Modelling and Solution for Assignment Problem

Liying Yang, Minghong Nie, Zhenwei Wu, and Yiyong Nie

Abstract**—**In this paper, the mixed-integer linear programming (MILP) of minimax assignment is formed, and a solution called *Operations on Matrix* is presented and compared with the solutions of exhaustion and MILP. Theoretical analyses and numerical tests show that the operations on matrix are efficient well-implied enumeration for both minimax and global-minimum assignment problems.

Keywords**—**Assignment problem, method of exhaustion, mixed-integer linear programming (MILP), operations on matrix.

I. Introduction

THE global-minimum assignment problem is described, see, e.g. [1], by language as: There are n people and n tasks. Each person completes and only completes a task. The payment in each person for a task is given. The problem is, which person completes which task such that the total payment is minimum?

Let x_{ij} be the n^2 0-1 decision variables, where $x_{ij}=1$ represents person i for task j, and $x_{ij}=0$, otherwise. The global-minimum assignment problem may be described as follows,

$$\begin{aligned} \min \quad & z=\sum_{i=1}^{n}\sum_{j=1}^{n}c_{ij}x_{ij}, \\ st. \quad & \sum_{i=1}^{n}x_{ij}=1 \quad (j=1,\dots,n), \\ & \sum_{j=1}^{n}x_{ij}=1 \quad (i=1,\dots,n), \\ & x_{ij}=0 \text{ or } 1, \quad i,j\in N\equiv\{1,\dots,n\}, \end{aligned} \tag{1}$$

where c_{ij} represents the payment in person i for task j, elements c_{ij} give an $n\times n$ real cost matrix. The problem (1) is an integer linear programming (ILP). If the constraint condition $x_{ij}=0$ or 1 is replaced by $0\le x_{ij}\le 1$ in (1), then it becomes the relaxed linear programming (RLP). We may naturally solve the problem (1) by ILP methods, for instance by the isometric surface method, see [2]. All the ILP methods are derived form the RLP methods, and there are n^2 decision variables in (1). The flops to solve once RLP for (1) are usually $O((n\times n)^{3.5})=O(n^7)$, so that, from the point of view of computational complexity, the ILP methods for (1) aren't better than the Hungarian algorithm, see, e.g. [1].

If $-c_{ij}$ represents the profit to gain by person i for task j in (1), then the global-minimum assignment problem may be regarded as the assignment problem for maximum gross profit. Let $C=(c_{ij})$ be the negative profit matrix, and solve the global-minimum assignment problem for the negative profit matrix. The negative value of objective function is the maximum gross profit.

If the "payment" is understood as cost depletion, then the objective function of the global-minimum assignment problem (1) is reasonable. However, if the "payment" is understood as time, then the objective function of (1) is not always reasonable. The "total time" to expend in people for their tasks is not interested here. It is important to minimize the maximal time to expend in some person for his task. Therefore the objective function of (1) should be modified. That is,

$$\begin{aligned} \min \quad & z=\max_{i,j}\{c_{ij}x_{ij}\}, \\ st. \quad & \sum_{i=1}^{n}x_{ij}=1 \quad (j=1,\dots,n), \\ & \sum_{j=1}^{n}x_{ij}=1 \quad (i=1,\dots,n), \\ & x_{ij}=0 \text{ or } 1, \quad i,j\in N\equiv\{1,\dots,n\}. \end{aligned} \tag{2}$$

This is the mathematical model of the minimax assignment problem. It is an integer programming, where the objective function is nonlinear. There is no evident relation between the solutions of (2) and (1).

Assignment problem, linear programming (LP), ILP, MILP and their applications have been researched for many years, see, e.g. [1,3,4,6,7,8].

In this paper, the method of exhaustion for the integer programming (2) is analyzed first and foremost. Then the MILP of the minimax assignment problem (2) is formed, and the isometric surface method for the MILP is discussed. A solution called *Operations on Matrix* is presented for the minimax assignment problem (2) is presented and applied to solving the global-minimum assignment problem (1). Finally, the analyses of computational complexity and numerical tests show that the operations on matrix are efficient well-implied

Manuscript received Jan. 7, 2008: Revised version received June 23, 2008. This work was supported by the National High Technology Research and Development Program "863"-Program (2007AA04Z250-1).

L. Y. Yang is with the State Key Laboratory of Robotics, Shenyang Institute of Automation (SIA), Chinese Academy of Sciences (CAS), and the Graduate School, CAS. e-mail: (yangliying@sia.cn).

M. H. Nie is with Shenyang Digital China Limited. Shenyang, 110004, P.R.China. email: (niemh@digitalchina.com).

Z. W. Wu is with the State Key Laboratory of Robotics, SIA, CAS, No.114, Nanta St., Shenyang, 110016, P.R.China. email: (wuzw@sia.cn).

