Identification of Continuous Time Systems with Direct and Feedback Nonlinearities

CONSTANTIN MARIN, DAN SELISTEANU,
DORIN SENDRESCU, VIRGINIA FINCA, DAN MANCAS
Department of Automatic Control
University of Craiova, Str. A. I. Cuza No.13
ROMANIA
cmarin@automation.ucv.ro http://www.automation.ucv.ro

Abstract: - This paper presents a procedure for the identification of two types of a continuous-time linear system interconnected by direct and feedback memoryless nonlinearities. The first case is the continuous time Hammesrstein system and the second is a specific case of the continuous time Wiener system. The direct and feedback nonlinear elements, described by bounded unknown functions, are expressed as a linear combination of some base functions. Both the parameters of the linear system and of the nonlinear elements representation are identified. To improve the representation of the nonlinear functions, the set of basis functions is iteratively refined. It is possible to identify the dominant nonlinearities, applying the singular value decomposition to the input matrix. In our approach, the linear dynamic subsystem is described by a transfer function of a given order and the distribution based identification method is applied. The DCHI (Distribution based Continuous time Hammerstein system Identification equations (DCHI) and the DCNFI (Distribution based of Continuous time Nonlinear Feedback Identification) equation are obtained. The consistency of the identification is analyzed and experimental results are presented.

Key-Words: - Identification, Nonlinear systems, Distributions.

1 Introduction

Identification of continuous-time systems has a great practical importance because the actual physical processes are characterized by continuous-time models.
In many applications, the structure and parameters of those continuous-time models must be known.
For linear models there are numerous frequency-domain and time-domain identification methods for both continuous-time and discrete-time models. Also there are many results for discrete-time nonlinear systems identification.
Unfortunately there are no direct correlations between the parameters of the physical continuous-time systems and its discretized model.
There are well known different approaches for the identification of nonlinear systems, split into two main categories: nonparametric and parametric methods.
Nonparametric methods are mainly frequency-domain based, including techniques for identifying Volterra kernels or time-domain constructing state-space realizations.
The parametric methods usually are time-domain using both structured and unstructured models. As presented in [1], [2], structured models are expressed as interactions of the linear and nonlinear subsystems.
One of these structures is the so called Hammerstein model, represented by series interconnection of a static nonlinear map N, followed by a linear dynamic model L as in Figure 1, [13], where the intermediate variable $z(t)$ is not measurable. Only information accessible for identification purposes are the input $u(t)$ and the output $y(t)$.

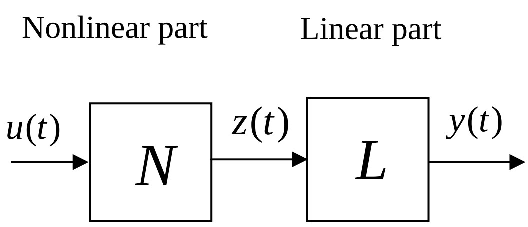

Fig. 1. The Hammerstein model

The Hammerstein models, proposed first by Narendra and Gallman in 1966, [3], are successfully utilized to model a large class of nonlinear systems.
As mentioned in [4], in the 1970s and 1980s, the

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problem of identifying Hammerstein systems was investigated by a number of authors [5], [6], where the identification algorithms assume that the unknown memoryless nonlinear characteristic is a polynomial of a finite and known order.
Other parametric identification methods describe the static nonlinearity by a finite number of basis functions superposition. Usually the linear system is discrete time ARX model.
A continuous time Hammerstein system identification algorithm are presented in [7], [8], [13]. These papers assume that a priori information about both subsystems, that means the nonlinear characteristic is a polynomial of a finite known degree and the order of the linear dynamic system is known too.
The main problem in all these approaches is the nonlinear map representation. Generally it can be expressed as a linear combination of a given set basis functions.
If the selected basis functions do not fit to the unknown nonlinearity structure, the validation errors are high, especially for large ranges of input variations.
Without prior knowledge of the nonlinear mapping form, a large number of basis functions are necessary.
To circumvent these difficulties, based on [13], in this paper an adaptive base identification procedure for continuous-time Hammerstein systems is proposed. It follows the procedure presented in [9] which is applied for discrete time systems only, using the subspace identification method.
To improve the representation of the nonlinear functions, the set of basis functions is iteratively refined. It is possible to identify the dominant nonlinearities, applying a singular value decomposition to the input matrix [9], [10].
In our approach, [13], the linear dynamic subsystem is described by a transfer function of a given order and the distribution based identification method is applied [11], [12].
Another structure approached in this paper is the so called CNF (Continuous time Nonlinear Feedback) systems, [15], represented by a feedback interconnection between a feedforward linear dynamic model L and a feedback static nonlinear map N, as in Figure 2. Both the error $\varepsilon(t)$ and the feedback variable $z(t)$ are not measurable.
Also the only pieces of information accessible for identification purposes are the input $u(t)$ and the output $y(t)$.

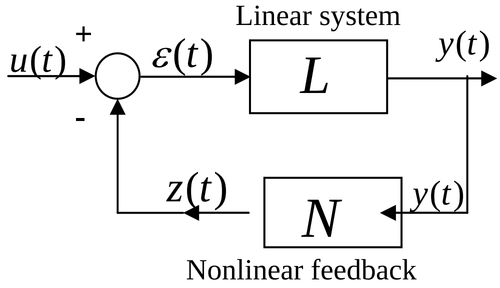

Fig. 2. The Continuous time Nonlinear Feedback system structure.

