Formulas for the Fourier Series of Orthogonal Polynomials in Terms of Special Functions

Nataniel Greene

***Abstract*— An explicit formula for the Fourier coefficients of the Legendre polynomials can be found in the Bateman Manuscript Project. However, formulas for more general classes of orthogonal polynomials do not appear to have been worked out. Here we derive explicit formulas for the Fourier series of Gegenbauer, Jacobi, Laguerre and Hermite polynomials. The methods described here apply in principle to an class of polynomials, including non-orthogonal polynomials.**

***Index Terms*— Fourier series, Gegenbauer polynomials, Jacobi polynomials, Laguerre polynomials, Hermite polynomials.**

An explicit formula for the Fourier coefficients of Legendre polynomials appears in the Bateman Manuscript Project [3, 4, 5, 6]. One finds that

$$P_n(x) = \sum_{k=-\infty}^{\infty} \widehat{P}_n(k)\, e^{ik\pi x} \tag{1}$$

where

$$\begin{aligned} \widehat{P}_n(k) &= \frac{1}{2}\int_{-1}^{1} P_n(x) e^{-ik\pi x} dx \\ &= \begin{cases} \frac{(-i)^n}{\sqrt{2k}} J_{n+\frac{1}{2}}(\pi k), & k \neq 0 \\ \delta_{0,n} & k = 0 \end{cases} \end{aligned} \tag{2}$$

The formula follows from either Bateman [5] p. 122 or from [4] p. 213 after setting $\lambda = \frac{1}{2}$.

However, explicit formulas for the Fourier coefficients for Gegenbauer, Jacobi, Laguerre and Hermite polynomials do not appear to have been worked out. This article derives formulas for the Fourier coefficients of orthogonal polynomials using two methods. One method utilizes the power series form of the polynomial and known explicit formulas for the power series coefficients. Another method uses a change of basis to Legendre polynomials and known explicit formulas for the connection coefficients. The methods apply even to non-orthogonal polynomials, provided that the power series coefficient or connection coefficients are known in explicit form.

N. Greene is with the City Univeristy of New York, Department of Mathematics and Computer Science, Kingsborough Community College, 2001 Oriental Boulevard, Brooklyn, NY 11235 USA (e-mail: ngreene.math@gmail.com).

I. Formulas Derived Using the Power Series form of the Polynomials

Begin with the power series form of a generic orthogonal polynomial of the form

$$p_n(x) = \sum_{j=0}^{n} a_j x^j. \tag{3}$$

or of the form

$$p_n(x) = \sum_{j=0}^{[n/2]} b_j x^{n-2j} \tag{4}$$

For Gegenbauer-λ polynomials, $\lambda \neq 0$, [7] p. 219,

$$b_j = \frac{1}{\Gamma(\lambda)} \frac{(-1)^j \Gamma(\lambda+n-j)\, 2^{n-2j}}{j!\,(n-2j)!} \tag{5}$$

and when $\lambda = 0$,

$$b_j = \frac{(-1)^j \Gamma(n-j)\, 2^{n-2j}}{j!\,(n-2j)!}. \tag{6}$$

For Jacobi-(α, β) polynomials

$$\begin{aligned} a_j &= \frac{(\alpha+1)_n}{2^n n!} \frac{(-n)_j (\alpha+\beta+n+1)_j}{(\alpha+1)_j\, j!} \\ &\quad \times {}_2F_1(-\beta-n, j-n; a+j+1; -1)(-1)^{\cdot j} \end{aligned} \tag{7}$$

For Laguerre-α polynomials [7] p. 240

$$a_j = \frac{(-1)^j}{j!} \binom{n+\alpha}{n-j} \tag{8}$$

and for Hermite polynomials [7] p. 250

$$b_j = n! \frac{(-1)^j\, 2^{n-2j}}{j!\,(n-2j)!} \tag{9}$$

Depending on which form is used, the kth Fourier coefficient of $p_n(x)$ is given either by

$$\begin{aligned} \widehat{p}_n(k) &= \frac{1}{2}\int_{-1}^{1} p_n(x) e^{-ik\pi x} dx \\ &= \frac{1}{2}\int_{-1}^{1} \sum_{j=0}^{n} a_j x^j e^{-ik\pi x} dx \\ &= \frac{1}{2}\sum_{j=0}^{n} a_j \int_{-1}^{1} x^j e^{-ik\pi x} dx \\ &= \frac{1}{2}\sum_{j=0}^{n} a_j \beta_j(i\pi k) \end{aligned} \tag{10}$$

