ , (14)

and $\eta^{(n)}$ is not the nth derivative of $\eta$. The invariance of the differential equation under the action of the prolonged vector field is invariant is well known. The general equation of order k denoted by

$$E(t,y,\dot{y},\ldots y^k) = 0\,, \tag{15}$$

and is invariant under the action of the kth prolonged vector field $V^{(k)}$ if and only if

$$V^{(k)}E(t,y,\dot{y},\ldots y^k)\big|_{E(t,y,\dot{y},\ldots,y^k)=0} = 0. \tag{16}$$

The system (17) separates into systems of partial differential equations in terms of $\xi(y,t)$ and $\eta(y,t)$ that can be solved by the method of superposition of linearly independent basis solutions $\xi_i(y,t)$ and $\eta_i(y,t)$ so that

$$V_i = \xi_i(y,t)\partial_t + \eta_i(y,t)\partial_y\,, \tag{17}$$

become the infinitesimal generator of the Lie point symmetries of (15). It is well known in the literature that the totality (dimension) of (17) defined the group dimensionality of the Lie point symmetry group of (15). When (15) is of order one, the totality of (17) is infinite and there is no known algorithm of obtaining them, while the dimension is less or equal eight if it is of order two or more.

*D. Definitions*

If the functions $\xi_i(y,t)$, $\eta_i(y,t)$ in (17) contains integral(s) of the dependent variable, the resulting infinitesimal generator is called nonlocal symmetry.[11],[12],[13] One type of the nonlocal symmetries is

$$Y = \{\textstyle\int \xi dt\}\partial_t + \eta\partial_y\,. \tag{18}$$

We note also that there are exponential nonlocal symmetries if the infinitesimal contained exponent of integral(s).[12],[15] If the infinitesimals $\xi_i(y,t)$ and $\eta_i(y,t)$ in (18) are dependent on the derivative of $x$ say $\xi(y,\dot{y},t)$ and $\eta(y,\dot{y},t)$ then the resulting infinitesimal generator is called contact symmetry. Note that the contact symmetries are also regarded as Lie point symmetries.

*E. Complete symmetry groups*

The concept of complete symmetry groups was generally accepted to mean the group of symmetries of differential equations which completely specify them on till recently. In this view Lie identified the symmetry groups of second-order differential equations to have not more than eight Lie point symmetries that specify them completely (any linearizable second-order differential equation has the maximum eight Lie point symmetry group).[16],[17],[18],[19] The literature in this issue is very rich, the work of Noether on the Kepler problem could only identified five variational symmetries ( also found by Lie analysis) [1],[2],[3] which could not specify the Kepler equation of motion. So there was a gap of not been able to obtain the complete symmetry groups for the Kepler problem in the sense of Lie. More recently, it was shown [7, 20] that complete symmetry groups and algebras are not unique and the concepts of "maximality" or "minimality" (the maximal or minimal set of symmetries required to specify the differential equation completely) of symmetry groups and algebras came to the fore. However for the purpose of this paper we intend to confine our discussion to emergence of nonlocal symmetries as the bye-products of the general quest for complete symmetry groups of the Kepler problem for which the forerunner is Krause (1994) [ we refer the interested reader to references in ref. [11],[10],[13]], who obtained the additional three symmetries (nonlocal type) and together with the five point symmetries obtained by either Noether theorem or Lie theorem, was able to specify the equation of motion of the Kepler problem completely.

## III. EXACT SYMMETRY TRANSFORMATIONS OF KEPLER PROBLEM IN TOW-DIMENSIONS

We firstly review the Kepler problem so as to note some of its interesting properties as below. The Kepler problem has the equation of motion given by

$$\ddot{\mathbf{y}} + \frac{\mu\mathbf{y}}{r^3} = 0\,,\ r = |\mathbf{y}|\,. \tag{19}$$

The system (19) possess the angular momentum vector $\mathbf{L}$ where

$$\mathbf{L} = \mathbf{y}\wedge\dot{\mathbf{y}}\,. \tag{20}$$

The vector product of (19) with (20) yields the relation

$$\left(\dot{\mathbf{y}}\wedge\mathbf{L}\right)^{\cdot} + \frac{\mu\mathbf{y}\wedge\mathbf{L}}{r^3} = 0\,. \tag{21}$$

Using $\dot{\mathbf{y}} = \dot{r}\mathbf{e}_r + r\dot{\mathbf{e}}_r$ we have that $\mathbf{y}\wedge\mathbf{L} = -r^3\dot{\mathbf{e}}_r$, so that (21) becomes

$$\left(\dot{\mathbf{y}}\wedge\mathbf{L}\right)^{\cdot} - \mu\dot{\mathbf{e}}_r = 0\,, \qquad (22)$$

and on integrating (22) we obtain the second conserved quantity called Laplace-Runge-Lenz (LRL) vector $\mathbf{J}$ given by

$$\left(\dot{\mathbf{y}}\wedge\mathbf{L}\right) - \mu\mathbf{e}_r = \mathbf{J} \qquad (23)$$

The third conserved vector of (19) is the Hamilton's vector obtained by Hamilton in 1845; videlicet

$$\mathbf{K} = \dot{\mathbf{y}} - \frac{\mu}{L}\hat{\mathbf{L}}\wedge\mathbf{e}_r\,,\; L = \left|\mathbf{L}\right|. \qquad (24)$$