Such a structure can express a control system whose transductor characteristic is unknown and nonlinear, possible because it changes in time.
As the authors know, this is the first time an identification algorithm is proposed in [15], for such a structure.
This is possible due to the distribution based identification methods as developed by the authors in [11], [12], [13], [14], [16].
There are many papers related to the identification of interconnections between a static nonlinearity and a linear dynamical system as Hammerstein, Wiener, Uryson.
Unfortunately these algorithms consider from the beginning discrete-time models for the dynamical part.

2 Continuous time Hammerstein model

Consider the time invariant state space representation of a single-input single-output nonlinear continuous-time Hammerstein system [8], [13], that cascades a static nonlinearity

$$z(t) = N_1[u(t)] \quad , \tag{1}$$

followed by an asymptotic stable linear dynamic system

$$\dot{x}(t) = A \cdot x(t) + b \cdot z(t),\ \sigma(A) \in \mathrm{C}^{-} \tag{2}$$

$$y(t) = c^T \cdot x(t) + d \cdot z(t), \tag{3}$$

where $x(t)$ is an n-dimensional state vector, $u(t)$ and $y(t)$ are the scalar control input and output of the overall system. A, b, c^T, d are respectively nxn, nxl, 1xn and 1x1, constant matrices.
Next, we assume that the memoryless nonlinear function (1) can be expanded as a linear combination of p basis functions $\{g_j(u)\}_{j=1:p}$, with

the coefficients $\{\gamma_j\}_{j=1:p}$ so,

$$z(t) = N_1[u(t)] = \sum_{j=1}^{p} \gamma_j \cdot g_j(u(t)) = \gamma^T \cdot g(u) , \quad (4)$$

where,

$$\gamma = [\gamma_1 ... \gamma_j ... \gamma_p]^T \quad (5)$$

$$g(u) = [g_1(u) ... g_j(u) ... g_p(u)]^T \quad (6)$$

Depending of the prior information about the nonlinearity, basis functions such as polynomials, splines, sigmoids, sinusoids, or radial basis functions can be used.

The linear part can be expressed by a transfer function with unitary proportional factor

$$H(s) = c^T \cdot (sI - A)^{-1} \cdot b + d \quad (7)$$

$$H(s) \triangleq \frac{Y(s)}{Z(s)} = \frac{b_m \cdot s^m + ... + b_i \cdot s^i + ... + b_1 \cdot s + 1}{a_n \cdot s^n + ... + a_i \cdot s^i + ... + a_1 \cdot s + 1}, \; n \geq m \quad (8)$$

$$K = H(0) = -c^T \cdot A^{-1} \cdot b + d = 1 \quad (9)$$

This does not limits the generality because the inaccessible intermediate variable $z(t)$ can be considered upstream or downstream of a proportional factor.

For example denoting

$$\gamma = K \cdot \gamma', \; \gamma' = [\gamma'_1 ... \gamma'_j ... \gamma'_p]^T , \quad (10)$$

a new intermediate variable $z'(t)$ can be considered as the output of the equivalent nonlinear part

$$z'(t) = N_1'[u(t)] = \sum_{j=1}^{p} \gamma'_j \cdot g_j(u(t)) = \gamma'^T \cdot g(u) . \quad (11)$$

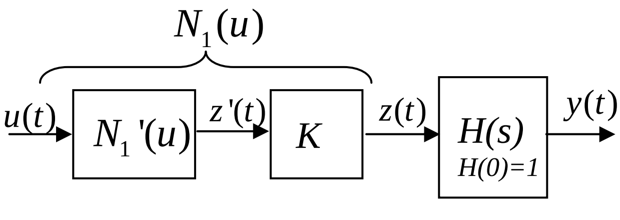

Fig. 3. The equivalent Hammerstein model

3 Continuous time CNF model

Consider the structure depicted in Figure 1 where the feedback static nonlinearity is described by an unknown function, [15]

$$z(t) = N_2[u(t)] . \quad , \quad (12)$$

The feedforward element is a single-input single-output linear continuous-time system, as (13), (14), asymptotic stable, having a state space representation

$$\dot{x}(t) = A \cdot x(t) + b \cdot \varepsilon(t), \; \sigma(A) \in \mathrm{C}^- \quad (13)$$

$$y(t) = c^T \cdot x(t) , \quad (14)$$

where $x(t)$ is an n-dimensional state vector, $u(t)$ and $y(t)$ are the scalar control input and output of the overall system. A, b, c^T, d are respectively nxn, nxl, 1xn and 1x1, constant matrices.

This feedforward element can represent a cascade connection between a continuous time controller and the controlled process in a control system structure.

The system error is

$$\varepsilon(t) = u(t) - z(t) . \quad (15)$$

As in the case of the above Hammerstein system, we assume, [15], that the memoryless nonlinear function (12) can be expanded as a linear combination of p basis functions $\{g_j(u)\}_{j=1:p}$, with the coefficients $\{\gamma_j\}_{j=1:p}$ so,

$$z(t) = N_2[y(t)] = \sum_{j=1}^{p} \gamma_j \cdot g_j(y(t)) = \gamma^T \cdot g(y) , \quad (16)$$

where,

$$\gamma = [\gamma_1 ... \gamma_j ... \gamma_p]^T \quad (17)$$

$$g(y) = [g_1(y) ... g_j(y) ... g_p(y)]^T \quad (18)$$

Depending of the prior information about the nonlinearity, basis functions such as polynomials, splines, sigmoids, sinusoids, or radial basis functions can be used.

