Evolutionary Processes Solved with Lie Series and by Picard Iteration Approach

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***Abstract*—** The solution of evolutionary Cauchy problems by means of Lie series expansion and its linkage to Picard iteration method, is presented. Thanks to a Taylor transform and to the introduction of a differential Lie-Groebner operator D, the initial generally non-linear and non-autonomous problem can be reduced to a linear one, whose solution is given in terms of the Lie operator $\exp(tD)$.

The Picard procedure applied to the Volterra integral equation that turns out from the initial problem, can rigorously introduce generalized Lie series since its steps are the partial sums of those series.

***Keywords*—** Lie series, Nonlinear Cauchy problems, Picard iteration, P.D.E. .

I. Introduction

FINDING solutions to partial differential equations (PDEs), in a general, fast and efficient way, is as important as it is a difficult task. In previous papers [1]-[8] we faced the problem to find out a unique method capable to solve evolutionary PDEs, both linear and nonlinear, both autonomous and non-autonomous, under the constraint of analyticity regarding the evolutionary operator, i.e. the transformed function of the unknown one were analytic, developable in a multiple power series of its arguments. We reached this goal by improving and extending a method, based on Lie series, proposed by W. Groebner and others in the 70's [9]-[11].

Inspired by those ideas, we started a systematic study both aimed at a better theoretical foundation and at a practical applicability of that method. So we fixed what can be considered a generalization of the Groebner's approach. The present paper is a further step in that direction. In it we study the integration of a linear or nonlinear PDE of the evolutionary type in the Cauchy's formulation of the problem. For the sake of simplicity of exposition, here we treat a two dimensional problem, but the method is easily extendible to higher dimensions, to systems of equations and to boundary value problems, as we partly already showed and partly are going to show in future papers. Anyway, in this work we utilize the classical Picard's iteration procedure in order to construct the unique solution, whose components will remain represented by Lie series.

In fact, as a standard result from nonlinear functional analysis, we know Picard's iteration gives a general theorem on the existence (and uniqueness) of the solution. Therefore we are going to revisit that approach in order to show that the Picard's procedure is constructive also because, by our point of view, it allows to write the solution in explicit form by means of Lie series.

Finally we stress from a practical point of view, that the improved method gives an approaching polynomial, in the frame of an integration by series, with the required precision, to the solution to the assigned Cauchy problem, even in situations when other methods fail. We want to remark that here, at the end, we shall find a link between the Picard's iteration and the series, belonging to a special class of the Lie's type, representative of the unique solution and by this approach, we reach, in a rigorous way, the fundamental result.

We have elsewhere, [7], also approached the same problem by a quite different point of view. There we have also found the convergence radius of the generalized Lie series implied in the representation of solution [7]. It will be useful to remember that result in what follows.

Resuming, we can say, apart from results of the last cited note: [7], two principal approaches are possible dealing with Lie series method: either

1) a heuristic one, [1]-[8]. Its foundation is a perturbation procedure (PP): we can consider a sequence of problems overall equivalent to the assigned Cauchy's problem and all resolvable a' la Groebner. That author concerned resolution of n initial value problem (i.v.p.) for a differential system of first order normal ordinary equations in finite number, [9]-[11]. The method approaches the solution and requires a final check of the results. It is an alternative pathway indeed in order to gain solution and in the sequel will be remembered where useful;

or

2) the use of a Picard's iteration method, which is allowed if the evolutionary operator is Lipschitzian or (in particular) analytic.

Picard's method, in addition, reveals that, in the case under study, its steps are partial sums of special Lie series, our generalized ones.

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By using one of the above procedures we want in particular to show that the Lie series method is, at the end, a linearizing method of integration. Indeed, even if the evolutionary operator is nonlinear, analytic in its arguments and also time dependent, we are able to change it into a linear and time independent operator. This is accomplished via a preliminary Taylor's transformation at a non singular point of its domain, and by a contemporary "symmetrization" procedure of the variables. The latter, as is well known, and as it will be clear in what follows, consists in treating the "*t*" variable on the right hand side (r.h.s.) functions of considered equations, on the same foot of the other variables by adding a very simple differential equation to the system.