The analysis of system (19) for its Lie point symmetries attracted large volume of articles in the literature. It is well known that Lie method produced five Lie point symmetry generator which was also demonstrated by Noether method of variational symmetry theory although these symmetry generators were unable to specify the Kepler problem in the art of complete symmetry analysis that followed which consequence brought to the for the notion of nonlocal symmetry.[10],[11],[13] The first five Lie point symmetry generators are given as follows

$$Y_1 = \partial_t\,, Y_2 = t\partial_t + \tfrac{2}{3}r\partial_r\,, Y_3 = y_2\partial_{y_3} - y_3\partial_{y_2}\,,$$

$$Y_4 = y_3\partial_{y_1} - y_1\partial_{y_3}\,, Y_5 = y_1\partial_{y_2} - y_2\partial_{y_1}\,. \qquad (25)$$

While the three additional nonlocal symmetry generators [11] are

$$V_1 = 2\{\textstyle\int y_1 dt\}\partial_t + y_1 r\partial_r\,, V_2 = 2\{\textstyle\int y_2 dt\}\partial_t + y_2 r\partial_r\,,$$

$$V_3 = 2\{\textstyle\int y_3 dt\}\partial_t + y_3 r\partial_r\,, \qquad (26)$$

where $r^2 = y_1^2 + y_2^2 + y_3^3$. One noticed that the above symmetry generators (25) and (26) separated into the following four symmetry transformations viz Translation symmetries (time and special); Dilation also called self similarity or scaling symmetries (time and special); Rotation symmetries; and the Nonlocal symmetries.

The scaling symmetry $Y_2$ described the Laplace-Runge-Lenz (LRL) vector of the Kepler problem which is the source of the orbit equation for system (19).[13],[23] However later works have established that these nonlocal symmetries are attainable by reduction of order developed by Nucci [13],[21], and more also it is well known that the reduction of order process is achieved by natural reduction variables of the system via the Ermanno-Bernoulli constants [21],[13,[22] and as well as Quasi-Ermanno-Bernoulli constants reported in ref. [14],[26],[27],[28] are used to reduced (19) to a system of oscillator(s) and a conservation law. We noted there that this is applicable to a number of dynamical systems. The reduced systems of (19) using the method of Nucci [13], Nucci and Leach [21] and the associated Lie symmetry generators are given by (27) and (28) respectively,

$$v_1'' + v_1 = 0\,, \quad v_2' = 0 \qquad (27)$$

where $v_1 = L^2 r^{-1} - \mu$; $v_2 = r^2\dot{\theta}$ and

$$\Gamma_1 = v_2\partial_2\,;\Gamma_2 = \partial_\theta\,;\Gamma_3 = v_1\partial_1\,;\; \Gamma_{4\pm} = e^{\pm i\theta}\partial_1\,;$$

$$\Gamma_{6\pm} = e^{\pm 2i\theta}[\partial_\theta \pm iv_1\partial_1]\,;\Gamma_{8\pm} = e^{\pm i\theta}[v_1\partial_\theta \pm iv_1^2\partial_1]\,, \qquad (28)$$

where $\partial_i = \partial/\partial v_i$. Obtaining the symmetry generators of the dynamical system (19) entails the backward translation from the Lie symmetry generators (28) of the reduced system (27) variables to the original variables of system (19), the scheme for doing this is available in [24] and many of which are largely nonlocal symmetries in the original variables (in some cases their representations are highly complicated). We only list the symmetries in the original variables for (28) below:

$$\Gamma_1 = 3t\partial_t + 2r\partial_r\,,$$

$$\Gamma_2 = \partial_\theta\,,$$

$$\Gamma_3 = 2\left[\mu\textstyle\int r dt - L^2 t\right]\partial_t + r\left(\mu r - L^2\right)\partial_r\,,$$

$$\Gamma_{4\pm} = 2\left[\textstyle\int re^{\pm i\theta}dt\right]\partial_t + r^2 e^{\pm i\theta}\partial_r\,,$$

$$\Gamma_{6\pm} = 2\left[\textstyle\int\left(\mu r + 3L^2\right)e^{\pm 2i\theta}dt\right]\partial_t + r\left(\mu r + 3L^2\right)e^{\pm 2i\theta}\partial_r + L^2 e^{\pm 2i\theta}\partial_\theta\,,$$

$$\Gamma_{8\pm} = 2\left[\textstyle\int\left\{2\dot{r}L^3 \pm ir\left(\mu - r^3\dot{\theta}^2\right)\left(\mu + r^3\dot{\theta}^2\right)\right\}e^{\pm i\theta}dt\right]\partial_t + r\left[2\dot{r}L^3 \pm ir\left(\mu - r^3\dot{\theta}^2\right)\left(\mu + r^3\dot{\theta}^2\right)\right]\partial_r + L^2\left(\mu - r^3\dot{\theta}^2\right)e^{\pm i\theta}\partial_\theta\,, \qquad (29)$$

in which the factor $L^2$ has been included to make the expressions look simpler. [11],[21],[24],[25]

We now calculate the exact symmetry transformations of (19) from (28) as following:

For the vector field $\alpha v_1\partial_1$ where $\alpha$ is arbitrary constant the flow of this vector field is the function $f(v_1, v_2, \theta) = (\bar{v}_1, \bar{v}_2, \bar{\theta})$ where

$$\frac{d\bar{v}_1}{d\lambda} = \alpha\bar{v}_1\;;\; \frac{d\bar{v}_2}{d\lambda} = 0\,; \frac{d\bar{\theta}}{d\lambda} = 0\,. \qquad (30)$$

Solving system (30) we have the following

$$\bar{v}_1 = e^{\alpha\lambda}v_1\;\;;\;\; \bar{v}_2 = v_2\,; \bar{\theta} = \theta\,. \qquad (31)$$