Y.Y. Nie, corresponding author, is with the State Key Laboratory of Robotics, SIA, CAS, No.114, Nanta St., Shenyang, 110016, P.R.China (phone: +86 24 23970276; fax: +86 24 23970021;email: nyy@sia.cn).

enumeration, see [5], for both minimax and global-minimum assignment problems.

II. THE METHOD OF EXHAUSTION FOR (2)

The basic idea of the method of exhaustion is to compare all the objective functions of feasible solutions directly or indirectly. After the comparison is completed, the optimal solution is found naturally. Unfortunately, number of the feasible solutions for general integer programming or mixed-integer programming increases with scale of problem too fast for any computer.

The feasible solution of the minimax assignment problem (2), for example, is an assignment Γ corresponding to n "1" and n^2-n "0" of $n\times n$ 0-1 matrix, where all the elements "1" are located in different rows and columns of matrix. The objective function z of each assignment Γ is the maximal element corresponding to "1" in the cost matrix $C=(c_{ij})$. The assignment Γ minimizing the objective function z is a solution of (2). The solution is not unique possibly, but the value of the optimal objective function is unique.

Let $\{j_1, j_2, ..., j_n\}$ be a permutation of $\{1,2,...,n\}$. Set $a_{1j_1}=1$, $a_{2j_2}=1$, $\dots$, $a_{nj_n}=1$ in $n\times n$ 0-1 matrix. Thus a feasible assignment Γ is formed, where the objective function $z_\Gamma=\max\{c_{1j_1}, c_{2j_2}, ..., c_{nj_n}\}$.

Each permutation $\{j_1, j_2, ..., j_n\}$ corresponds to an assignment Γ, so that the sum total of assignments is $n!$. When n is big enough, in Stirling's factorial formula, $n!\approx\sqrt{2\pi n}(n/e)^n$. This shows that the sum total of assignments increases in exponent with n.

It is not convenient to solve (2) by the method of exhaustion on PC when n exceeds by 10.

III. AN MILP MODEL

Introducing a real variable y_{n+1}, we can improve the minimax assignment problem (2) into MILP. The MILP equivalent to (2) is as follows,

$$\begin{aligned}
&\min\ \ z=y_{n+1},\\
&st.\quad \sum_{i=1}^{n}x_{ij}=1\quad (j=1,...,n),\\
&\qquad\ \sum_{j=1}^{n}x_{ij}=1\quad (i=1,...,n),\\
&\qquad\ y_{n+1}-c_{ij}x_{ij}\ge 0,\quad (i,j=1,...,n),\\
&\qquad\ x_{ij}=0 \text{ or } 1,\quad i,j\in N\equiv\{1,...,n\}.
\end{aligned}\tag{3}$$

The MILP (3) is equivalent to the problem (2) clearly, where there is only one real variable y_{n+1}, but there are more n^2 constraint inequalities in comparison with (2).

The isometric surface method for ILP, see [2], may be also applied to MILP, see [3]. In order to apply the isometric surface method for the MILP (3), (3) should be changed into the following canonical form,

$$\begin{aligned}
&\max\ \ z=-y_{n+1},\\
&st.\quad \sum_{i=1}^{n}x_{ij}\ge 0.85,\ -\sum_{i=1}^{n}x_{ij}\ge\text{-}1.15,\quad (j=1,...,n),\\
&\qquad\ \sum_{j=1}^{n}x_{ij}\ge 0.85,\ -\sum_{j=1}^{n}x_{ij}\ge\text{-}1.15,\quad (i=1,...,n),\\
&\qquad\ y_{n+1}-c_{ij}x_{ij}\ge 0,\quad (i,j=1,...,n),\\
&\qquad\ x_{ij}=0 \text{ or } 1,\quad i,j\in N\equiv\{1,...,n\}.
\end{aligned}\tag{4}$$

The isometric surface method for the MILP (4) is derived form the solution of the following RLP,

$$\begin{aligned}
&\max\ \ z=-y_{n+1},\\
&st.\quad \sum_{i=1}^{n}x_{ij}\ge 0.85,\ -\sum_{i=1}^{n}x_{ij}\ge\text{-}1.15,\quad (j=1,...,n),\\
&\qquad\ \sum_{j=1}^{n}x_{ij}\ge 0.85,\ -\sum_{j=1}^{n}x_{ij}\ge\text{-}1.15,\quad (i=1,...,n-1),\\
&\qquad\ y_{n+1}-c_{ij}x_{ij}\ge 0,\quad (i,j=1,...,n),\\
&\qquad\ x_{ij}\ge 0,\ -x_{ij}\ge\text{-}1.0,\quad (i,j=1,...,n).
\end{aligned}\tag{5}$$

There are n^2+1 variables and $3n^2+4n-2$ constraint inequalities in the RLP (5). The flops to solve once RLP (5) are more than $O(n^7)$. In order to get an optimal solution of the MILP (4), generally we need to solve the RLP (5) for n^2 times. Therefore the flops to solve the MILP (4) are generally $O(n^9)$ using the isometric surface method.