So the principal aspects of this paper could be shrunk into the following two:

1) We treat about the linearization of a problem which firstly is nonlinear analytic and time dependent.

2) Thanks to the Picard's iteration, we introduce generalized Lie series in order to explicit the solution. Those series represent the components of the solution of the initial value problem that a preliminary Taylor's transformation does equivalent to the assigned Cauchy problem concerning the original evolutionary equation. The solution of such an open differential system (i.e. with non limited number of equations), may be find also, as above affirmed, in a procedure of perturbation (PP) considering a sequence of approaching problems all finite and solvable according with Groebner's approach.

This last observation will turn useful as well, both in its theoretical and practical implications.

In fact by a (PP), automatic calculus is allowed. We ourselves developed a suitable software, in the Mathematica ® framework, in order to establish the practical feasibility and convenience of this method. The numerical problems connected with this procedure will be the subject of a foregoing paper. Anyway this method proves to be useful even when more standard techniques fail, and capable to solve nonlinear problems of the non autonomous type by simple means although in the frame of power series developments.

II. General problem and its reduction to linearity

The most general problem we must deal with, is the following Cauchy problem:

$$\begin{aligned} &\frac{dP}{dt} = A(P), \\ &P(0,x) = P_o(x) \end{aligned} \tag{1}$$

concerning with an evolutionary equation with *A* nonlinear differential operator of order μ in the spatial variable *x* and, in general, time dependent. The sole hypothesis made on *A(P)* is its analyticity, i.e. this function is supposed to be susceptible of a multiple powers series representation with respect to (w.r.t.) all its arguments in a multidimensional disk *domA*. The importance of analyticity will be stressed at different levels in the sequel. Let us also suppose, what is not restrictive, the analyticity regarding the spatial variable *x*, is confined to a compact, $\Im(x) \subset domA$, a neighbourhood of the origin.

Then, how are we going to treat the above problem?

1) we write, by a Taylor's transformation at an initial non singular point, e.g. $x = 0 \in \Im(x)$, an equivalent initial value problem (i.v.p.), concerning an *open* (i.e. with non limited number of eqs) differential system of normal first order differential equations;

2) we solve the i.v.p. by the Picard's iteration, that ensures a unique solution to exist and that also is a constructive pathway finding explicitly the solution by special powers series;

3) we recognize the solution as expressed by Lie series of a generalized type;

4) we write the unique solution of the original Cauchy problem (1) by its Taylor's anti- transformation.

As it is clear from the above considerations the strategy of Lie series is also recommendable in linear problems. Especially when more standard tools, e.g. semigroups theory, with the aim of an integration in closed form, are not available.

Now the basic idea of our approach is a Taylor's transformation which allows us to act on the linear space *S* of the Cauchy sequences on *C*, the complex field. That space includes the vector of the Taylor's coordinates of the unknown function, i.e. the sequence of the coefficients in its Taylor's representation at the initial point *x* = 0.

More in details, let us observe that, we are looking for an analytic function with respect to *x* on the compact $\Im(x) \subset domA$, which solves our problem (1). Then by a Taylor's transformation, at the point $x = 0 \in \Im(x)$, we deal, instead with problem (1), with the following i.v.p. (2) for an *open* differential system, which can be done autonomous, i.e. time independent, adding another eq. for the variable p_{-1} as reported in the following picture:

$$\begin{aligned} &\frac{dp_n}{dt} = \Theta_n(p_{-1}, p_{\mu+n}, ..., p_{o+n}, ..., p_o), \quad n \in N_o \\ &\frac{dp_{-1}}{dt} = 1 \\ &p_n = \frac{1}{n!}\left[\frac{\partial^n P}{\partial x^n}\right]_{x=0} ;\ \Theta_n = \frac{1}{n!}\left[\frac{\partial^n A(P)}{\partial x^n}\right]_{x=0} \\ &p_n(0) = a_n ;\ a_n = \frac{1}{n!}\left[\frac{\partial^n P_o}{\partial x^n}\right]_{x=0} ;\ p_o = (a_n)_{n=0}^{+\infty} \\ &p_{-1}(0) = 0. \end{aligned} \tag{2}$$