The second equation in (31) implies that $\bar{L} = L$ while the first equation implies that

$$\bar{L}^2\bar{r}^{-1} - \mu = C(L^2 r^{-1} - \mu)\,,$$

$$\frac{r}{\bar{r}} = \mu r L^{-2} + C(1 - \mu L^{-2} r)\,,$$

$$\bar{r} = H_1^{-1} r\,, \qquad (32)$$

where $H_1 = \mu r L^{-2} + C(1 - \mu L^{-2} r)$, $C = e^{\alpha\lambda}$. (33)

From $\bar{r}^2\dot{\bar{\theta}} = r^2\dot{\theta}$ we have that

$$\frac{d\bar{t}}{dt} = H_1^{-2}\,. \qquad (34)$$

Equations (33) and (34) constitute the exact symmetry transformations of (19) with the given generator $\Gamma_3 = v_1\partial_1$. We note that these symmetry transformations are global, that is

$$\bar{\mathbf{y}} = H_1^{-1}\mathbf{y}\,. \qquad (35)$$

We also note that when $\mathbf{y}$ is made three-dimensional, the symmetry transformations (35) is also true. For the vector field $\alpha v_2 \partial_2$, we have the flow as $f(v_1, v_2, \theta) = (\bar{v}_1, \bar{v}_2, \bar{\theta})$ where

$$\frac{d\bar{v}_1}{d\lambda} = 0 \ ; \ \frac{d\bar{v}_2}{d\lambda} = \alpha\bar{v}_2 \, ; \frac{d\bar{\theta}}{d\lambda} = 0 . \tag{36}$$

Solving system (36) we have the following

$$\bar{v}_1 = v_1 \ ; \ \bar{v}_2 = e^{\alpha\lambda} v_2 \, ; \bar{\theta} = \theta . \tag{37}$$

The second equation in (44) implies $\bar{L} = CL$ while the first equation implies that

$$\bar{L}^2 \bar{r}^{-1} - \mu = L^2 r^{-1} - \mu$$

$$\text{i.e. } C^2 L^2 \bar{r}^{-1} - \mu = L^2 r^{-1} - \mu$$

where $C = e^{\alpha\lambda}$, then

$$\frac{\bar{r}}{r} = C^2 \Rightarrow \bar{r} = C^2 r \ . \tag{37}$$

But $\bar{L} = CL$ implies that

$$\dot{\bar{\theta}} \bar{r}^2 = C\dot{\theta} r^2 \Rightarrow \bar{r}^2 \frac{d\bar{\theta}}{d\bar{t}} = C r^2 \frac{d\theta}{dt} ,$$

which implies that

$$\frac{d\bar{t}}{dt} = C^3 , \tag{38}$$

i.e. $\bar{t} = d + C^3 t$,

where $d$ is an arbitrary constant.

Consequently the exact symmetry transformations generated by the vector field $\Gamma_1 = v_2 \partial_2$ for the Kepler problem is given by equations (37) and (38). If $(y_1, y_2) = (r\cos\theta, r\sin\theta)$ denotes the Cartesian coordinates of $\mathbf{x}$ in the plane of motion then $\bar{\theta} = \theta$ implies that

$$\bar{\mathbf{y}} = C^2 \mathbf{y} , \tag{39}$$

which is the global symmetry transformation where $\mathbf{y} = y_1 \mathbf{i} + y_2 \mathbf{j}$ is the two dimensional Cartesian vector.

The vector field $\alpha\partial_\theta$ has the flow $f(v_1, v_2, \theta) = (\bar{v}_1, \bar{v}_2, \bar{\theta})$ where

$$\frac{d\bar{v}_1}{d\lambda} = 0 \ ; \ \frac{d\bar{v}_2}{d\lambda} = 0 \, ; \frac{d\bar{\theta}}{d\lambda} = \alpha . \tag{40}$$

Solving system (40) we have the following,

$$\bar{v}_1 = v_1 \ ; \ \bar{v}_2 = v_2 \, ; \bar{\theta} = \theta + \alpha\lambda . \tag{41}$$

Since $\bar{v}_2 = v_2 \Rightarrow \bar{L} = L$ we have that

$$\bar{L}^2 \bar{r}^{-1} - \mu = L^2 r^{-1} - \mu \ \Rightarrow \ \bar{r} = r , \tag{42}$$

and $\bar{t} = t$ . (43)

While the global symmetry transformations are given by $\bar{\mathbf{y}} = A\mathbf{y}$ and (43). The rotation symmetry transformations are given by $\bar{\mathbf{y}} = A\mathbf{y}$, where $y_1 = r(\cos(\theta + \alpha\lambda))$, $y_1 = r(\cos(\theta + \alpha\lambda))$; That is

$$\bar{\mathbf{y}} = \begin{pmatrix} \bar{y}_1 \\ \bar{y}_2 \end{pmatrix} = \begin{pmatrix} \cos\alpha\lambda & -\sin\alpha\lambda \\ -\sin\alpha\lambda & \cos\alpha\lambda \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} . \tag{44}$$

If the matrix $A$ is arbitrary it implies that all rotation symmetry transformations are ensured. Applying the same manner of calculations we obtain the exact symmetry transformations for the vector field

i.e. $(\alpha_1 \cos\theta + \alpha_2 \sin\theta)\partial_1$

$$\bar{\mathbf{y}} = H_4^{-1} \mathbf{y} \, ; \frac{d\bar{t}}{dt} = H_4^{-2} \tag{45}$$

where

$$H_4 = 1 + \lambda L^{-2} \boldsymbol{\alpha} \cdot \mathbf{x} , \ C = e^{\alpha\lambda} ,$$

$$\boldsymbol{\alpha} \cdot \mathbf{y} = \alpha_1 y_1 + \alpha_2 y_2 \ .$$

We note here also that this is true for the case when $\mathbf{x}$ is in three-dimensions.