What is more, the n^2+1 hyperplanes to determine optimal solution of the RLP (5) constitute ill-condition linear equations. Generally we cannot get the solution of (5) whatever using the isometric plane method, see [4], or MATLAB simplex method. Thus it can be seen that some ILP or MILP of decision problem are difficult to solve.

IV. THE OPERATIONS ON MATRIX FOR (2)

The operations on matrix for the minimax assignment problem (2) find the solution directly from the cost matrix $C_n=(c_{ij})$ and the objective function, here the subscript n of cost matrix C_n represents the order of matrix.

We arrange the n^2 payments c_{ij} in order from small to large, where some payments are possibly equal each other. So that these n^2 payments are arranged in order from small to large into the r $(1\le r\le n^2)$ real numbers with different levels $R_1, R_2, ..., R_r$, $(R_1<R_2<...<R_r)$, where there are s_k payments c_{ij} equal to R_k $(1\le k\le r)$. Clearly $s_k\ge 1$ and $\sum_{k=1}^{r}s_k=n^2$. Let R be a real number larger than or equal to R_r, R may be also regarded as the sign ∞. If $r=1$, namely all the $c_{ij}=R_r$, then there are $n!$ solutions for (2), the objective function value of every solution is R_r. There is no harm in supposing $r>1$.

It is easy to see $R_r=\max_i\max_j\{c_{ij}\}=\max_j\max_i\{c_{ij}\}$ and. $R_1=\min_i\min_j\{c_{ij}\}=\min_j\min_i\{c_{ij}\}$. But $R^r_{\min}=\max_i\min_j\{c_{ij}\}$ and

$R^c_{\min} = \max_j \min_i \{c_{ij}\}$ aren't equal each other possibly. Let $R^o_{\max} = \max\{R^r_{\min}, R^c_{\min}\}$. We have,

Proposition 1. The optimal objective function value of the minimax assignment problem (2), z^*, is larger than or equal to $R^o_{\max}$.

[Proof] Each person completes and only completes a task because of the constraint conditions of (2). The payment to expend in the last person for his task is larger than or equal to $R^r_{\min}$, and larger than or equal to $R^c_{\min}$ as well. Therefore the payment $z^* \geq R^o_{\max}$. □

All the elements larger than $R^o_{\max}$ in the cost matrix $C_n = (c_{ij})$ are covered by R. This covering is called basic covering. Via Proposition 1, after the basic covering of C_n, $R^o_{\max}$ is the largest uncovering element, and there is an uncovering element in each row and each column at least. If there exists a feasible assignment Γ in all the uncovering elements, then Γ is an optimal solution of (2).

Proposition 2. Assume that, after the basic covering of C_n, there are s ($s \geq 2$) uncovering elements in each row and each column at least. Removing the row and column where some uncovering element is located at the cross point, we get the residual matrix of $n-1$ order, C_{n-1}, where there are $s-1$ uncovering elements in each row and each column at least.

[Proof] It is sufficient to prove for $s = 2$. Assume that, after the basic covering of C_n, there are two uncovering elements c_{ij_1} and c_{ij_2} ($c_{ij_1}, c_{ij_2} \leq R^o_{\max}$) in the i-th row. Since there are two uncovering elements in the j_1-th column or the j_2-th column of C_n at least, there is an uncovering element less than or equal to $R^o_{\max}$ in each row and each column of C_{n-1} at least, after the i-th row and the j_1-th column or the i-th row and the j_2-th column of C_n are removed.

Similarly, assume that, after the basic covering of C_n, there are two uncovering elements $c_{i_1 j}$ and $c_{i_2 j}$ in the j-th column. There is an uncovering element less than or equal to $R^o_{\max}$ in each row and each column of C_{n-1} at least, after the j-th column and the i_1-th row or the j-th column and the i_2-th row of C_n are removed. □

Set $t := n, C_t := C_n$ in the beginning, and make the basic covering of the matrix $C_t = (c_{ij})$. Find the sequence number of $R^o_{\max}$ in series $R_1, R_2, ..., R_r$, and let $R_k = R^o_{\max}$. The elements larger than or equal to $R^t_{k+1} = R_{k+1}$ in the cost matrix C_t have been covered by R. The algorithm, operations on matrix, for the minimax assignment problem (2) consists of the following steps:

① Investigate the uncovering elements in every row and column of C_t. Suppose there exists row or column where only one element is uncovered, and the uncovering element is c_{pq}. Record the minimal c_{pq} (optimal assignment in single entry), and remove the p-th row and the q-th column of the matrix C_t. The residual matrix of $t-1$ order is C_{t-1}. Set $t := t-1$, $C_t := C_{t-1}$, and turn to Step ① or ②.