Integrating (1) is equivalent to solve (2) and vice versa and we shall utilize this fundamental observation in the sequel. We want also to notice that the possibility of transforming a non autonomous problem into an autonomous one, by means of this very simple trick, which often however could render nonlinear an original linear system, is particularly convenient in this context when linear and nonlinear problems are treated in the same way. As is known this position is very common in the Lagrangian dynamics field. Then let us rewrite our i.v.p. (2) in a more compact form:

$$\begin{aligned} &\frac{dp}{dt} = Dp \\ &p(0) = p_{\hat{o}}\,,\ p_{\hat{o}} = \left(0,(a_n)_{n=0}^{+\infty}\right), \end{aligned} \tag{3}$$

which involves the sequence p of the Taylor's coordinates of P with the addition of p_{-1}, i.e. $p \equiv \left(p_{-1},(p_n)_{n=0}^{+\infty}\right) \in S$, being:

$$D = \frac{\partial}{\partial \pi_{-1}} + \sum_{n=0}^{+\infty} \Theta_n(\pi_{k+n},...,\pi_{o+n},...,\pi_o,\pi_{-1})\frac{\partial}{\partial \pi_n}$$

the Lie-Groebner operator, where π_j are complex numbers which have no linkage to the unknown functions p_j, i.e. they are parameters. Their variability has only one constraint: all Θ_n must be defined on those values. Better saying, in all functions on the r.h.s. of the eqs (3), suitable parameters take place of the previous arguments in order to build the D operator.

It is important to notice that the above compact form is allowed by the following rule:

Remark : *The linear operator D acts on p through the following steps:*

- *firstly, an arbitrary sequence, say* $\pi \equiv \left(\pi_{-1},(\pi_n)_{n=0}^{+\infty}\right) \in S$, *is substituted to* $p \equiv \left(p_{-1},(p_n)_{n=0}^{+\infty}\right) \in S$*;*
- *then, having written the image* D_π*, the terms of the p sequence in it take place of the corresponding ones of* π*, i.e.:*

$$Dp \equiv [D\pi]_{\pi\to p} = \Theta(p)$$

where $\Theta(p) = \left(\Theta_n(p_{-1},p_o,...,p_{o+n},...,p_o)\right)_{n=0}^{+\infty}$*, i.e.* Θ *depends on p. This means that the n+1-th term of the sequence depends on the finite sequence (or numerical vector):* $(p_{-1},p_o,...,p_{o+n},...,p_{\mu+n})$*. Equivalently the components of* Θ *depend on sequences:*

$$(p_{-1},p_o,...,p_{o+n},...,p_{\mu+n},0,...) \in S.$$

In the sequel beside D, a preeminent role is played by the exponential operator:

$$e^{tD} = 1 + tD + \frac{t^2}{2!}D^2 + \frac{t^3}{3!}D^3 + ... + \frac{t^n}{n!}D^n + ...$$

Then it needs preliminarily more deeply to speak about the above two operator D and e^{tD}:

We can gain for them a rigorous definition by a perturbation strategy.

As said above, in a procedure of perturbation (PP), we can consider, at every step m, only the first m + 1 equations of the equivalent system (2) to the assigned Cauchy problem (1), and write the pertinent m-th approaching i.v.p. in a compact form:

$$\begin{aligned} &\frac{dp_{(m)}}{dt} = D_{(m)}p_{(m)} \\ &D = \frac{\partial}{\partial \pi_{-1}} + \sum_{n=0}^{+\infty} \Theta_n(\pi_{k+n},...,\pi_{o+n},...,\pi_o,\pi_{-1})\frac{\partial}{\partial \pi_n} \\ &p_{(m)} = (p_{-1},p_o,...,p_{o+n},...,p_{\mu+m}) \\ &p_{(m)}(0) = p_{(m)\hat{o}}\,,\ p_{(m)\hat{o}} = \left(0,(a_n)_{n=0}^{m}\right) \end{aligned}$$

Now the following fundamental statement holds:

as $m \to +\infty$*, the sequence* $\left(D_{(m)}\right)_{m=0}^{+\infty}$ *is a Cauchy sequence, then such is* $\left(e^{tD_{(m)}}\right)_{m=0}^{+\infty}$.