## IV. ON THE KEPLER AND GENERALIZED KEPLER PROBLEMS IN 2-DIMENSIONS

In the treatment of the exact symmetries of the Kepler problem above we considered the cases where the natural variables for both the Kepler and generalized Kepler problems are respectively $v_1 = L^2 r^{-1} - \mu$, $v_2 = L$ and $v_1 = A^2 r^{-1} - \mu$, $v_2 = A$. We now consider the cases where the respective natural variables are constant multiples of the positive quadratic powers (or inverse quadratic powers) of the angular momentum, $u_1 = r^{-1} - \mu L^{-2}$, $u_2 = L$; and $u_1 = r^{-1} - \mu A^{-2}$, $u_2 = A$ respectively. And correspondingly we note that their generalizations videlicet $w_i = L_i^2 (r^{-1} - \mu L_i^{-2})$ for the Kepler problem and $w_i = A_i^2 (r^{-1} - \mu A_i^{-2})$ for the generalized Kepler problem reduction variables followed. We now in the following compute their exact symmetries using these new variables.

### A. The Kepler problem

In the reduced system for the Kepler problem the vector field $\alpha u_1 \partial_1$, possesses the flow $f(u_1, u_2, \theta) = (\bar{u}_1, \bar{u}_2, \bar{\theta})$ which satisfies the relations

$$\frac{d\bar{u}_1}{d\lambda} = \alpha\bar{u}_1 , \frac{d\bar{u}_2}{d\lambda} = 0 , \frac{d\theta}{d\lambda} = 0 . \tag{46}$$

The solutions to relations in (46) are

$$\bar{u}_1 = C u_1 , \bar{u}_2 = u_2 , \bar{\theta} = \theta \tag{47}$$

where $C = e^{\alpha\lambda}$ .

But $\bar{u}_2 = u_2$ and $\bar{\theta} = \theta \ \Rightarrow \ \bar{L} = L$, so that we have from the first relation in (46) that

$$\frac{1}{\bar{r}} - \mu L^{-2} = C\left(\frac{1}{r} - \mu L^{-2}\right).$$

$$\text{i.e.} \quad \bar{r} = H^{-1} r , \tag{48}$$

where $H = \left(\mu L^{-2} r + C(1 - \mu L^{-2} r\right)$, and $\frac{d\bar{t}}{dt} = H^{-2}$.

For the vector field $\alpha u_2 \partial_2$ we have the flow as the function $f(u_1, u_2, \theta) = (\bar{u}_1, \bar{u}_2, \bar{\theta})$ which satisfies the relations

$$\frac{d\bar{u}_1}{d\lambda} = 0, \frac{d\bar{u}_2}{d\lambda} = \alpha\bar{u}, \frac{d\theta}{d\lambda} = 0. \quad (49)$$

The solutions to the relations in (49) are respectively given by

$$\bar{u}_1 = u_1, \bar{u}_2 = Cu_2, \bar{\theta} = \theta. \quad (50)$$

The second expression in (50) implies $\bar{L} = CL$, thus the first expression gives

$$\frac{1}{\bar{r}} - \mu C^{-2} L^{-2} = \frac{1}{r} - \mu L^{-2}$$

i.e. $\bar{r} = H^{-1} r$, (51)

where $H = \left(1 + (C^{-2} - 1) r \mu L^{-2}\right)$

and from $\bar{L} = CL$ we have that

$$\bar{r}^2 \dot{\bar{\theta}} = C r^2 \dot{\theta} \Rightarrow \frac{d\bar{t}}{dt} = C^{-1} H^{-2}.$$

In the case of the vector field $\alpha\partial_\theta$ the flow is the function $f(u_1, u_2, \theta) = (\bar{u}_1, \bar{u}_2, \bar{\theta})$ such that

$$\frac{d\bar{u}_1}{d\lambda} = 0, \frac{d\bar{u}_2}{d\lambda} = 0, \frac{d\theta}{d\lambda} = \alpha \quad . \quad (52)$$

So that we have the solutions of the expressions in (52) given by

$$\bar{u}_1 = u_1,\ \bar{u}_2 = u_2,$$
$$\bar{\theta} = \theta + \alpha\lambda \ . \quad (53)$$

Following the usual substitutions the first and the second relations in (53) respectively produced the following transformations

$$\bar{r} = r, \quad (54)$$
$$\bar{t} = d + t.$$

The global transformation is denoted by $\bar{\mathbf{y}} = \mathbf{y}$, from which one obtains the rotation symmetry transformations denoted by (55) (for arbitrary matrix *B*)

$$\bar{\mathbf{y}} = B\mathbf{y}, \quad (55)$$

with $\bar{y}_1 = r\left(\cos(\theta + \alpha\lambda)\right),\ \bar{y}_2 = r\left(\sin(\theta + \alpha\lambda)\right)$.

For the case of the vector field $\alpha e^{i\theta}\partial_1$ the flow is a function $f(u_1, u_2, \theta) = (\bar{u}_1, \bar{u}_2, \bar{\theta})$. We only consider the cosine part of this symmetry and note that the sine part is easily deductive from this calculation. That is the flow satisfy the following relation (56) such that the cosine part is given by

$$\frac{d\bar{u}_1}{d\lambda} = \alpha\cos\theta, \frac{d\bar{u}_2}{d\lambda} = 0, \frac{d\theta}{d\lambda} = 0. \quad (56)$$

Solving (56) we have the following relations

$$\bar{u}_1 = u_1 + \alpha\lambda\cos\theta, \bar{u}_2 = u_2, \bar{\theta} = \theta. \quad (57)$$

On adapting the same substitutions as before we obtain

$$\bar{r} = H^{-1} r \quad (58)$$
$$\frac{d\bar{t}}{dt} = H^{-2}$$

where $H = (1 + \alpha\lambda r\cos\theta)$.