② Investigate the uncovering elements in every row and column of C_t. Suppose that each row or column has two uncovering elements at least. Turn to Step ③.

③ The optimal objective function value of C_t is R^t_k. Each row or column of C_t has two uncovering elements at least. Deflate C_t with one order according to Proposition 2. There are many selections of recorded uncovering element c_{pq} during the deflation. The selective sequence is as follows,

I) Removed the p-th row or the q-th column contains the least uncovering elements,

II) The number of elements contained in the p-th row and the q-th column is minimum,

III) Record the minimal c_{pq} (optimal assignment in single entry).

Set $t := t-1, C_t := C_{t-1}$ after deflation. Turn to Step ① or ② if $t \geq 2$.

If $\max\{c_{pq} \mid t = n, n-1, \cdots, 1\}$ is larger than $R_k = R^o_{\max}$, then the R_k covering doesn't make a feasible assignment. Consider the R_k covering for $k := k+1$, and perform again the next covering circle of Step ① - ③, until $\max\{c_{pq} \mid t = n, n-1, \cdots, 1\}$ is equal to R_k (the optimal objective function value).

Proposition 3. Assume that the n^2 payments of cost matrix C_n are arranged in order from small to large into the r real numbers with different levels $R_1, R_2, ..., R_r$, where there are s_i payments c_{ij} equal to R_i ($i = 1, \cdots, r$), and that, after the basic covering of C_n, the sequence number of $R^o_{\max}$ in series $R_1, R_2, ..., R_r$ is k ($1 \leq k \leq r$). Then using $r-k+1$ covering circles at most, the operations on matrix find a solution of the minimax assignment problem (2).

[Proof] Since, for the first covering circle, there is an uncovering element in each row and each column at least, two uncovering elements are recorded at least in the first covering circle. If n uncovering elements are recorded in the first covering circle, then the operations on matrix find an optimal solution with the basic covering. Otherwise, increasing a covering circle, for example $k := k+1$, the level of uncovering elements is increased to R_{k+1}, and the number of uncovering elements is increased to $\sum_{i=1}^{k+1} s_i$. Since $\sum_{i=1}^{r} s_i = n^2$, the operations on matrix find a solution of (2) using $r-k+1$ covering circles at most. □

We shall prove that the operations on matrix give the optimal solution of the problem (2) in well-implied enumeration. An enumerative algorithm is called *well-implied enumeration*, see [5], that means it can obtain the optimal solution in fact with only a few enumerated feasible solutions, where "only a few" is understood as the number of enumerated feasible solutions is a polynomial function of dimension of solution space and number of constraints, and "in fact" is understood as some probability to obtain the optimal solution is equal to 1. The operations on matrix, via Proposition 3, obtain a solution of the problem (2) with only a few enumerated feasible solutions. It is necessary to prove that some probability to give the optimal solution with operations on matrix is equal to 1.

The following hypotheses concerning probability calculation are to conform to reality.

U1) All the elements of cost matrix C are equally likely distribution;

U2) After R covering of C, all the elements are divided into two classes, covering elements and uncovering elements, there is no difference between the elements of each class.

Proposition 4. Assume that the number of uncovering elements of cost matrix C, where there is an uncovering element in each row and each column at least, is m. Then $n \le m \le n^2$, and the probability $p(Z_1, m)$ to obtain the optimal solution with operations on matrix for C is equal to 1 when $m = n$ or $n^2 - n \le m \le n^2$.

[Proof] Since, for the n-order cost matrix C, there is an uncovering element in each row and each column at least, clearly $n \le m \le n^2$. If $m = n$, then $p(Z_1, n) = 1$ because the n uncovering elements are distributed on different row and column of matrix. Step ① of operations on matrix can give the optimal solution.

If $m \ge n^2 - n$, then $p(Z_1, m) = 1$ as well. This is easily to prove by inductive method. In fact, $p(Z_1, m \mid n^2 - n \le m \le n^2) = 1$ holds clearly for the 3-order cost matrix with 6 uncovering elements at least. Suppose that the conclusion holds for the k-order cost matrix. Let $n = k+1$. If there is only one uncovering element in some row or column of cost matrix, then, after deflation of matrix with operations on matrix, there is one covering element at most in the k-order cost matrix, and the conclusion holds clearly for the k-order matrix. If there are two uncovering elements in each row and each column of cost matrix at least, then, after deflation of matrix with operations on matrix, there are $(k+1)^2 - 3(k+1) + 2 = k^2 - k$ uncovering elements at least for the k-order cost matrix, and the conclusion holds as well. □

It is possible that there is no feasible solution of the assignment problems for the basic covering of the cost matrix C when $n < m < n^2 - n$. Therefore it is possible that $p(Z_1, m) < 1$ for $n < m < n^2 - n$.