 Manuscript received April 4, 2008; Revised received July 29, 2008

or by:

$$\begin{aligned}
\widehat{p_n}(k) &= \frac{1}{2}\int_{-1}^{1} p_n(x)e^{-ik\pi x}dx \\
&= \frac{1}{2}\int_{-1}^{1}\sum_{j=0}^{[n/2]} b_j x^{n-2j}e^{-ik\pi x}dx \\
&= \frac{1}{2}\sum_{j=0}^{[n/2]} b_j \int_{-1}^{1} x^{n-2j}e^{-ik\pi x}dx \\
&= \frac{1}{2}\sum_{j=0}^{[n/2]} b_j \beta_{n-2j}(i\pi k) \qquad (11)
\end{aligned}$$

where $\beta_m(z)$ is a special function defined in Abramowitz and Stegun [1] p. 228 by

$$\beta_m(z) = \int_{-1}^{1} x^m e^{-zx}dx,\ m = 0, 1, 2, ... \qquad (12)$$

The function $\beta_m(z)$ can be stated in terms of the incomplete gamma function as follows:

$$\beta_m(z) = \Gamma(m+1, z) - \Gamma(m+1, -z)$$

where

$$\Gamma(\nu, z) = \int_z^{\infty} t^{\nu-1}e^{-t}dt.$$

is known as the incomplete gamma function. When $\nu = n$, a positive integer, the incomplete gamma function has an elementary formula given in Spanier and Oldham [9] p. 438, by:

$$\Gamma(n, z) = (n-1)!e^{-z}e_{n-1}(z) \qquad (13)$$

where

$$e_n(z) = \sum_{j=0}^{n} \frac{z^j}{j!} \qquad (14)$$

is referred to as the exponential polynomial function, [9] p. 239.

Therefore the function $\beta_m(z)$ has an elementary explicit formula which can be used for calculations:

$$\beta_m(z) = z^{-m-1}m!\left(e^z e_m(-z) - e^{-z}e_m(z)\right) \qquad (15)$$

II. Formulas Derived Using a Change of Basis to Legendre polynomials

An alternative method uses the fact that change-of-basis coefficients, known as connection coefficients, connecting Gegenbauer, Jacobi, Laguerre and Hermite polynomials to Legendre polynomials have been worked out. The method has been formulated using Legendre polynomial because an explicit formula for the Fourier coefficients of a Legendre polynomial happens to be known. In principle, the method of connection coefficients can be applied to any polynomial whose Fourier coefficients are already known in explicit form.

Begin with a generic orthogonal polynomial $p_n(x)$ and write either:

$$p_n(x) = \sum_{j=0}^{n} c_{j,n}P_j(x) \qquad (16)$$

or

$$p_n(x) = \sum_{j=0}^{[n/2]} d_{j,n}P_{n-2j}(x) \qquad (17)$$

where $c_{j,n}$ and $d_{j,n}$ are connection coefficients. For Gegenbauer-λ polynomials, Rainville [8] p. 284,

$$d_{j,n} = \frac{\left(\lambda - \frac{1}{2}\right)_k (\lambda)_{n-k}(1+2n-4k)}{k!\left(\frac{3}{2}\right)_{n-k}}.$$

For Jacobi-(α, β) polynomials,

$$\begin{aligned}
c_{j,n} &= \frac{\Gamma(j+1)\Gamma(n+j+\alpha+\beta+1)}{\Gamma(n+\alpha+\beta+1)\Gamma(j+\alpha+1)} \qquad (18)\\
&\times \frac{\Gamma(n+\alpha+1)}{\Gamma(2j+1)(n-j)!} \\
&\times {}_3F_2\left(\begin{array}{c} -n+j, n+j+\alpha+\beta+1, j+1 \\ j+\alpha+1, 2j+2 \end{array}\middle| 1\right).
\end{aligned}$$

This follows from the general formula in Askey [2] p. 62.