Globally we have that the first relation in (58) transformations as

$$\bar{\mathbf{y}} = H^{-1}\mathbf{y} \quad (59)$$
$$H = (1 + \lambda\boldsymbol{\alpha}\cdot\mathbf{y})$$

where $\boldsymbol{\alpha}\cdot\mathbf{y} = \alpha_1 y_1 + \alpha_2 y_2$.

### *B. The Generalized Kepler problem*

Following the same procedures for the Kepler problem diligently we have for the generalized Kepler problem the following transformations for the respective vector fields given as in the table1 below.

**Table1.** Exact symmetry transformations of the generalized Kepler problem in 2-dimensions.

| Vector fields : exact symmetry transformations |
|---|
| a) $\alpha u_1\partial_1$ : $\bar{r} = H^{-1}r \Rightarrow \bar{\mathbf{y}} = H^{-1}\mathbf{y}$<br>$H = \mu A^{-2} r + C(1 - \mu A^{-2} r) \Rightarrow H = \mu A^{-2}\mathbf{y} + C(1 - \mu A^{-2}\mathbf{y})$<br>$C = e^{\alpha\theta}, A = \left(r/g\right)^{1/2} r^{-2} L, \quad \frac{d\bar{t}}{dt} = \sqrt{\frac{\bar{r}g(r)}{rg(\bar{r})}}$ |
| b) $\alpha u_2\partial_2$ : $\bar{r} = H^{-1}r \Rightarrow \bar{\mathbf{y}} = H^{-1}\mathbf{y}$<br>$H = (1 + (C^{-2} - 1)\mu A^{-2} r) \Rightarrow H = (1 + (C^{-2} - 1)\mu A^{-2}\mathbf{y})$<br>$\frac{d\bar{t}}{dt} = C^{-1}\sqrt{\frac{\bar{r}g(r)}{rg(\bar{r})}}$ |
| c) $\alpha\partial_\theta$ : $\bar{r} = r$<br>$\Rightarrow \bar{\mathbf{y}} = B\mathbf{y}$ the rotation symmetry transformations<br>$\bar{t} = d + t$ |
| d) $\alpha e^{i\theta}\partial_1$ : $\bar{r} = H^{-1}r \Rightarrow \bar{\mathbf{y}} = H^{-1}\mathbf{y}$<br>$H = (1 + \alpha\lambda r\cos\theta) \Rightarrow H = (1 + \lambda\boldsymbol{\alpha}\cdot\mathbf{y})$<br>$\frac{d\bar{t}}{dt} = \sqrt{\frac{\bar{r}g(r)}{rg(\bar{r})}}$. |

We observed from the foregoing that the exact symmetry transformation of the vector field $\alpha w\partial_w$ where $w$ is the natural variable for reducing the radial component of the equation of motion to oscillator is invariant while that of the vector field $\alpha u_2\partial_2$ for the conservation law of the reduced systems is not when we take the constant multiples of the natural reduction variable(s) with the angular momentum for the exact symmetry transformation computations. We also saw that the exact symmetry transformations for the vector fields $\alpha\partial_\theta$ and $\alpha e^{i\theta}\partial_1$ are unaltered.

## V. EXACT SYMMETRY TRANSFORMATIONS OF KEPLER PROBLEM IN THREE-DIMENSIONS

The reduced system for (19) in three-dimensions have been known and is given by[14], [24],[25]

$$\begin{aligned} &u_1'' + u_1 = 0 \\ &u_2'' + u_2 = 0 \\ &u_3' = 0 \end{aligned} \tag{60}$$

where

$$\begin{aligned} u_1 &= \left(r^{-1} - L^{-2}\mu\right)\sin\theta - L^{-2}r^2\dot r\dot\theta\cos\theta, \\ u_1' &= -L^{-2}r^2\dot r\dot\phi\sin\theta, \\ u_2 &= L^{-1}r^2\dot\phi\sin\theta\cos\theta, \\ u_2' &= -L^{-1}r^2\dot\theta \\ u_3 &= r^2\dot\phi\sin^2\theta\ . \end{aligned} \tag{61}$$

We have reported [14],[28] that the exact symmetry transformations of dynamical systems in three-dimensions can be obtained from the Lie symmetries of the reduced systems. We list here the Lie symmetry generators of the reduced system (19). They consist of sixteen generators, one viz $\Gamma_1$ for the conservation law $u_3' = 0$ and the fifteen Lie symmetry generators for the pair of harmonic oscillators (60). They are as follows

$$\begin{aligned} &\Gamma_1 = u_3\partial_3,\ \Gamma_2^{jk} = u_j\partial_k, \\ &\Gamma_3 = \partial_\phi,\ \Gamma_{4\pm}^{j} = e^{\pm i\phi}\partial_j, \\ &\Gamma_{5\pm} = e^{\pm 2i\phi}(\partial_\phi + i\mathbf{u}\cdot\partial), \\ &\Gamma_{6\pm}^{j} = e^{\pm i\phi}u_j(\partial_\phi + i\mathbf{u}\cdot\partial) \end{aligned} \tag{62}$$

where $j,k = 1,2; \partial_j = \partial/\partial u_j$ and $\mathbf{u}\cdot\partial = u_1\partial_1 + u_2\partial_2$.