The linear part can be expressed by a transfer function with the proportional factor K

$$H(s) = c^T \cdot (sI - A)^{-1} \cdot b \quad (19)$$

$$H(s) \triangleq \frac{Y(s)}{Z(s)} = \frac{b_m \cdot s^m + ... + b_i \cdot s^i + ... + b_1 \cdot s + b_0}{a_n \cdot s^n + ... + a_i \cdot s^i + ... + a_1 \cdot s + 1}, \; n > m \quad (20)$$

$$K = H(0) = -c^T \cdot A^{-1} \cdot b\,. \tag{21}$$

The task of identification is to identify both the feedback nonlinear function and the linear model from input-output measurement data $\{u(t), y(t)\}$ on a finite time interval $t \in [t_0, t_1]$.
The internal variables $\{\varepsilon(t),\ z(t)\}$ are inaccessible for measurements.
In our approach it is possible uniquely to identify all these parameters without the unitary gain condition on the dynamical element.
Considering the transfer function (20) and the nonlinearity expressed by (16), the structure of the CNF model utilised for identification is illustrated in Figure 4.
$\gamma^T \cdot g(y)$

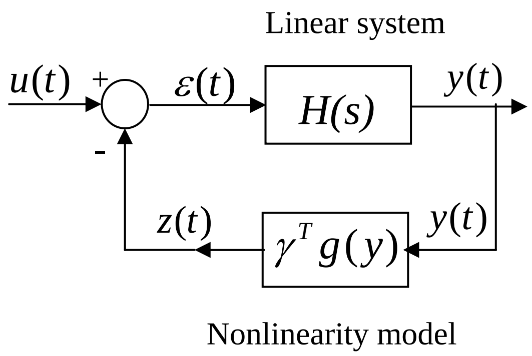

Fig. 4. The CNF model structure for identification

4 Distribution based continuous time Hammerstein system identification equations (DCHI)

The input-output differential equation of the system, based on (4) and (8) is

$$\sum_{i=1}^{n} a_i y^{(i)}(t) + y(t) = \sum_{i=1}^{m}\sum_{j=1}^{p}(b_i\gamma_j)\cdot g_j^{(i)}(u(t)) + \sum_{j=1}^{p}\gamma_j g_j(u(t)) \tag{22}$$

Let us denote by Φ_n the fundamental space from distribution theory, of the real testing functions φ_k, with continuous derivatives at least up to the order n, and compact support T_k, where, $T_k = [t_k^a, t_k^b] \subseteq \mathbb{R}$

$$\varphi_k : \mathbb{R} \to \mathbb{R}, t \to \varphi(t);\ \varphi(t) = 0, \forall t \in \mathbb{R} - T_k\,, \tag{23}$$

Let

$$q : \mathbb{R} \to \mathbb{R}, t \to q(t)$$

be a function that admits a Riemann integral on any compact interval T from $\mathbb{R}$.

Using this function, a distribution

$$Q(\varphi_k) : \Phi_n \to \mathbb{R},\ \varphi_k \to Q(\varphi_k)$$

can be built by the relation

$$Q(\varphi_k) = \int_{\mathbb{R}} q(t)\cdot\varphi_k(t)\cdot dt\,,\ \forall\varphi_k \in \Phi_n\,. \tag{24}$$

The i-order derivative, of $q(t)$, $q^{(i)}(t),\ i = 0:n$, generates, for $\forall\varphi_k \in \Phi_n$, a distribution

$$Q_i(\varphi_k) = \int_{\mathbb{R}} q^{(i)}(t)\cdot\varphi_k(t)\cdot dt = \int_{\mathbb{R}} (-1)^i q(t)\cdot\varphi_k^{(i)}(t)\cdot dt \tag{25}$$

where the derivative burden is undertaken by the known testing function φ_k. For the zero derivatives, we write

$$Q(\varphi_k) = Q_0(\varphi_k)\,.$$

Considering $q(t)$ as being the output signal $y(t)$, the functionals (24), (25) become

$$Y^i(\varphi_k) = \int_{\mathbb{R}} (-1)^i y(t)\cdot\varphi_k^{(i)}(t)\cdot dt\,, i = 0:n \tag{26}$$

Also, considering

$$q(t) = g_j^{(i)}(u(t))\,,$$

functionals (24), (25) become

$$U_j^i(\varphi_k), i = 0:m;\ j = 1:p$$

$$U_j^i(\varphi_k) = \int_{\mathbb{R}} (-1)^i g_j(u(t))\cdot\varphi_k^{(i)}(t)dt\,, i = 0:m; j = 1:p \tag{27}$$

For a given testing function φ_k, by using (26), (27), the differential equation (22) is transformed to an algebraic equation

$$\sum_{i=1}^{n} a_i Y^i(\varphi_k) + Y^0(\varphi_k) = \sum_{i=1}^{m}\sum_{j=1}^{p}(b_i\gamma_j)U_j^{(i)}(\varphi_k) + \sum_{j=1}^{p}\gamma_j U_j^{(0)}(\varphi_k) \quad (28)$$