For Laguerre-α polynomials, Rainville [8] p. 216,

$$\begin{aligned}
c_{j,n} &= \frac{(-1)^j(1+\alpha)_n(2j+1)}{2^j(n-j)!\left(\frac{3}{2}\right)_n(1+\alpha)_j} \qquad (19)\\
&\times {}_2F_3\left(\begin{array}{c} -\frac{1}{2}(n-j), -\frac{1}{2}(n-j-1) \\ \frac{3}{2}+j, \frac{1}{2}(1+\alpha+j), \frac{1}{2}(2+\alpha+j) \end{array}\middle| \frac{1}{4}\right).
\end{aligned}$$

For Hermite polynomials, Rainville [8] p. 196,

$$d_{j,n} = \frac{(-1)^j n!\, {}_1F_1\left(-j; \frac{3}{2}+n-2j; 1\right)(2n-4j+1)}{j!\left(\frac{3}{2}\right)_{n-2j}}. \qquad (20)$$

The kth Fourier coefficient of $p_n(x)$ can then be expressed for $k \neq 0$ as

$$\begin{aligned}
\widehat{p_n}(k) &= \frac{1}{2}\int_{-1}^{1} p_n(x)e^{-ik\pi x}dx \\
&= \frac{1}{2}\int_{-1}^{1}\sum_{j=0}^{n} c_{j,n}P_j(x)e^{-ik\pi x}dx \\
&= \frac{1}{2}\sum_{j=0}^{n} c_{j,n}\int_{-1}^{1} P_j(x)e^{-ik\pi x}dx \\
&= \frac{1}{2}\sum_{j=0}^{n} c_{j,n}\frac{(-i)^j}{\sqrt{2k}}J_{j+\frac{1}{2}}(\pi k) \qquad (21)
\end{aligned}$$

and for $k = 0$,

$$\begin{aligned}
\widehat{p_n}(0) &= \frac{1}{2}\sum_{j=0}^{n} c_{j,n}\delta_{0,j} \\
&= \frac{c_{0,n}}{2}.
\end{aligned}$$

Alternatively, for $k \neq 0$,

$$\begin{aligned} \widehat{p}_n(k) &= \frac{1}{2}\int_{-1}^{1} p_n(x)e^{-ik\pi x}dx \\ &= \frac{1}{2}\int_{-1}^{1}\sum_{j=0}^{[n/2]} d_{j,n}P_{n-2j}(x)\,e^{-ik\pi x}dx \\ &= \frac{1}{2}\sum_{j=0}^{[n/2]} d_{j,n}\int_{-1}^{1} P_{n-2j}(x)\,e^{-ik\pi x}dx \\ &= \frac{1}{2}\sum_{j=0}^{[n/2]} d_{j,n}\frac{(-i)^{n-2j}}{\sqrt{2k}}J_{n-2j+\frac{1}{2}}(\pi k) \qquad (22) \end{aligned}$$

and for $k = 0$,

$$\begin{aligned} \widehat{p}_n(0) &= \frac{1}{2}\sum_{j=0}^{[n/2]} d_{j,n}\delta_{0,n-2j} \\ &= \frac{d_{[n/2],n}}{2}. \end{aligned}$$

III. Formulas for Orthogonal Polynomials Over a Subinterval

Unlike Gegenbauer and Jacobi polynomials which share the orthogonality interval of $[-1,1]$ with the complex exponentials $e^{ik\pi x}$, Hermite polynomials are orthogonal over $(-\infty,\infty)$ and Laguerre polynomials are orthogonal over the interval $[0,\infty)$. We offer here formulas for expressing a Hermite or Laguerre polynomial over a finite subinterval $[a,b]$ as a Fourier series. The choice of subinterval can depend on the application in question, however for Hermite and Laguerre polynomials there is an interval which may be of particular interest for approximation purposes called the oscillatory region.