The symmetry representations of (62) in the original variables are very much complicated than that of section 2 above. We now compute the symmetry transformation generated by the vector field $\alpha\Gamma_2^{11} = \alpha u_1\partial_1$ for the Kepler problem. The symmetry transformation generated by this vector field is the transformation $f$ given by

$$\begin{aligned} &(\bar u_j, \bar\phi) = f(u_j, \phi) \text{ where} \\ &\bar u_1 = Cu_1, \bar u_2 = u_2,\ \bar u_3 = u_3, \\ &\bar\phi = \phi, C = e^{\alpha\lambda} \end{aligned} \tag{63}$$

from which it follows that

$$\bar u_1' = Cu_1',\ \bar u_2' = u_2',\ \bar L = L\ . \tag{64}$$

From $\bar u_2 = u_2$ we have that

$$\bar L^{-1}\bar r^2\dot{\bar\phi}\sin\bar\theta\cos\bar\theta = L^{-1}r^2\dot\phi\sin\theta\cos\theta\,. \tag{65}$$

Also we have that

$$\bar u_2' = u_2' \Rightarrow \bar L^{-1}\bar r^2\dot{\bar\theta} = L^{-1}r^2\dot\theta\,. \tag{66}$$

And from $L^2 = r^4(\dot\theta^2 + \dot\phi^2\sin^2\theta)$ we obtained

$$u_2^2\sec^2\theta + (u_2')^2 = 1 \tag{67}$$

Equations (65), (66), (67) imply that $\bar\theta = \theta$. Thus from the invariance of $u_2$ and $u_2'$ in (63) and (64) we note that

$$\sec\bar\theta = \sec\theta,$$

i.e.

$$\bar\theta = \theta\,. \tag{68}$$

The relations in (54) imply that

$$u_1 = \left(r^{-1} - \mu L^{-2}\right)\sin\theta - u_1'\theta'\cot\theta\,. \tag{69}$$

Since $L$, $\theta'$ and $\cot\theta$ are invariants of this transformation, the first relation in (63) becomes

$$\left(r^{-1} - \mu L^{-2}\right)\sin\theta - \bar u_1'\theta'\cot\theta = C\left(r^{-1} - \mu L^{-2}\right)\sin\theta - Cu_1'\theta'\cot\theta, \tag{70}$$

which reduces to

$$\left(\bar r^{-1} - \mu L^{-2}\right) = C\left(r^{-1} - \mu L^{-2}\right); \tag{71}$$

i.e. $\bar r = H_2^{-1}r$,

where $H_2 = \mu L^{-2}r + C(1 - \mu L^{-2}r)$.

The relation $\bar u_2' = u_2'$ in (64) implies that

$$\bar L^{-1}\bar r^2\dot{\bar\theta} = L^{-1}r^2\dot\theta$$

$$\text{i.e. } \frac{d\bar t}{dt} = \left(\bar r/r\right)^2 = H_2^{-2} \tag{72}$$

In view of equations (71), (72) and the relations $\bar\theta = \theta$, $\bar\phi = \phi$ in (63) and (64) the required exact symmetry transformation of the Kepler problem in three-dimensions for the vector field $\alpha u_1\partial_1$ is given by

$$\bar{\mathbf{y}} = H_2^{-1}\mathbf{y};\ \frac{d\bar t}{dt} = H_2^{-2}\ . \tag{73}$$

Thus, we hereby depict in the following Table2. some of the vector fields with their corresponding exact symmetry transformations below:

**Table2.** Vector fields and exact symmetry transformations in 3-dimensions (note the correction of misprints in [28])

| **Vector fields : exact symmetry transformations** |
|---|
| $\alpha\Gamma_2^{22} = \alpha u_2\partial_2$: $\bar r = Cr$, $\bar t = d + C^2 t$ ($d$ is constant) and $\bar{\mathbf{y}} = C\mathbf{y}$ is the Global exact symmetry. |

| |
|---|
| $\alpha\Gamma_2^{12} = \alpha u_1\partial_2 :$ $\bar r = r$, $\dfrac{d\bar t}{dt} = \dfrac{\theta'}{\theta' + \alpha\lambda L^{-2}\dot r\sin\theta}$, where $'$ denotes derivation with respect to $\phi$. |
| $\alpha\Gamma_2^{21} = \alpha u_2\partial_1 :$ $\bar r = H^{-1}r$, $\dfrac{d\bar t}{dt} = H^{-2}$ where $H = \left[1+\lambda\alpha r\cos ec^2\theta\cos\theta(1-\theta'^2\cot\theta)\right]$. |
| $\alpha\Gamma_4^1 = \alpha e^{i\phi}\partial_1$ : (The sine part is deductive from this cosine part obviously. $\bar r = H^{-1}r$, $\dfrac{d\bar t}{dt} = H^{-2}$ where $H = \left[1+\alpha\lambda r\cos ec\theta(\cos\phi-\theta'\cot\theta\sin\phi)\right]$ |
| $\alpha\Gamma_3 = \alpha\partial_\phi :$ $\bar r = r \Rightarrow \bar{\mathbf{y}} = B\bar{\mathbf{y}}$ $\bar t = d+t$ |

We note [28] that the rotation symmetry transformations of the Kepler problem in three-dimension are obtainable from the vector field $\alpha\Gamma_3 = \alpha\partial_\phi$ which has the rotation symmetries denoted by $\bar{\mathbf{y}} = B\mathbf{y}$, $B$ is a scalar 3x3-matrix. That is by setting

$$\mathbf{y} = (r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta)$$

as the spherical coordinates of the motion we have the rotation symmetry transformation about the $z$-axis as

$$\begin{pmatrix}\bar y_1\\ \bar y_2\\ \bar y_3\end{pmatrix} = \begin{pmatrix}\cos\alpha\lambda & -\sin\alpha\lambda & 0\\ \sin\alpha\lambda & \cos\alpha\lambda & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}y_1\\ y_2\\ y_3\end{pmatrix}. \qquad (74)$$

If $B$ is an arbitrary rotation matrix then the rotation symmetry is globally defined.