Defining the linear part parameter vectors

$$a = [a_n \dots a_i \dots a_1]^T ; b = [b_m \dots b_i \dots b_1]^T \quad (29)$$

and taking account of (5), (28) becomes,

$$\mathbf{Y}(\varphi_k)\cdot a + Y^0(\varphi_k) = \mathbf{G}^0(\varphi_k)\cdot\gamma + b^T\cdot\mathbf{G}(\varphi_k)\cdot\gamma \quad (30)$$

where

$$\mathbf{Y}(\varphi_k) = [Y^n(\varphi_k)\dots Y^i(\varphi_k)\dots Y^1(\varphi_k)],\ 1\times n \quad (31)$$

$$\mathbf{G}^i(\varphi_k) = [U_1^i(\varphi_k)\dots U_j^i(\varphi_k)\dots U_p^i(\varphi_k)],\ 1\times p\,. \quad (32)$$

$$\mathbf{G}(\varphi_k) = [\mathbf{G}^1(\varphi_k)^T \dots \mathbf{G}^i(\varphi_k)^T \dots \mathbf{G}^m(\varphi_k)^T]^T,\ m\times p \quad (33)$$

Equation (30) expresses relations between parameters a, b, γ and the functionals $\mathbf{Y}(\varphi_k)$, $\mathbf{G}^i(\varphi_k)$, $Y^0(\varphi_k)$ evaluated for one experiment, generated by the testing function φ_k.

Repeating this experiment for N testing functions $\{\varphi_k\}_{k=1:N}$ and defining the matrices,

$$\mathbf{Y} = [\mathbf{Y}(\varphi_1)^T \dots \mathbf{Y}(\varphi_k)^T \dots \mathbf{Y}(\varphi_N)^T]^T,\ N\times n \quad (34)$$

$$Y^0 = [Y^0(\varphi_1)\dots Y^0(\varphi_k)\dots Y^0(\varphi_N)]^T,\ N\times 1 \quad (35)$$

$$\mathbf{G}^0 = [\mathbf{G}^0(\varphi_1)^T \dots \mathbf{G}^0(\varphi_k)^T \dots \mathbf{G}^0(\varphi_N)^T]^T, N\times p \quad (36)$$

$$\mathbf{W}(b) = [\mathbf{G}(\varphi_1)^T b \dots \mathbf{G}(\varphi_k)^T b \dots \mathbf{G}(\varphi_N)^T b]^T, N\times p \quad (37)$$

the Distribution based Continuous time Hammerstein system Identification equation (DCHI) is obtained

$$\mathbf{Y}\cdot a - \mathbf{G}^0\cdot\gamma - \mathbf{W}(b)\cdot\gamma + Y^0 = 0\,. \quad (38)$$

An equivalent form of (38) is

$$\mathbf{Y}\cdot a - \mathbf{G}^0\cdot\gamma - \mathbf{V}(\gamma)\cdot b + Y^0 = 0\,. \quad (39)$$

where $\mathbf{V}(\gamma)$ is similar to $\mathbf{W}(b)$.

If b is known, from (38), a linear equation in parameters

$$\theta_{a\gamma} = [a^T \ \ \gamma^T]^T$$

can be obtained.

Also if γ is known, then $\theta_{ab} = [a^T \ \ b^T]^T$ can be easy obtained. If both vectors b and γ are unknown, the following vectors are defined,

$$c_i = b_i\cdot\gamma,\ i = 0:m,\ c_0 = \gamma \quad (40)$$

$$\theta = [a^T \ c_0{}^T \ c_1{}^T \ \dots \ c_i{}^T \ \dots \ c_m{}^T]^T \quad (41)$$

From (30) one obtain,

$$\mathbf{H}(\varphi_k)\cdot\theta = -Y^0(\varphi_k) \quad (42)$$

where $\mathbf{H}(\varphi_k)$ is a $1\times q$ matrix, $q = (n + m + m\cdot p)$,

$$\mathbf{H}(\varphi_k) = [\mathbf{Y}(\varphi_k)\ \mathbf{G}^0(\varphi_k)\ \mathbf{G}^0(\varphi_k)\ \dots\ \mathbf{G}^0(\varphi_k)\ \dots\ \mathbf{G}^0(\varphi_k)] \quad (43)$$

Considering N testing functions $\{\varphi_k\}_{k=1:N}$ from (43) the general DCHI equation is built,

$$\mathbf{H}\cdot\theta = \mathbf{F} \quad (44)$$

where

$$\mathbf{H} = [\mathbf{H}(\varphi_1)^T \dots \mathbf{H}(\varphi_k)^T \dots \mathbf{H}(\varphi_N)^T]^T,\ N\times q \quad (45)$$

$$\mathbf{F} = [-Y^0(\varphi_1)\dots -Y^0(\varphi_k)\dots -Y^0(\varphi_N)]^T,\ N\times 1 \quad (46)$$

The least squares estimation

$$\hat\theta = [\hat a^T \ \hat c_0{}^T \ \hat c_1{}^T \ \dots \ \hat c_i{}^T \ \dots \ \hat c_m{}^T]^T \quad (47)$$

is given by

$$\hat\theta = (\mathbf{H}^T\cdot\mathbf{H})^{-1}\cdot\mathbf{H}^T\cdot\mathbf{F}\,. \quad (48)$$