The following lemma found in Spanier and Oldham [9] p. 217, describes the oscillatory region for Hermite polynomials.

Lemma 1: The n zeros, $\left[\frac{n}{2}\right]$ minima, and $\left[\frac{n}{2}\right]$ maxima of $H_n(x)$, for $n > 1$, are all located in the interval $\left[-\sqrt{2n},\sqrt{2n}\right]$. Outside of this interval $|H_n(x)|$ increases monotonically, bounded globally by

$$|H_n(x)| < 1.09\sqrt{2^n n! e^{x^2}} \qquad (23)$$

Since the intervals described above are nested as n increases, it is apparent that the resolving power of an $N+1$ term Hermite partial sum lies within the oscillatory region $\left[-\sqrt{2N},\sqrt{2N}\right]$.

The next lemma describes the oscillatory region for Laguerre polynomials.

Lemma 2: The n zeros, $\left[\frac{n}{2}\right]$ minima, and $\left[\frac{n}{2}\right]$ maxima of $L_n^{\alpha}(x)$, $n > 1$, where $\alpha > -1$, are all located in the interval $[0,\beta]$ where

$$\begin{aligned} \beta &= 2n+\alpha+1+\sqrt{(2n+\alpha+1)^2+\frac{1}{4}-\alpha^2} \qquad (24) \\ &\sim 4n+2\alpha+2 \end{aligned}$$

Outside of this interval $|L_n^{\alpha}(x)|$ increases monotonically, bounded globally by

$$|L_n^{\alpha}(x)| < \frac{(\alpha+1)_n}{(1)_n}e^{x/2} \qquad (25)$$

when $\alpha \geq 0$ and

$$|L_n^{\alpha}(x)| < \left(2-\frac{(\alpha+1)_n}{(1)_n}\right)e^{x/2} \qquad (26)$$

when $-1 < \alpha < 0$.

The sources for this lemma are [9] p. 209 for $\alpha = 0$, [10] for the value of β when $\alpha \neq 0$ and [1] p. 786 for the bounds on $L_n^{\alpha}(x)$.

Since the intervals described above are nested as n increases, it is apparent that the resolving power of a Laguerre partial sum lies within the oscillatory region $[0,\beta]$. This suggests that the subinterval $[0,\beta]$ could be a useful one to consider for approximation purposes.

The power series method described above can be used to develop formulas for the Fourier coefficients of an orthogonal polynomial over a subinterval. Define first a special function $\beta_m^{a,b}(z)$ which generalizes the function $\beta_m(z)$.

Definition 3:

$$\beta_m^{a,b}(z) = \int_a^b u^m e^{-zu}du \qquad (27)$$

Then $\beta_m^{a,b}(z)$ has an explicit form given by

$$\beta_m^{a,b}(z) = z^{-m-1}\Gamma(m+1,az) - z^{-m-1}\Gamma(m+1,bz) \qquad (28)$$

which for whole number m has the elementary representation:

$$\beta_m^{a,b}(z) = z^{-m-1}m!\left(e^{-az}e_m(az) - e^{-bz}e_m(bz)\right). \qquad (29)$$

Lemma 4: Let $[a,b]$ be a subinterval of the orthogonality interval. Let $x = \varepsilon\xi + \delta$ where

$$\varepsilon = \frac{b-a}{2}$$

and

$$\delta = \frac{b+a}{2}$$

so that $\xi \in [-1,1]$ when $x \in [a,b]$. Then

$$\int_{-1}^{1}(\varepsilon\xi+\delta)^m e^{-i\pi k\xi}dx = \frac{1}{\varepsilon}e^{ik\delta/\varepsilon}\beta_m^{a,b}\left(\frac{i\pi k}{\varepsilon}\right). \qquad (30)$$