Proposition 5. Assume that the number m of uncovering elements of cost matrix C_n satisfies $m \ge n(n+1)/2$, and that, during the deflation of each t-th order matrix ($t = n, n-1, \cdots$) with operations on matrix, t uncovering elements are removed at most , then the probability to obtain the optimal solution by operations on matrix $p(Z_1, m) = 1$.

[Proof] It is easy to prove the proposition by inductive method. In fact, the equality $p(Z_1, m \mid m \ge n(n+1)/2) = 1$ holds for $n = 2$ clearly. Suppose that the proposition holds for $n = s-1$, namely $p(Z_1, m_{s-1} \mid m_{s-1} \ge s(s-1)/2) = 1$, where m_{s-1} represents the number of uncovering elements of the ($s-1$)-th order matrix. Let $n = s$, and the number of uncovering elements of the s-th order matrix $m_s \ge s(s+1)/2$. Since s uncovering elements are removed at most when the s-th order matrix is deflated with operations on matrix, the number of uncovering elements of the ($s-1$)-th order deflated matrix $m_{s-1} \ge \; s(s+1)/2 - s = s(s-1)/2$. Therefore $p(Z_1, m_s \mid m_s \ge s(s+1)/2) = 1$. □

It is easy to see that, for any covering of cost matrix C_n, the exit of operations on matrix is always Step ① whether the covering has feasible solution of problem (2) or not. Assume that there are s uncovering elements in each row and each column of some covering at least, thus the number of uncovering elements $m \ge sn$, where $s \ge 1$. If $s = 1$ holds from start to finish during the deflations of matrix with operations on matrix, then the solution is clearly correct. However, if $s > 1$ occurred during the deflations, then the solution is not always correct. For example,

$$C = \begin{pmatrix} 10001 \\ 01001 \\ 11110 \\ 11110 \\ 01001 \end{pmatrix},$$

where 1 represents uncovering element, and 0 covering element. There is a feasible solution at least for matrix C and there are 14 uncovering elements. $s = 2$. Four uncovering elements are removed when the matrix is deflated with Step ③ of operations on matrix, and there are 4 selections of recorded uncovering element. If row 1 and column 5 are removed, then the operations on matrix cannot give feasible solution. With the exception of this, the operations on matrix find the feasible solution. Therefore, when $s > 1$, the operations on matrix mistake possibly the covering that has solution for the covering that has no solution.

In order to avoid possibly mistaken about this case, the operations on matrix should deal specially with the cost matrix. That is, the recorded element is modified artificially as the sign ∞, then the operations on matrix are applied to the modified cost matrix again. This special treatment guarantees that the operations on matrix give always a correct solution of the problem (2).

Proposition 6. Assume that there are s uncovering elements in each row and each column of cost matrix C_n at least, where $s \ge 1$. If the cost matrix is modified artificially when the conclusion of no feasible solution is given by operations on matrix for $s > 1$, then the probability to obtain the correct solution of problem (2) by operations on matrix $p(Z_2, m, s) = 1$.

[Proof] Since the recorded element becomes into ∞ and is covered for each artificial modification, there are $s-1$

uncovering elements at least in each row and each column of cost matrix modified artificially. If $s-1>1$ and the conclusion of no feasible solution is still given by operations on matrix, then the cost matrix is artificially modified again. Therefore, the result to perform operations on matrix for some cost matrix modified artificially is that a feasible solution is obtained, or that $s=1$ holds from start to finish during the deflations and no feasible solution is obtained. The operations on matrix give a correct solution always, namely $p(Z_2, m, s)=1$. □

Proposition 7. The operations on matrix with modifying artificially cost matrix for the minimax assignment problem (2) are well-implied enumeration. Using $r-k+1$ covering circles at most, the operations on matrix find the optimal solution, where r and k are specified as in Proposition 3.

[Proof] If there is feasible solution of the problem (2) for the basic covering of cost matrix C, then, via Propositions 4 - 6, the optimal solution is found by the first covering circle. Otherwise, the next covering circle is performed. Increasing a covering circle, one level of uncovering elements is increased. Using $r-k+1$ covering circles at most, via Proposition 3, a feasible solution is found. Suppose that the feasible solution is found at the earliest j-th covering circle. Then there is no feasible solution before the j-th covering circle. That is, in order to find feasible solution, $j-1$ levels of R covering for cost matrix should be increased from the level $R^o_{\max}$ of basic covering. Therefore the optimal objective function value equals R_{k+j-1}, where $R_k = R^o_{\max}$. The solution found by operations on matrix in the j-th covering circle is the optimal solution. □

To perform Step ① - ③ for the t-order matrix, the flops are $O(t^2)$; t decreases from n to 1, so that, to perform a covering circle, the flops are $O(n^3)$. To perform $r-k+1$ covering circles at most, the optimal solution is found. Therefore, to neglect the times of artificial modifications, the flops to solve the minimax assignment problem (2) by operations on matrix are $O(n^5)$ at most. The computational complexity of the operations on matrix is greatly lower than the computational complexity to solve the MILP (4).