We have reported that the exact symmetries of the remaining six vector fields are computable as well following the same method diligently. We note also that the Hamilton vector $\mathbf{K}$ for the Kepler problem [25] is given by $\mathbf{K} = \dot{\mathbf{y}} - \mu L^{-2}r^{-1}(\mathbf{L}\wedge\mathbf{y})$ (This is a constant multiple of the expression for $\mathbf{K}$ given in ref.[14]). This expression for $\mathbf{K}$ yields the relation [14],[26],[27]

$$K_\pm = K_1 \pm iK_2 = (\upsilon_1 \pm i\upsilon_1')e^{\pm i\phi}, \qquad (75)$$

where

$$\upsilon_1 = \dot r\sin\theta + r(1-\mu L^{-2}r)\cos\theta\dot\theta,$$

$$\upsilon_1' = (1-\mu L^{-2}r)\sin\theta\dot\phi. \qquad (76)$$

We note that one could consider instead of (60), the same system of equations with $u_1$ replaced with $\upsilon_1$, and its Lie symmetries to obtain the exact symmetry transformations of the original system.[14],[26],[27],[28],[29]

However when we utilized the reduction variables which are constant multiple of the one above by the angular momentum say

$$v_1 = \left(L^2r^{-1}-\mu\right)\sin\theta - r^2\dot r\dot\theta\cos\theta,$$

$$v_2 = r^2\dot\phi\sin\theta\cos\theta,$$

$$v_3 = r^2\dot\phi\sin^2\theta$$

instead for the above computations the symmetry transformations for the corresponding vector field $\alpha w\partial_w$, we obtain the exact symmetry transformations given by

$$\bar{\mathbf{y}} = H_2^{-1}\mathbf{y}; \ \frac{d\bar t}{dt} = H_2^{-2} \qquad (77)$$

where $H_2 = \mu L^{-2}\mathbf{y} + C(1-\mu L^{-2}\mathbf{y})$,

which are unchanged (invariant) compared with (73). For the other vector fields we obtain the following exact symmetry transformations in table3 below.

**Table3.** Vector fields and exact symmetry transformations in 3-dimensions when the reduction variable is a constant multiple of the quadratic powers of the angular momentum

| **Vector fields : exact symmetry transformations** |
|---|
| $\alpha\Gamma_2^{22} = \alpha v_2\partial_2 :$ $\bar r = r$, $\bar t = d+t$ ($d$ is constant) and $\bar{\mathbf{y}} = \mathbf{y}$ is the Global exact symmetry transformations only exist at the identity. |
| $\alpha\Gamma_2^{12} = \alpha v_1\partial_2 :$ $\bar r = r$, $\dfrac{d\bar t}{dt} = \dfrac{\theta'}{\theta' + \alpha\lambda\dot r\sin\theta}$ where $'$ denotes derivation with respect to $\phi$. |
| $\alpha\Gamma_2^{21} = \alpha v_2\partial_1 :$ $\bar r = H^{-1}r$, $\dfrac{d\bar t}{dt} = H^{-2}$ where $H = [1+\alpha\lambda L^{-2}r\cot\theta\cos ec\theta(1+\theta'^2\cot\theta)]$. |
| $\alpha\Gamma_4^1 = \alpha e^{i\phi}\partial_1$ : (The sine part is deductive from this cosine part obviously), $\bar r = H^{-1}r$, $\dfrac{d\bar t}{dt} = H^{-2}$;where $H = \left[1+\alpha\lambda L^{-2}r\cos ec\theta(\cos\phi-\dot\theta\dot\phi^{-1}\cot\theta\sin\phi)\right]$. |
| $\alpha\Gamma_3 = \alpha\partial_\phi :$ $\bar r = r$ $\Rightarrow \bar{\mathbf{y}} = B\mathbf{y}$ (rotation symmetry transformations) $\bar t = d+t$. |

## VI. CONCLUDING REMARKS

The exact symmetry transformations of the vector field $\alpha w\partial_w$ where *w* is the natural variable for reducing the radial component of the equation of motion to oscillator is invariant while that of the other vector fields may not when we take the constant multiples of the natural variable(s) with the angular momentum for the exact symmetry transformation computations in 2-dimensions. One could varify from the comparison of equations (32)-(35) and (48), (37)-(39) and (51), (42)-(44) and (54), (45) and (58). We have also noted that in the case for 3-dimensions the situation is not different for the vector field $\alpha w\partial_w$, but for other vector fields the symmetry transformations vary as the table2 and table3 clearly shown.

We have demonstrated [26],[28] that the exact symmetry transformations of the Kepler problem can be calculated from the symmetries of its reduced systems rather than just obtaining the symmetry generators (vector fields) that are often complicated in their representations as they are in their nonlocal symmetry forms in system (29) above. Hitherto (c.f. [26],[28]) the exact symmetry transformations computation as demonstrated above is new. We report here that we have devised and utilized this computational method to obtain the exact symmetries of other dynamical systems that are reducible to systems of oscillator(s) and conservation law(s). Consequently the complicated nonlocal symmetry representations of dynamical systems are simply realizable in their simple explicit forms as shown using the Kepler problem as a vehicle. In our recent works the Kepler problem with drag, the generalized Kepler problem, the MICZ problem (though much more cumbersome than that of the former) and a host of other dynamical systems with complicated nonlocal symmetries have proven to admit this computational method for obtaining their exact symmetries in both two- and three-dimensions. These are subject for further discussions. The computation of the exact symmetry transformations of the generalized Kepler problem in 3-dimensions and that of the Kepler problem with drag are in progress.