In fact in the *Sup*-norm in *S*, operators $D_{(m)}$ have finite norms, and holds the Lipschitz's condition for analytic exponential operators:

$$\begin{aligned} &\left\| e^{tD_{(m)}} - e^{tD_{(m_1)}} \right\| < K \left\| D_{(m)} - D_{(m_1)} \right\|,\ m \ge m_1\,, \\ &\left\| D_{(m)} \right\| = Sup \left\| D_{(m)}\pi' \right\|,\ \pi' \in S : \left\| \pi' \right\| \le 1\,; \end{aligned}$$

Now if $\pi \in S$, $D_{(m)}\pi - D_{(m_1)}\pi$ is the sequence:

$$\begin{aligned} &(0,...,0,\Theta_{m_1+1}(\pi_{k+m_1+1},...,\pi_{o+m_1+1},...,\pi_o,\pi_{-1}),..., \\ &\Theta_m(\pi_{k+m},...,\pi_{o+m},...,\pi_o,\pi_{-1}),0...)\,. \end{aligned}$$

Since:

$$\Theta_n(\pi_{k+n},...,\pi_{o+n},...,\pi_o,\pi_{-1}) \to 0, \text{as} \to +\infty$$

then for every $\varepsilon > 0$ there is a pair

$$(m,m_1): Sup\left(D_{(m)}\pi - D_{(m_1)}\pi\right) < \varepsilon$$

Then the existing limits of two sequences:

$$\left(D_{(m)}\right)_{m=0}^{+\infty}\,,\ \left(e^{tD_{(m)}}\right)_{m=0}^{+\infty}$$

define: D, e^{tD}.

A formal consequence of the existence of the two above operators: D and e^{tD}, may be that, being D linear, it is a standard result:

$$e^{tD}\,e^{-tD}\left(\frac{d}{dt}p - Dp\right) = 0 \;\Rightarrow\; e^{tD}\frac{d}{dt}\left(e^{-tD}p\right) = 0$$

which, after integration, reads:

$$e^{-tD}p - p_{\hat{o}} = 0 \qquad\qquad \Rightarrow \qquad p = e^{tD}p_{\hat{o}}$$

i.e. the operator e^{tD} gives the formal solution to i.v.p. (3), a well known implication.

But here our aim is different from the above pathway.

In a less formal path let us observe that we need to integrate the Volterra's integral equation:

$$p = p_{\hat{o}} + \int_0^t D\,p\,d\xi \tag{4}$$

which can be derived by integration from i.v.p. (3).

It is also well known that, if the integral linear operator:

$$\pounds = \int_0^t D * d\xi$$

is a contraction mapping, then an iteration procedure, which in the sequel will be recognized as the same in the Picard's one, may be followed in order to integrate the above functional equation. Then a unique solution exists to the above Volterra's equation (4). In addition, we can opportunely choose $t \le T$ in order to satisfy the sufficient condition of integrability of (4) by a Neumann series:

$$\|\pounds\| < 1\,.$$

Let us finally observe that often in applied sciences, e.g. in quantum mechanics, it is convenient, dealing with similar problems, in a first time, to limit ourselves to a formal introduction of the exponential operator e^{tD} and to require only at the end a check that the series:

$$p = e^{tD}p_{\hat{o}} \tag{5}$$

is really the solution to the assigned i.v.p. (3). We shall return on final check of result in the last paragraph of present note, being this possibility strictly linked to analyticity of *A*.

In this work, as already said, we want to follow a more rigorous procedure based on the integration of the above Volterra's equation (4), of course pleonastic will be the final check of result.

Resuming, we can affirm that being D linear two paths are possible: the former, formal, which requires a preliminary introduction of the exponential operator e^{tD}. The latter, a consequence of the contraction mapping theorem, which reveals itself to be not different from Picard's iteration procedure (PI) and the same conducts to a final introduction of the exponential operator e^{tD} in order to write in compact form the result. This latter is a constructive procedure of (5), the solution to i.v.p. (3). In every instance firstly or lastly, the introduced operator e^{tD} is a generalized exponential operator because D is a symbolic non finite sum of terms, all linear differential operators.

III. Proofs of Main Statements

In order to legitimate each of the above propositions, we shall prove:

1) $e^{tD}p_{\hat{o}}$ is an absolutely convergent series.