This lemma will be utilized for the explicit evaluation of the kth Fourier series coefficient of $p_n(\varepsilon\xi+\delta)$ expanded in terms of the local variable ξ. Begin with the power series form of a polynomial expressed either as

$$p_n(\varepsilon\xi+\delta) = \sum_{j=0}^{n} a_j(\varepsilon\xi+\delta)^j. \qquad (31)$$

or as

$$p_n(\varepsilon\xi+\delta) = \sum_{j=0}^{[n/2]} b_j(\varepsilon\xi+\delta)^{n-2j}. \qquad (32)$$

Then

$$\begin{aligned} \widehat{p}_n(k) &= \frac{1}{2}\int_{-1}^{1} p_n(\varepsilon\xi+\delta)e^{-ik\pi\xi}d\xi \\ &= \frac{1}{2}\int_{-1}^{1}\sum_{j=0}^{n} a_j(\varepsilon\xi+\delta)^j e^{-ik\pi\xi}d\xi \\ &= \frac{1}{2}\sum_{j=0}^{n} a_j\int_{-1}^{1}(\varepsilon\xi+\delta)^j e^{-ik\pi\xi}d\xi \\ &= \frac{1}{2\varepsilon}e^{ik\delta/\varepsilon}\sum_{j=0}^{n} a_j\beta_j^{a,b}\left(\frac{i\pi k}{\varepsilon}\right). \qquad (33) \end{aligned}$$

Alternatively,

$$\begin{aligned}
\widehat{p}_n(k) &= \frac{1}{2}\int_{-1}^{1} p_n(\varepsilon\xi+\delta)e^{-ik\pi\xi}d\xi \\
&= \frac{1}{2}\int_{-1}^{1}\sum_{j=0}^{[n/2]} b_j\left(\varepsilon\xi+\delta\right)^{n-2j}e^{-ik\pi\xi}d\xi \\
&= \frac{1}{2}\sum_{j=0}^{[n/2]} b_j\int_{-1}^{1}\left(\varepsilon\xi+\delta\right)^{n-2j}e^{-ik\pi\xi}d\xi \\
&= \frac{1}{2\varepsilon}e^{ik\delta/\varepsilon}\sum_{j=0}^{[n/2]} b_j\beta_{n-2j}^{a,b}\left(\frac{i\pi k}{\varepsilon}\right) \qquad (34)
\end{aligned}$$

IV. CONCLUSION

The power series technique and the connection coefficient technique can be applied to obtain explicit formulas for the Fourier coefficients of any class of polynomial. Once formulas for the Fourier coefficients are known, it is then straightforward to write an explicit Fourier series of the corresponding orthogonal polynomial partial sum. Although our concern here has been orthonormal polynomials, the techniques apply to any class of polynomials: orthogonal or non-orthogonal. This is provided that the power series coefficients or the connection coefficients can be found in explicit form. Some of the formulas described here were found to be useful in our own work and we present them with the hope that others may find them useful as well.

REFERENCES

[1] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, Dover 1972.

[2] R. Askey, *Orthogonal Polynomials and Special Functions*, CBMS-NSF Regional Conf. Ser. in Appl. Math. 21, SIAM, Philadelphia, 1975.

[3] H. Bateman, *Higher Transcendental Functions*, Vol. 1, McGraw Hill, New York, 1953.

[4] H. Bateman, *Higher Transcendental Functions*, Vol. 2, McGraw Hill, New York, 1953..

[5] H. Bateman, *Tables of Integral Transforms*, Vol. 1, McGraw Hill, New York, 1954.

[6] H. Bateman, *Tables of Integral Transforms*, Vol. 2, McGraw Hill, New York, 1954..

[7] W. Magnus, F. Oberhettinger, and R. P. Soni, Formulas and Theorems for the Special Functions of Mathematical Physics, Springer Verlag, New York, 1966.

[8] E. Rainville, *Special Functions*, Chelsea Publishing Company, New York, 1960.

[9] J. Spanier and K. B. Oldham, *An Atlas of Functions*, Hemisphere Publishing Company, Washington,1987.

[10] G. Szego, *Orthogonal Polynomials*, American Mathematical Society, Providence, Rhode Island, 1939