## REFERENCES


[1] G.W. Bluman and S. Kumei. *Symmetries and Differential Equations. Springer- Verlag New York, Inc. (1989).*

[2] P. J. Olver. *Applications of Lie Groups to Differential Equations. 2nd ed. Springer- Verlag New York, Inc. (1993).*

[3] L.V. Ovsiannikov. *Group Analysis of Differential Equations. Translation edited by W.F. Ames. Academic Press New York, (1982)*

[4] H. Stephani. Edited by M. MacCallum. *Differential equations: Their solution using Symmetries. Cambridge University Press . New York, (1989).*

[5] CRC Handbook of Lie Group Analysis of Differential Equations. Vol. 1: "*Symmetries, Exact Solutions and Conservation Laws", Editor: Ibragimov N.H.,CRC press, Boca Raton, (1994).*

[6] CRC Handbook of Lie Group Analysis of Differential Equations. Vol. 3: "*New Trends in Theoretical Developments",Editor: Ibragimov N. H., CRC press, BocaRaton, (1996).*

[7] K. Andriopoulos, P.G.L. Leach and G.P. Flessas. "*Complete symmetry Groups of Ordinary Differential Equations and Their Integrals: Some Basic consideration"s. J. Math. Anal. and Appl. 262(2001). Pp 256-273.*

[8] P.G.L. Leach and S.E. Bouquet. "*Symmetries and Integrating factors". J. Nonlinear Math. Phys. 9 N2 (2002). Pp 73-91.*

[9] K. Andriopoulos and P.G.L. Leach. "*The Economy of complete symmetry groups for linear higher dimensional systems". J. Nonlinear Math. Phys. 9 N2 (2002). Pp 10-23.*

[10] P.G.L. Leach and G.P. Flessas. "*Generalizations of the Laplace-Runge-Lenz vector". J. Nonlinear Math. Phys. 10 N 3 (2003). Pp 340-423.*

[11] J. Krause. "*On the complete symmetry group of the classical Kepler system". J. Math. Phys. 35 N11 (1994). Pp 5734-5748.*

[12] 12] K.S. Govinder and P.G.L. Leach. "*A group theoretic approach to a class of second-order ordinary differential equations not possessing Lie point symmetries". J. Phys A: Math. Gen. 30 (1997). Pp 2055-2068.*

[13] M.C. Nucci. "*The complete Kepler group can be derived by Lie group analysis". J. Math. Phys. 37 N4 (1996).Pp 1772-1775.*

[14] F.I. Arunaye and H. White. "*On the Ermanno-Bernoulli and Quasi- Ermanno-Bernoulli constants for linearizing dynamical systems". WSEAS J. Transactions on Mathematics Vol. 7, issues 3, (2008). Pp 71-77.*

[15] K.S. Govinder and P.G.L. Leach. "On the determination of non-local symmetries*". J. Phys. A: Math. Gen. 28(1995). Pp 5349-5359.*

[16] M. Aguirre and J. Krause. "$SL(3,R)$ as the group of symmetry transformations for all one-dimensional linear systems*". J. Math. Phys. 29 N1 (1988). Pp 9-15.*

[17]A. González-López. "Symmetries of linear systems of second-order differential equations*". J. Math. Phys. 29 N5 (1988). Pp 1097-1105.*

[18] W. Sarlet, F.M. Mahomed and P.G.L. Leach. "Symmetries of non-linear differential equations and linearization*". J. Phys. A: Math. Gen. 20 (1987).Pp 277-292.*

[19] C.E. Wulfman and B.G. Wybourne. "The Lie group of Newton's and Lagrange's equations of the Harmonic oscillator*". J. Phys A: Math. Gen. 9 N4 (1976). Pp 507- 518.*

[20] P.G.L. Leach, M.C. Nucci and S. Cotsakis. "Symmetry, Singularities and Integrabilities in complex Dynamics V; Complete symmetry Groupof certain Relativistic spherically symmetric systems*" J. Nonlinear Math. Phys. 8 N4 (2001). Pp 475-490.*

[21] M. C. Nucci and P.G.L. Leach. "Determination of Nonlocal symmetries by the Technique of reduction of order*". J. Math. Anal. and Appl. 251 (2000). Pp 871-884.*

[22] P.G.L. Leach and M.C. Nucci. "Reduction of the classical MICZ-Kepler problem to a two-dimensional linear isotropic harmonic oscillator*". J. Math. Phys. 45 N9 (2004). Pp 3590-3604.*

[23] P.G.L. Leach. "The first integrals and orbit for the Kepler problem with drag*". J. Phys. A: Math. Gen. 20 (1987). Pp 1997-2002.*

[24] M.C. Nucci and P.G.L. Leach. "The harmony in the Kepler and related problems*". J. Math. Phys. 42 N2 (2001).Pp 746-764.*

] P.G.L. Leach, K. Adriopoulos and M.C. Nucci. "The Ermanno-Bernoulli constants and representation of the complete symmetry groups of the Kepler problem*". J Math. Phys. 44 N9(2003). Pp 4090- 4106.*

[26] F.I. Arunaye. **"**Computing exact symmetries of dynamical systems from their reduced system of equations can be

Interesting II". *(To be published in WSEAS J. Transactions on Mathematics)*

[27] F.I. Arunaye and H. White. "On the Ermanno-Bernoulli and Quasi- Ermanno-Bernoulli constants for linearizing dynamical systems". *(presented at the 9th Int. conf. WSEAS )*

[28] F.I. Arunaye. *WSEAS Book: RECENT ADVANCES in SYSTEMS, COMMUNICATIONS & COMPUTERS: Selected Papers from WSEAS Conference in Hangzhou, China (April 6-8, 2008). Pp.83.*

[29] F.I. Arunaye and H. White. Recent Advances On Applied Mathematics. Proceedings of the American Conference on Applied Mathematics (MATH'08), WSEAS press, www.wseas.org Cambridge, Massachusetts, USA, March 24-26, 2008.