V. TO APPLY OPERATIONS ON MATRIX TO SOLUTION OF (1)

Since the minimal c_{pq} in Step ① and Step ③ is recorded, namely the *optimal assignment in single entry* is adopted into the algorithm, the solution of the problem (2) obtained with operations on matrix is often the solution of the problem (1). However, the operations on matrix may not necessarily solve the global-minimum assignment problem (1) if the cost matrix is not transformed into the relative cost matrix before R covering.

In the cost matrix $C_n=(c_{ij})$, every element of each row subtracts the least element of the row (row conventional number), and every element of each column subtracts the least element of the column (column conventional number). The new cost matrix $B_n=(b_{ij})$, after such a transformation, is a nonnegative matrix, and there is one element 0 at least in each row and each column. Corresponding to the element $b_{ij}=0$, the sum of the i-th row conventional number and the j-th column conventional number equals the payment c_{ij}. Via the Hungarian algorithm, see, e.g. [1], the cost matrix B_n is equivalent to C_n for the assignment problem (1). The element of B_n represents the relative payment. If all the 0-elements of B_n can constitute a feasible assignment, then, the sum of the elements of C_n corresponding to the assignment, namely the sum of row and column conventional numbers, is the optimal objective function value.

Now the operations on matrix with the optimal assignment in single entry are applied to the relative cost matrix B_n. The basic covering is the covering of nonzero elements. If a feasible assignment can be found by the basic covering of B_n, then the assignment is a solution of the problem (1). Otherwise, using positive element covering of the lowest level, the operations on matrix give a solution of the problem (2) for B_n, where the least uncovering elements are contained as many as possible. Via the optimal assignment in single entry, this solution is generally the solution of the problem (1). The objective function value of the problem (1) equals the sum of corresponding elements of B_n plus the sum of row and column conventional numbers.

Notice that the solution of the problem (2) given by the operations on matrix with the optimal assignment in single entry for the relative cost matrix B_n is not always the solution of the problem (1). For example, the deflated relative cost matrix of some covering circle is as follows,

$$B_2=\begin{pmatrix}3 & 2\\ 2 & 0\end{pmatrix},\quad B_3=\begin{pmatrix}3 & 3 & 1\\ 3 & 2 & 0\\ 1 & 0 & 0\end{pmatrix}.$$

The operations on matrix find the correct solutions of the minimax assignment problem (2) for B_2, B_3, and both objective function values are equal to 2. But these solutions are not the solutions of the global-minimum assignment problem (1).

Since the operations on matrix with the optimal assignment in single entry will be applied to 2-order deflated relative cost matrix ultimately, the absolute payment sum of the last two assigned elements should be verified after the global-minimum assignment problem (1) is solved by the operations on matrix. There is small probability to make a mistake, even if the verification has been done.

In order to estimate an upper bound of the probability $p(E)$, where the event E represents to make a mistake by the operations on matrix for the problem (1), a hypothesis is added besides the hypotheses U1) and U2).

U3) Provided that the solution of operations on matrix for the problem (2) is obtained with nonbasic covering of relative cost matrix, there are n incomplete uncovering elements in the relative cost matrix such that the sum of them is less than the sum of solution elements of the problem (2).

The number of these events is less than or equal to $r-k+1$, a maximum of covering circles given by Proposition 3.

Hypothesis U3) amplifies the probability of event E, but simplifies the estimation of $p(E)$.

Proposition 8. With the relative cost matrix, the operations on matrix for the global-minimum assignment problem (1) are a well-implied enumeration, the probability to find the optimal solution with $r-k+1$ covering circles at most tends to 1 with increasing n.

[Proof] The operations on matrix are applied to the relative cost matrix, via Proposition 7, find the optimal solution of the minimax assignment problem (2) with $r-k+1$ covering circles at most. When the covering circles are larger than 1, it is possible to make a mistake since the optimal solution of problem (2) is regarded as the optimal solution of problem (1). According to Hypothesis U3), provided that the solution of problem (2) is obtained with incomplete 0-elements of relative cost matrix, there are n incomplete uncovering elements such that the sum of them is less than the sum of solution elements of the problem (2). In the $r-k+1$ events at most, the maximal probability event is that there are $n-1$ elements equal to 0 except one covering element. We now calculate the probability to form a feasible solution of problem (1) with these n elements. According to Hypotheses U1) and U2), there are $n^2C_{n-1}^{n\times n-1}$ distributions for one positive covering element and $n-1$ 0-elements in B_n, where $n!n$ distributions are different feasible solutions of the problem (1). So that the probability to form feasible solution of the global-minimum assignment problem (1) with these n elements is $\tau_2 = n!n/(n^2C_{n-1}^{n\times n-1}) = ((n-1)!)^2(n^2-n)!/(n^2-1)!$. The probability of event E satisfies $p(E) \le (r-k+1)\tau_2$. With Stirling's factorial formula to estimate $(r-k+1)\tau_2$, right side of $p(E)$. When n is big enough, $(r-k+1)\tau_2 \le 2\pi n^2(n-1)\sqrt{n/(n+1)}((n-1)/(n+1))^{n-1}$ $(n/(n+1))^{n(n-1)}/e^{n-1}$. Therefore, with the operations on relative cost matrix B_n for problem (1), the probability to make a mistake $p(E)$ tends to 0 with increasing n. □