2) the sequence: $(\tilde{p}_n)_{n=0}^{+\infty}$ of vectors which represent the steps to be run in integration a' la Picard of (5) is a Cauchy sequence and converges to a $\tilde{p} \in S$.

That is:

$$\tilde{p}_n = \tilde{p}_o + \int_0^t D\,\tilde{p}_{n-1}\,d\xi$$
$$\ldots$$
$$\tilde{p}_o \equiv p_{\hat{o}}\,.$$

are the steps of Picard's iteration (PI) which approach a $\tilde{p} \in S$.

3) the partial sums of $e^{tD}p_{\hat{o}}$ are the functions $\tilde{p}_n$, then: $\tilde{p} = e^{tD}p_{\hat{o}}$.

4) functions $\tilde{p}_n$ are also the partial sums of the Neumann series which solves the above Volterra's equation (4) in the unknown *p*. That proves $\tilde{p} = p$ in a convergence disk, which at least includes the neighbourhood:

$$[0, T[\;:\;\|\pounds\| < 1$$

Finally we have:

$$p = e^{tD}p_{\hat{o}}\,,$$

as a sequence of Lie Series.

Theorem: $e^{tD}p_{\hat{o}}$ *is a convergent series*.

Proof of the absolute convergence:

$$\|t(D\,p_{\hat{o}} - D\,\hat{p})\| \le tk\|p_{\hat{o}} - \hat{p}\|$$

$$\left\|\frac{t^2}{2!}(D^2\,p_{\hat{o}} - D^2\,\hat{p})\right\| = \left\|\frac{t^2}{2!}(D(D\,p_{\hat{o}}) - D(D\,\hat{p}))\right\| \le$$

$$\le \frac{t^2}{2!}k\|(D\,p_{\hat{o}}) - (D\,\hat{p})\| \le$$

$$\le \frac{t^2}{2!}k^2\|p_{\hat{o}} - \hat{p}\|$$

...

$$\left\|\frac{t^n}{n!}(D^n\,p_{\hat{o}} - D^n\,\hat{p})\right\| \le \frac{t^n}{n!}k^n\|p_{\hat{o}} - \hat{p}\|$$

...

$$\Rightarrow$$

if $\hat{p} = 0$, $\sum_{n=0}^{+\infty}\frac{t^n}{n!}\|D^n\,p_{\hat{o}}\| < +\infty$

Then therefore the series $e^{tD}p_{\hat{o}}$ converges in norm, then absolutely in some neighbourhood of $t = 0$.

Furthermore we easily obtain:

$$\tilde{p} = \sum_{k=0}^{n}\frac{t^k}{k!}D^k p_{\hat{o}} \ ,$$

that is the function which represents the *n-th* step of Picard's iteration, but also it is the *n+1-th* partial sum of the exponential series $e^{tD}p_{\hat{o}}$, then the sum of the series is the limit function, say $\tilde{p}$, of the sequence: $(\tilde{p}_n)_{n=0}^{+\infty}$.

Theorem: *Provided that* $(\tilde{p}_n)_{n=0}^{+\infty}$*, the sequence of PI, is a Cauchy sequence, then it converges to* $e^{tD}\tilde{p}_o \equiv e^{tD}p_{\hat{o}}$.

Proof:

We have:

$$\frac{d\tilde{p}_n}{dt} = D\,\tilde{p}_{n-1} \Rightarrow \frac{d\tilde{p}_n}{dt} = D\,\tilde{p}_{n-1} + D\,\tilde{p}_n - D\,\tilde{p}_n \Rightarrow$$

$$e^{tD}(e^{-tD}\frac{d\tilde{p}_n}{dt} - e^{-tD}D\,\tilde{p}_n) = D\,\tilde{p}_{n-1} - D\,\tilde{p}_n$$

$$e^{tD}\frac{d}{dt}e^{-tD}\tilde{p}_n = D\,\tilde{p}_{n-1} - D\,\tilde{p}_n \Rightarrow$$

$$e^{-tD}\tilde{p}_n = \tilde{p}_o + \int_0^t e^{-\xi D}(D\,\tilde{p}_{n-1} - D\,\tilde{p}_n)\,d\xi$$