The estimation $\hat a$ results directly from (40)Taking into consideration (47), but $\hat\gamma$ and

$$\hat b_i,\ i = 1:m,$$

are given by relations,

$$\hat\gamma = \hat c_0\,;\ \hat b_i = [\hat c_0{}^T\cdot\hat c_0]^{-1}\cdot\hat c_0,\ i = 1:m\,. \quad (49)$$

5 Distribution based continuous time nonlinear feedback identification equations (DCNFI)

The input-output differential equation of the system, based on (12) and (20) is

$$\sum_{i=1}^{n} a_i y^{(i)}(t) + y(t) = \sum_{i=0}^{m} b_i u^{(i)}(t) - \sum_{i=0}^{m} b_i z^{(i)}(t) \tag{50}$$

or

$$\sum_{i=1}^{n} a_i y^{(i)}(t) + y(t) = \sum_{i=0}^{m} b_i u^{(i)}(t) - \sum_{i=0}^{m} b_i [N(y(t))]^{(i)} \tag{51}$$

The model of CNF system considering the approximation (16) is

$$\sum_{i=1}^{n} a_i y^{(i)}(t) + y(t) = \sum_{i=0}^{m} b_i u^{(i)}(t) + \sum_{i=0}^{m}\sum_{j=1}^{p} (b_i\gamma_j) g_j^{(i)}(y(t)) + \xi(t) \tag{52}$$

where $\xi(t)$ is a noise expressing the modelling errors.

Let us denote by Φ_n the fundamental space from distribution theory, of the real testing functions φ_k, with continuous derivatives at least up to the order n, having a compact support T_k, where, $T_k = [t_k^a, t_k^b] \subseteq \mathbb{R}$

$$\varphi_k : \mathbb{R} \to \mathbb{R}, t \to \varphi(t);\ \varphi(t) = 0, \forall t \in \mathbb{R} - T_k, \tag{53}$$

Let

$$q : \mathbb{R} \to \mathbb{R},\ t \to q(t) \tag{54}$$

be a function that admits a Riemann integral on any compact interval T from $\mathbb{R}$. Using this function, a distribution

$$Q(\varphi_k) : \Phi_n \to \mathbb{R},\ \varphi_k \to Q(\varphi_k) \tag{55}$$

can be built by the relation

$$Q(\varphi_k) = \int_{\mathbb{R}} q(t)\cdot\varphi_k(t)\cdot dt,\ \forall\varphi_k \in \Phi_n. \tag{56}$$

The i-order derivative, of $q(t)$, $q^{(i)}(t)$, $i = 0:n$, generates, for $\forall\varphi_k \in \Phi_n$, a distribution

$$Q_i(\varphi_k) = \int_{\mathbb{R}} q^{(i)}(t)\cdot\varphi_k(t)\cdot dt = \int_{\mathbb{R}} (-1)^i q(t)\cdot\varphi_k^{(i)}(t)\cdot dt \tag{57}$$

where the derivative burden is undertaken by the known testing function φ_k. For the zero derivatives, we write

$$Q(\varphi_k) = Q_0(\varphi_k).$$

Considering $q(t)$ as being the output signal $y(t)$ and the input signal $u(t)$, the functionals (56), (57) become respectively

$$Y^i(\varphi_k) = \int_{\mathbb{R}} (-1)^i y(t)\cdot\varphi_k^{(i)}(t)\cdot dt,\ i = 0:n \tag{58}$$

$$U^i(\varphi_k) = \int_{\mathbb{R}} (-1)^i u(t)\cdot\varphi_k^{(i)}(t)\cdot dt,\ i = 0:m\cdot \tag{59}$$

Also, considering

$$q(t) = g_j^{(i)}(y(t)), \tag{60}$$

the functionals (56), (57) become

$$G_j^i(\varphi_k), i = 0:n;\ j = 1:p$$

$$G_j^i(\varphi_k) = \int_{\mathbb{R}} (-1)^i g_j(y(t))\cdot\varphi_k^{(i)}(t)dt,\ i = 0:n;\ j = 1:p \tag{61}$$

For a given testing function φ_k, by using (58), (61), the differential equation (52) is transformed to an algebraic equation

$$\sum_{i=1}^{n} a_i Y^i(\varphi_k) + Y^0(\varphi_k) = \sum_{i=0}^{m} b_i U^i(\varphi_k) - \sum_{i=0}^{m}\sum_{j=1}^{p} (b_i\gamma_j) G_j^i(\varphi_k) + \Xi(\varphi_k) \tag{62}$$

where $\Xi(\varphi_k)$ is the distribution image of the noise $\xi(t)$ pointed out by the testing φ_k,

$$\Xi(\varphi_k) = \int_{\mathbb{R}} \xi(t)\cdot\varphi_k(t)\cdot dt,\ \forall\varphi_k \in \Phi_n \tag{63}$$