VI. Numerical tests

Via comparison of computational complexity, the operations on matrix for the minimax assignment problem (2) or the global-minimum assignment problem (1) are far efficiency in comparison with method of exhaustion and MILP model. We need only to test the operations on matrix for problems (2) and (1) with higher order cost matrix. The cost matrix C_n is generated by computer in the numerical tests.

The tests of the operations on matrix for problems (2) and (1) are completed on a PC. The program is run using MATLAB7.0 under WindowsXP. Including the time to generate cost matrix, the computational CPU time required on a PC is given in seconds.

Example 1. The matrix elements

$$c_{ij}^n = 10+5i+5j \ (i \ne j),\ c_{ii}^n = 5+10i; \quad n = 10,15,20,25,30,35,40,45,50. \tag{6}$$

The optimal objective function value of problem (2) is clearly $15+5n$. There are two optimal solutions (for even number n) at most. The diagonal elements from left lower to right upper are always an optimal assignment.

The solutions between problems (1) and (2) are totally different for the cost matrix with elements (6). Via the relative cost matrix, the unique solution of (1) is the diagonal elements from left upper to right lower. The optimal objective function value is $10n+5n^2$.

Example 2. The matrix elements

$$c_{ij}^n = 10+5i+5j \ (i \ne j),\ c_{ii}^n = 15+10i; \quad n = 10,15,20,25,30,35,40,45,50. \tag{7}$$

The optimal objective function value of problem (2) is $15+5n$ (for even number n) or $20+5n$ (for odd number n). There is unique optimal solution for even number n. There are 4 optimal solutions for odd number n. The diagonal elements from left lower to right upper are always an optimal assignment.

Via the relative cost matrix, there are many solutions of problem (1) for the cost matrix with elements (7). Many feasible assignments without diagonal elements are all optimal solutions, and the optimal objective function value is $15n+5n^2$.

Example 3. The matrix elements

$$c_{ij}^n = 10+5i+5j; \quad n = 10,15,20,25,30,35,40,45,50. \tag{8}$$

The optimal objective function value of problem (2) is $15+5n$. The optimal solution is unique, namely the diagonal elements from left lower to right upper.

Via the relative cost matrix, there are $n!$ solutions of problem (1) for the cost matrix with elements (8). Any feasible assignments are all optimal solutions of (1), and the optimal objective function value is $15n+5n^2$.

Let n=5 and n=6 in order investigate the solutions of operations on matrix for Examples 1-3. Table 1 gives the row and column assignments and the payments to solve the problem (2). Notice that there are two solutions for n=6 and Example 1, and that there are four solutions for n=5 and Example 2. The solutions given by Table 1 are not diagonal assignments from left lower to right upper, while they accord with the optimal assignment in single entry. The optimal assignment in single entry, of course, is possible only for multiple solutions of (2). Usually there are multiple solutions for the minimax assignment problem (2). The solutions of Examples 2 and 3 found directly by the operations on matrix for the problem (2) are also for the problem (1), but the situation of Example 1 is not so.

Table 2 gives all the computation time to solve the problem (2) for Examples 1-3 of $n=10 \to 50$, including the time to generate cost matrix and to display results. The maximal expended time is about 0.125s for n=45 and Example 2 since there is one covering circle of $R_k = 245$ more performed. It is easy to see from Table 2 that efficient operations on matrix are

Table 1: Solutions of the problem (2)

	n =5					n =6					
E. 1	40 5–1	40 4–2	35 3–3	40 2–4	40 1–5	45 6–1	45 5–2	45 1–6	45 2–5	35 3–3	45 4–4
E. 2	40 5–1	40 4–2	40 1–5	35 2–3	45 3–4	45 6–1	45 5–2	45 4–3	45 3–4	45 2–5	45 1–6
E. 3	40 5–1	40 4–2	40 3–3	40 2–4	40 1–5	45 6–1	45 5–2	45 4–3	45 3–4	45 2–5	45 1–6

Table 2: Time to solve the problem (2)

n	*10*	*15*	*20*	*25*	*30*	*35*	*40*	*45*	*50*
E. 1	0.031	0.031	0.031	0.063	0.063	0.063	0.078	0.109	0.109
E. 2	0.031	0.063	0.031	0.094	0.063	0.109	0.094	0.125	0.094
E. 3	0.031	0.031	0.031	0.063	0.063	0.063	0.078	0.078	0.094