$$\tilde{p}_n = e^{tD}\tilde{p}_o + e^{tD}\int_0^t e^{-\xi D}(D\,\tilde{p}_{n-1} - D\,\tilde{p}_n)\,d\xi$$

$$\tilde{p}_n \to e^{tD}\tilde{p}_o, \ \text{as} \ n \to +\infty$$

In fact:

$$\left\|e^{tD}\int_0^t e^{-\xi D}(D\,p_{n-1} - D\,p_n)\,d\xi\right\| \le k\|B\|\,\|p_{n-1} - p_n\|,$$

if B is the linear similarity operator: $e^{tD}\int_0^t e^{-\xi D} * d\xi$,

since $\|e^{tD}\| < e^{t\|D\|} < +\infty$, then $\|B\| < +\infty$.

Theorem: *Since D is a Lipschitzian mapping, then* $(\tilde{p}_n)_{n=0}^{+\infty}$ *is a Cauchy sequence.*

Proof:

If $n > m$,

$$\frac{d^n\tilde{p}_n}{dt^n} = D^n\,\tilde{p}_o\,, \Rightarrow$$

$$\frac{d^n\tilde{p}_m}{dt^n} = 0 \Rightarrow$$

$$\frac{d^n}{dt^n}(\tilde{p}_n - \tilde{p}_m) = D^n\,(\tilde{p}_o - 0)$$

integrating n times and thanks to the linearity of D, we have:

$$\tilde{p}_n - \tilde{p}_m = \int_0^t\int_0^{\xi_{n-1}}...\int_0^{\xi_1}D^n\,(\tilde{p}_o - 0)\,ds_1...ds_{n-1}\,ds_n$$

Being D a Lipschitzian linear mapping:

$$\|\tilde{p}_n - \tilde{p}_m)\| < \frac{t^n}{n!}k^n\|\tilde{p}_o - 0\|^n$$

approaching 0 as $n \to +\infty$. Q.E.D.

The space of Cauchy sequences is a Banach space so the Cauchy sequence $(\tilde{p}_n)_{n=0}^{+\infty}$ converges to a $\tilde{p}$, consequently it exists and it is unique.

Theorem: *Picard iterations integrate the above Volterra equation.*

Proof: As is well known, a Neumann series integrates the Volterra's equation (5):

$$p = p_{\hat{o}} + \pounds\,p_{\hat{o}} + \pounds^2\,p_{\hat{o}} + ...$$

and it converges in

$$[0,T[\ :\ \|\pounds\| < 1$$

The partial sums of the above series are the steps in Picard's iteration (PI):

$S_o = p_{\hat{o}}$ the starting point in PI being $\tilde{p}_o \equiv p_{\hat{o}}$;

$S_1 = p_{\hat{o}} + \pounds\, p_{\hat{o}}$ the first step in PI;

$$S_2 = p_{\hat{o}} + \pounds\, p_{\hat{o}} + \pounds^2\, p_{\hat{o}} =$$
$$p_{\hat{o}} + \int_0^t D\, p_{\hat{o}}\, d\xi_1 + \int_0^t D \int_0^{\xi_2} D\, p_{\hat{o}}\, ds_1\, ds_2$$
$$= p_{\hat{o}} + \int_0^t D\,(p_{\hat{o}} + \int_0^{\xi_2} D\, p_{\hat{o}}\, ds_1)\, ds_2$$

the second step in PI, and so on.

IV. The solution *P*

The sequence *p* represents the solution to (3)

$$p = \sum_{n=0}^{+\infty} \frac{t^n}{n!} \left[D^n\, \pi \right]_{\pi \to p_{\hat{o}}},$$

or in compact form

$$p = [e^{tD}\, \pi]_{\pi \to p_{\hat{o}}}$$

Then the solution to problem (1) is obtained by the Taylor's anti-transformation:

$$\begin{aligned} P &= [e^{tD}\, \pi]_{\pi \to p_{\hat{o}}} \bullet z \\ z &= \left(x^n \right)_{n=0}^{+\infty} \end{aligned} \tag{6}$$

This series has coefficients which are Lie series and converges in $\Im(x)$, neighbourhood of x=0, within *domA*, the convergence disk of *A(P)*.