Defining the linear part parameter vectors

$$a = [a_n \ldots a_i \ldots a_1]^T;\ b = [b_m \ldots b_i \ldots b_1\, b_0]^T \tag{64}$$

and taking account of (17), (62) becomes,

$$Y^0(\varphi_k) = -\mathbf{Y}(\varphi_k)\cdot a + \mathbf{U}(\varphi_k)\cdot b - b^T\cdot\mathbf{G}(\varphi_k)\cdot\gamma + \Xi(\varphi_k) \tag{65}$$

where

$$\mathbf{Y}(\varphi_k) = [Y^n(\varphi_k)...Y^i(\varphi_k)...Y^1(\varphi_k)],\ 1\times n \tag{66}$$

$$\mathbf{U}(\varphi_k) = [U^m(\varphi_k)...U^i(\varphi_k)...U^0(\varphi_k)],\ 1\times(m+1) \tag{67}$$

$$\mathbf{G}^i(\varphi_k) = [G_1^i(\varphi_k)...G_j^i(\varphi_k)...G_p^i(\varphi_k)],\ 1\times p$$

$$\mathbf{G}(\varphi_k) = [\mathbf{G}^m(\varphi_k)^T ...\mathbf{G}^i(\varphi_k)^T ...\mathbf{G}^0(\varphi_k)^T]^T,\ (m+1)\times p \tag{68}$$

Equation (65) expresses relations between parameters a, b, γ and the functionals $\mathbf{Y}(\varphi_k)$, $\mathbf{U}(\varphi_k)$, $\mathbf{G}^i(\varphi_k)$, $Y^0(\varphi_k)$ and the residuum $\Xi(\varphi_k)$ evaluated for one experiment, generated by the testing function φ_k. It can be expressed as,

$$Y^0(\varphi_k) = \mathbf{F}[a, b, \gamma, \varphi_k] + \Xi(\varphi_k) \tag{69}$$

Repeating this experiment for N testing functions $\{\varphi_k\}_{k=1:N}$ and defining the matrices,

$$\mathbf{Y} = [\mathbf{Y}(\varphi_1)^T ... \mathbf{Y}(\varphi_k)^T ... \mathbf{Y}(\varphi_N)^T]^T,\ N\times n \tag{70}$$

$$\mathbf{U} = [\mathbf{U}(\varphi_1)^T ... \mathbf{U}(\varphi_k)^T ... \mathbf{U}(\varphi_N)^T]^T, N\times(m+1) \tag{71}$$

$$\mathbf{Y^0} = [Y^0(\varphi_1)...Y^0(\varphi_k)...Y^0(\varphi_N)]^T,\ N\times 1 \tag{72}$$

$$\mathbf{\Xi} = [\Xi(\varphi_1)...\Xi(\varphi_k)...\Xi(\varphi_N)]^T,\ N\times 1 \tag{73}$$

$$\mathbf{W}(b) = [\mathbf{G}(\varphi_1)^T b...\mathbf{G}(\varphi_k)^T b...\mathbf{G}(\varphi_N)^T b]^T, N\times p \tag{74}$$

the Distribution based of Continuous time Nonlinear Feedback Identification equation (DCNFI) is obtained

$$\mathbf{Y^0} = -\mathbf{Y}\cdot a + \mathbf{U}\cdot b - \mathbf{W}(b)\cdot\gamma + \mathbf{\Xi}. \tag{75}$$

An equivalent form of (75) is

$$\mathbf{Y^0} = -\mathbf{Y}\cdot a + \mathbf{U}\cdot b - \mathbf{V}(\gamma)\cdot b + \mathbf{\Xi}\ . \tag{76}$$

where $\mathbf{V}(\gamma)$ is similar to $\mathbf{W}(b)$.

If b is known, from (75), a linear equation in parameters $\theta_{a\gamma} = [a^T\ \ \gamma^T]^T$ can be obtained.

Also if γ is known, then $\theta_{ab} = [a^T\ \ b^T]^T$ can be easy obtained. If both vectors b and γ are unknown, the following vectors are defined,

$$c_i = b_i\cdot\gamma,\ i = 0:m \tag{77}$$

$$\theta = [a^T\ b^T\ c_m{}^T\ c_{m-1}{}^T\ \ ...\ c_i{}^T\ \ ...\ c_1{}^T\ c_0{}^T]^T \tag{78}$$

From (65) one obtain,

$$Y^0(\varphi_k) = \mathbf{H}(\varphi_k)\cdot\theta + \Xi(\varphi_k) \tag{79}$$

where $\mathbf{H}(\varphi_k)$ is a $1\times q$ matrix, where

$$q = (n + m + 1 + (m+1)\cdot p), \tag{80}$$

$$\mathbf{H}(\varphi_k) = [-\mathbf{Y}(\varphi_k)\ \mathbf{U}(\varphi_k)\ \mathbf{G}^m(\varphi_k)\ \ ...\ \mathbf{G}^i(\varphi_k)\ ...\ \mathbf{G}^0(\varphi_k)] \tag{81}$$

Considering N testing functions $\{\varphi_k\}_{k=1:N}$ from (81) the general DCNFI equation is built,

$$\mathbf{Y^0} = \mathbf{H}\cdot\theta + \mathbf{\Xi} \tag{82}$$

where

$$\mathbf{H} = [\mathbf{H}(\varphi_1)^T ...\mathbf{H}(\varphi_k)^T ...\mathbf{H}(\varphi_N)^T]^T,\ N\times q \tag{83}$$

$$\mathbf{F} = [Y^0(\varphi_1)...Y^0(\varphi_k)...Y^0(\varphi_N)]^T = \mathbf{Y^0},\ N\times 1 \tag{84}$$