Table 3: Solutions of the problem (1)

	n =5					n =6					
E. 1	25 2–2	35 3–3	45 4–4	15 1–1	55 5–5	25 2–2	35 3–3	45 4–4	55 5–5	15 1–1	65 6–6
E. 2	25 2–1	30 1–3	40 4–2	55 5–4	50 3–5	25 2–1	30 1–3	40 4–2	55 5–4	65 6–5	55 3–6
E. 3	20 1–1	30 2–2	40 3–3	50 4–4	60 5–5	20 1–1	30 2–2	40 3–3	50 4–4	60 5–5	70 6–6

Table 4: Solutions of the problem (1) for negative profit matrix

	n =5					n =6					
E. 1	-40 1–5	-45 5–2	-30 3–1	-45 4–3	-40 2–4	-45 1–6	-50 6–2	-30 3–1	-45 4–3	-55 5–4	-45 2–5
E. 2	-25 1–1	-35 2–2	-45 3–3	-55 4–4	-65 5–5	-25 1–1	-35 2–2	-45 3–3	-55 4–4	-65 5–5	-75 6–6
E. 3	-20 1–1	-30 2–2	-40 3–3	-50 4–4	-60 5–5	-20 1–1	-30 2–2	-40 3–3	-50 4–4	-60 5–5	-70 6–6

completely fit to solve online both minimax and global-minimum assignment problems.

The tests for Example 1 of n =100 and 1000 have been done, and the time is 0.250s and 161.828s respectively. The tests for Example 2 of n =101 and 1001 have been done as well, and the time is 0.422s and 253.750s respectively. The time to solve the 1000-order problem (2) with the operations on matrix doesn't exceed 5 minutes. However, the time to solve on PC the 1000-order MILP problem (4) with the isometric surface method is over 20 minutes, see [3], while the 1000-order MILP problem corresponds to only the 31-order assignment problem. ($\sqrt{1000} \approx 31$.)

Table 3 gives the row and column assignments and the payments to solve the problem (1), via the relative cost matrix, with the operations on matrix. And Table 4 gives the row and column assignments and the payments to solve the problem (1) for negative profit matrix. The optimal objective function value equals the sum of every payment. The results conform to completely the notes of Examples 1-3. In Table 4, the maximum gross profit equals the negative optimal objective function value.

REFERENCES

[1] H.W. Kuhn, "The Hungarian Method for the Assignment Problem," Naval Research Logistics Quart, Vol.2 (1955), 83-97.

[2] Y.Y.Nie, L.J.Su and C.Li, "An Isometric Surface Method for Integer Linear Programming", Inter. J. Computer Math., Vol.80 (2003) No.7, 835-844.

[3] LY.Yang, JD.Han, LJ.Su and YY.Nie, "Isometric Surface Method and Its Numerical Tests for Mixed-Integer Linear Programming", ISAST Transactions on Computers and Software Engineering, to appear.

[4] Y.Y.Nie and S.R.Xu, "An Isometric Plane Method for Linear Programming", J. Computational. Math., Vol.9 (1991) No.3, 262-272.

[5] Y.Y.Nie, X.Song, L.J.Su, J.Yu, M.Z.Yuan, "Well-Implied and Near-Implied Enumerations", Information and Control, Vol.34 (2005) No.3, 296-302. (in Chinese.)

[6] Moustapha Diaby, "The traveling salesman problem: A linear programming," *WSEAS Transactions on Mathematics*, v 6, n 6, June, 2007, p 745-754.

[7] Rabih A. Jabr, "A binary integer programming approach to worse-case linear circuit tolerance analysis," *WSEAS Transactions on Circuits and Systems*, v 6, n 8, August, 2007, p 525-531.

[8] Xinli Xu, Yu Tong, Xiangli Wang, Shiyan Ying, Wanliang Wang, "Production plan optimization assignment based on multi-agent," *WSEAS Transactions on Systems*, v 5, n 6, June, 2006, p 1468-1475.

L.Y. Yang is presently pursuing her Doctoral Degree at the State Key Laboratory of Robotics, Shenyang Institute of Automation, Chinese Academy of Sciences, Shenyang, China. Her research interests include autonomous planning of multi-moblie robots in dynamic environment, mixed integer linear programming.

M.H. Nie is a general manager of Shenyang Digital China Technology Development Company. His research interests include management information system (MIS) and logistics system.

Z.W. Wu is a professor at the State Key Laboratory of Robotics, Shenyang Institute of Automation, Chinese Academy of Sciences, Shenyang, China. His research interests include robot control and intelligent control. He has published more than 30 papers in various journals.

Y.Y. Nie is a professor at the State Key Laboratory of Robotics, Shenyang Institute of Automation, Chinese Academy of Sciences, Shenyang, China. His research interests include applied mathematics and computer aided engineering. He has published more than 70 papers in various journals.