V. Existence of the solution

It is very important to notice that:

it is required the exchange theorem, [8], when the Lie operator e^{tD} *acts on the image obtained by (* $Dp \equiv \Theta(p)$ *).*

In fact since the definition of the image *Dp* is given by $\Theta(p)$, it is $e^{tD}Dp = e^{tD}\Theta(p)$. Now we notice that while the former term, *Dp*, allows the action of

the Lie operator, e^{tD}, *directly* on *p*; being The commutator $[e^{tD}, D] = 0$; the same action demands the analyticity w.r.t. its arguments of every term of the sequence Θ in order to can apply the exchange property (which writes the image of an analytic function by e^{tD} as the value of the same function on the variable transformed by the Lie exponential operator).

We remark the necessity of the exchange theorem (descending from the analyticity hypothesis on *A*) also in order to verify, as final check, that

$$p = [e^{tD}\, \pi]_{\pi \to p_{\hat{o}}}$$

is the solution to the equivalent initial value problem i.v.p.(2).

In fact

$$\begin{aligned} &\frac{d}{dt}[e^{tD}\, \pi]_{\pi \to p_{\hat{o}}} = \Theta\left([e^{tD}\, \pi]_{\pi \to p_{\hat{o}}} \right) \Rightarrow \\ &\Rightarrow [e^{tD}\, D\pi]_{\pi \to p_{\hat{o}}} = \Theta\left([e^{tD}\, \pi]_{\pi \to p_{\hat{o}}} \right) \Rightarrow \\ &\Rightarrow [e^{tD}\, \Theta(\pi)]_{\pi \to p_{\hat{o}}} = \Theta\left([e^{tD}\, \pi]_{\pi \to p_{\hat{o}}} \right) \end{aligned}$$

is true by the exchange theorem.

Finally let us observe that integrating an evolutionary equation in linear instance, is an easy task if we know the set of eigenfunctions of *A*; a generator of the functional space in which we search the solution. If we do not, we must again introduce the Lie exponential operator, which is a sort of passe-partout.

Again we remark that the method of Lie series *linearizes* the assigned *non-linear* problem, and it is also useful in non autonomous instances, when the alternative path in linear problems is the Tanabe's method [12] and in nonlinear cases the Magnus' integration [13].

Resuming: *P* expressed by the Lie series (6), is the unique solution to problem (1), in the more general case of a nonlinear evolutionary time dependent operator. It is analytic with respect to *x* within the convergence disk around *x* = 0, *domA* of *A(P)*, and w.r.t. *t* within [0; *T*[at least. In truth the possibility of an analytic continuation is expected on t-axis within *domA*, how our study, [7], concludes.

In other words Lie series have not only a local validity, as method here used seem to suggest. Our reader may glace at the cited note, [7].

The method which we have supplied foundations by the Picard's general tool is based on an integration by special powers series and therefore asks for computer help in order to handle the generalized Lie series involved. We think it could be appreciated for its simplicity in approaching both linear as well as nonlinear, both autonomous as well as non autonomous evolutionary problems.

VI. Conclusion

Whilst in previous papers we dealt with the problem of solving an evolutionary problem with analytic operator by means of the integration of a sequence of approaching all finite problems to the equivalent initial value problem (2), here we answer to this subject by means of the Picard's classical iteration method. By this tool we can write an approaching sequence to the unknown solution, using a consequence of the principle of the contraction mappings. The Picard's iteration tool furnishes mainly, as is well known, a theorem on the existence (and uniqueness) of a solution to the evolutionary Cauchy problem. This because such a theorem holds for the initial value problem, to be integrated in *S*, the space of Cauchy sequences on *C*, the complex numbers field, which a Taylor's transformation at a non singular point (e.g. $x = 0$) associates as equivalent to the assigned Cauchy problem. This transformation is peculiar of our approach and allows to reduce, to linear and autonomous systems, any assigned analytical problems which had not these characters. Finally we show the linkage of Picard's procedure with the Lie series finding again the fundamental result, which expresses the components of the unique solution in terms of special Lie series.

The true radius of convergence of the Lie series involved, has been studied, [7], in showing a wider validity of the representation within *domA*.

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