The least squares estimation

$$\hat{\theta} = [\hat{a}^T\ \ \hat{c}_0{}^T\ \ \hat{c}_m{}^T\ \ ...\ \hat{c}_i^T\ \ ...\ \hat{c}_0{}^T]^T \tag{85}$$

is given by

$$\hat{\theta} = (\mathbf{H}^T\cdot\mathbf{H})^{-1}\cdot\mathbf{H}^T\cdot\mathbf{F}\,. \tag{86}$$

The estimations $\hat{a}$ and $\hat{b}$ result directly from (86) and (78). Taking into consideration (77), $m+1$ expressions $\hat{\gamma}_i$ for the same vector γ are obtained,

$$\hat{\gamma}_i = \hat{c}_i / \hat{b}_i,\ i = 0:m\,. \tag{87}$$

The least squares approximation of the vector γ is given by the relation,

$$\hat{\gamma}^T = (\hat{b}^T\cdot\hat{b})^{-1}\cdot\hat{b}^T\cdot[\hat{c}_m\,\hat{c}_1\cdots\hat{c}_0]^T\,; \tag{88}$$

$$\hat{\gamma} = \sum_{i=0}^{m} \hat{b}_i \cdot \hat{c}_i / \sum_{i=0}^{m} \hat{b}_i^2 \tag{89}$$

6 Algorithm for the base refining

As we can see, from (86) and (87) simultaneously there are estimated a, b, γ assuming given the set of base functions (18).

The choice of them is not an easy task. If the base functions are properly chosen, that means the identification model is consistent, for the case of free noise estimation, instead of (87), would have

$$\hat{\gamma} = \hat{c}_i / \hat{b}_i \Rightarrow \hat{c}_i = \hat{b}_i \cdot \hat{\gamma}\ , \quad i = 0:m\,. \tag{90}$$

because as the true value of the vector γ is unique. This determines the set of vectors $\hat{c}_i,\ i = 0:m$ from (77) to be collinear, that means

$$\hat{\alpha}_i = 0,\ i = 1:m\,, \tag{91}$$

where $\hat{\alpha}_i$ is the angle between $\hat{c}_0$ and $\hat{c}_i,\ i = 0:m$,

$$\hat{\alpha}_i = \measuredangle\{\hat{c}_i, \hat{c}_0\} = \arccos\{(\hat{c}_i^T \cdot \hat{c}_0\} / \|\hat{c}_i\| / \|\hat{c}_0\|\} \tag{92}$$

If the base functions (18) contain some free parameters, they can be optimized by minimizing the criterion,

$$J = \sum_{i=1}^{m} (\hat{\alpha}_i)^2 \tag{93}$$

which is the identification consistency criterion.

The selective refinement algorithm proposed in [9], can be directly applied to above DCNFI. To do this, consider an initial set of p' basis functions (18)

$$g'(y) = [g_1'(y) \ldots g_j'(y) \ldots g_{p'}'(y)]^T\,. \tag{94}$$

Building DCNFI as (86), the vectors $\hat{c}_0'; \hat{c}_i', \hat{b}_i', i = 1:m$ are evaluated. Define

$$\hat{\gamma}_i' = \hat{c}_i' / \hat{b}_i', i = 0:m;\ \hat{b}_0' = 1 \tag{95}$$

$$\Gamma' = [\hat{\gamma}_0' \hat{\gamma}_1' .. \hat{\gamma}_i' .. \hat{\gamma}_m]^T, (m+1) \times p'\,. \tag{96}$$

Performing the singular value decomposition of Γ',

$$\Gamma' = M' \cdot \Sigma' \cdot N' \tag{97}$$

we retain the $q' \ll p'$ largest singular values in Σ'. So it is possible to approximate

$$\Sigma' \approx \hat{\Sigma}' = L' \cdot R', rank\{\hat{\Sigma}'\} = q'\,, \tag{98}$$

where the $(m+1) \times q'$ matrix L' has a full column rank and the $(m+1) \times q'$ matrix R' has a full row rank. As mentioned in [9], the retained q' largest singular values can be incorporated into either L' or R'. We can write,

$$\Gamma' \cdot g'(u) = M' \cdot \Sigma' \cdot N' \approx M' \cdot \hat{\Sigma}' \cdot N' =$$

$$= (M' \cdot L') \cdot (R' \cdot N' \cdot g'(u)) = \Gamma'' \cdot g''(u)\,. \tag{99}$$

In such away $g''(u)$ can be viewed as dominant nonlinearities, as the rank of the nonlinear map.

Following the same procedure, the nonlinear map $g''(u)$ can now be refined to $g'''(u)$ and so on.

7 Experimental results

The identification procedure developed in this paper has been implemented in Matlab. Considering the structure from Figure 1, the linear part is described by the transfer function

$$H(s) = \frac{3s + 1}{4s^2 + 0.2s + 1} \tag{100}$$

and the feedback nonlinear element is described by

$$z = r_1 \cdot \operatorname{atan}(y / r_2) * 2 / \pi \tag{101}$$

where r_1 and r_2 are unknown parameters. In the simulation, they were considered

$$r_1 = 0.1;\ r_2 = 1\,.$$

The input output variables obtained from this system and utilised for identification are depicted in Figure 5.

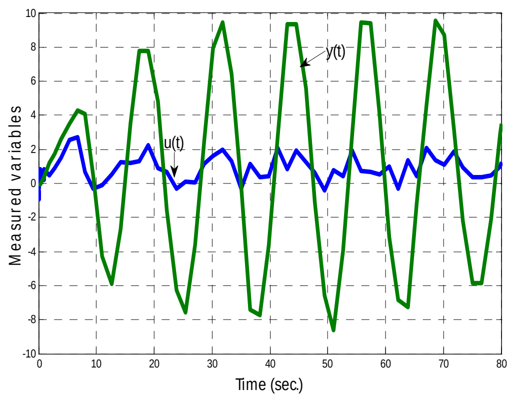

Fig. 5. The measured variables for identification.

Twelve types of testing functions $\varphi(t)$, characterized by a bounded support

$$T = [t_a, t_b], t_a < t_b \tag{102}$$

are considered.

All of these accomplish the condition

$$\varphi(t)=0,\ \forall t\in(-\infty,t_a]\cup[t_b,\infty). \tag{103}$$

The nonzero restriction, is of the form

$$\varphi(t)=\alpha\cdot\beta(t_a,t_b)\cdot\Psi(t,t_a,t_b)\in\Phi_n \tag{104}$$

where

$$\Psi(t)=\Psi(t,t_a,t_b)\in \mathrm{C}^n[(t_a,t_b)] \tag{105}$$

is one of the four types, with $p\ge n+1$.

1.Exponential:

$$\Psi(t)=\exp[\,|t_a\cdot t_b|/(t-t_a)\cdot(t-t_b)\,];$$

2. Sinusoidal:

$$\Psi(t)=\sin^p[\pi\cdot(t-t_b)/(t_b-t_a)],$$

3. Polynomial :

$$\Psi(t)=(t-t_a)^p\cdot(t-t_b)^p,$$

4. Product :

$$\Psi(t)=f_a(t)\cdot f_b(t),$$

where

$$f_a\in \mathrm{C}^n[(t_a,t_b)],\ f_b\in \mathrm{C}^n[(t_a,t_b)],\ p_a;p_b\ge n+1$$

$$f_a^{(k)}(t_a)=0,k=0:p_a;\ f_b^{(k)}(t_b)=0,k=0:p_b.$$

For each of the four types, three variants can be implemented with respect to the coefficient

$$\beta=\beta(t_a,t_b).$$

In (104), α is a scaling factor.

a. Free amplitude:

$$\beta(t_a,t_b)=1,\ \forall t_a,t_b$$

b. Normalized peak:

$$\beta(t_a,t_b)=1/\max_{t\in T}|\Psi(t,t_a,t_b)|,\ \forall t_a,t_b$$

c. Normalized area:

$$\beta(t_a,t_b)=1/\int_{t_a}^{t_b}\Psi(t,t_a,t_b),\ \forall t_a,t_b$$

The nonlinear feedback function is approximated by a six order Bezier function [17]

$$z(y)=\sum_{j=1}^{6}\gamma_j\cdot B_{j-1}^{5}\left(\frac{y-y_{\min}}{y_{\max}-y_{\min}}\right) \tag{106}$$

where $B_j^n(s)$ is the j-Bernstein function of the degree n,

$$B_j^n(s)=\frac{n!}{j!\cdot(n-j)!}\cdot s^j\cdot(1-s)^{n-j},\ s\in[0,1] \tag{107}$$

The following results are obtained:

$$a=[a_2\,a_1]=[4\ 0.2];\ \hat{a}=[\hat{a}_2\,\hat{a}_1]=[3.678\quad 0.846];$$

$$b=[b_1\,b_0]=[3\quad 1];\ \hat{b}=[\hat{b}_1\,\hat{b}_0]=[2.878\quad 0.967];$$

$$\gamma=[\ \gamma_1\quad \gamma_2\quad \gamma_3\quad \gamma_4\quad \gamma_5\quad \gamma_6\ ]=$$
$$=[-0.0937\,-0.0895\,-0.0705\,0.0705\ 0.0895\ 0.0937];$$

$$\hat{\gamma}=[\ \hat{\gamma}_1\quad \hat{\gamma}_2\quad \hat{\gamma}_3\quad \hat{\gamma}_4\quad \hat{\gamma}_5\quad \hat{\gamma}_6\ ]=$$
$$=[-0.0881\ -0.0935\ -0.0653\ 0.0731\ 0.0913\ 0.0987];$$

Figure 6 shows the real and identified feedback nonlinearities.

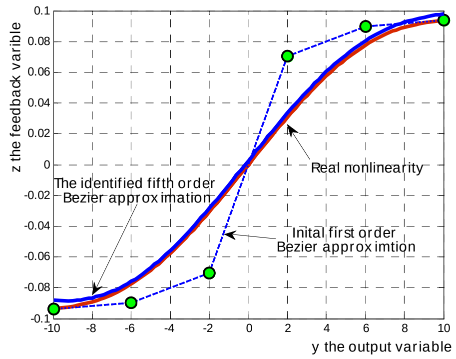

Fig. 6. The real and identified feedback nonlinearities.

5 Conclusions

Using techniques from distribution theory it is possible to obtain algebraic equations with respect to unknown parameters for continuous time systems with nonlinear feedback.. The base functions can now be refined using the method of singular value decomposition and the consistency condition.

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