

# Implementation of the systems approach in mathematical modeling, dynamic simulation and visualization using MS Excel Spreadsheet

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**Abstract**—One of the most important tasks in teaching of mathematics and science is to use a suitable method for developing of theoretically acquired knowledge.

Drawing on their own experience, the authors have found that the best method to be used is the system approach. System approach together with modeling and computer simulation can be understand as one of the most important method in teaching of mathematics and science and simultaneously can be use as suitable method for practical developing of theoretically acquired knowledge.

The two case studies, described in the paper, present possibilities of implementation of system approach in creation of mathematical model as well as computer simulation model.

The first case study presents animation of the shape of quadratic function depending on change of parameter  $c$  in MS Excel worksheet and chart.

The second case study presents creation of simulation model of damped harmonic oscillator. The solution is demonstrated progressively – it starts from the physical and mathematical analysis of the problem, then the numerically solution of the mathematical model in MS Excel is presented and it finishes with the presentation of the visualization of motion of the oscillator in MS Excel chart.

**Keywords**—Animation, computer simulation, damped harmonic oscillator, mathematical modeling, quadratic function, system approach, visualization

## I. INTRODUCTION

THE term “developing of theoretically acquired knowledge” can be connected with the terms “system”, “model”, “simulation”, “multidisciplinary approach”, which are important in current approach to scientific, technological and professional practice.

Many universities are realizing that modeling and simulation is becoming an important tool in finding the strategy for solving and understanding numerous and diverse problems. The examples can be found e.g. in [1], [2], [3], [4].

In this paper we first briefly introduce the theory of system approach, modeling and simulation as a method of multidisciplinary and system approach to education.

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Secondly we introduce two case studies illustrating step by step process of development of computer simulation and visualization of mathematics functions.

## II. THE SYSTEM APPROACH, MODELING AND SIMULATION

### A. System approach and education

System approach, scientific modeling and computer simulation are the terms which are closely related.

System approach enables description of the real processes, without which mathematical model and computer simulation model cannot be realized.

From the education point of view the system approach can be applied:

- In the field of sciences - in case of models and simulations of science processes and phenomena;
- In the technical field - in case of the process control of machines and simple robots;
- In the humanities and social studies - in case of the processes and phenomena associated with this issue;
- In management - in case of the management processes and quality control processes.

The other information on system approach in education can be found e.g. in [1], [2], [3] or [4].

### B. Multidisciplinary approach

System approach, modeling and simulation of real processes are also closely related to multidisciplinary approach.

Multidisciplinary approach generally means that specialized disciplines are applied in a study of real process. These disciplines provide partial analysis of the process. These mono-disciplinary analyses are integrated to overall solution by integrating the solver who has basic multi-disciplines knowledge.

### C. Modeling

Modeling is a method that is often used in professional and scientific practice in many fields of human activity.

The main goal of modeling is not only describing the content, structure and behavior of the real system representing a part of the reality but also describing the processes.

The process can be understood as series of transformations that changes the input values to output values. From the system

point of view the process is dynamic system in which the values of the characteristic of the system elements are changed under the influence of the external elements.

The models are always only approaching of the reality, because the real systems are usually more complex than the models are. The system homomorphism is applied in the process of modeling, which means that each element and interaction between the elements of the model corresponds to one element and interaction of the modeled real system or real process, but the reverse is not true. The model is always to be understood as simplification of the original. If the relation of isomorphism is between the model and real system the original model we could not distinguish between the model and the original, which is discussed e.g. in [5] and [6].

The first step in the process of computer simulation is creation of conceptual model of the studied real system / real process. Conceptual model can be represented in different way. The most used representations are:

- Mathematical equitation;
- Static or dynamic 2D scatter charts.

*Mathematical equitations* establishes mathematical model of the studied real system. The model can be obtained either theoretically based on basic physical properties of the system, or numerically by means of the measured values. Determination of parameters of theoretical model developed from empirical data is called system identification.

*Scatter charts (scatter plot)* is a type of mathematical diagram using Cartesian coordinates to display values for two variables for a set of data in 2D types of the chart. The scatter plot takes two sets scalar variables and uses them for two axes in 2D space. Time depending variables create dynamic scatter chart representing time depending conceptual model of the dependency of system outputs on its inputs. The two case studies illustrating step by step process of development of computer simulation of dynamic mathematical model is presented in the paper.

#### D. Simulation

The process of modeling is closely related to the simulation. Simulation can be understood as process of executing the model. Simulation enables representation of the modeled real system or real process and its behavior in real time by means of computer. The simulation enables also visualization and editing of the model.

A typical simulation model can be written both through specialized programming languages that were designed specifically for the requirements of simulations, or the simulation model can be created in standard programming languages.

From the above considerations, it is clear that simulation is a process that runs on the computer. In some publications, therefore, can be found the term "computer simulation". It generally is valid that computer simulation is a computer-implemented method used for exploring, testing and analysis of properties of the conceptual (mathematical or process) models that describe the behavior of the real systems or real

process which cannot be solved using standard analytical tools, see e.g. [6].

The simulation models represented by executable computer program have to be isomorphic with the conceptual model that is a representation. It means that the mathematical model and simulation model have to represent the real system, its elements, internal interactions and external interaction with the environment in the same way.

Two case studies demonstrated the dynamic simulation and visualization of the mathematical functions in MS Excel is presented in the paper. – see e.g. [5], [6].

#### E. Significant function of the simulation

Simulation has from the scientific point of view several functions – see e.g. [5].

We will focus in this paper two of them and they are:

- replacing the real process;
- development of educational process.

##### 1) Replacement of the real process

This is an important and indispensable feature of simulations and simulation model because it allows realize a situation of the process that cannot be investigated conventionally. The main advantage of simulations is that simulations model allows providing rather big number of the process steps in relatively short time, changing of input parameters and its visualization and optimization of the process.

##### 2) Development educational process

The simulation is very useful from educational point of view. Using the simulation model and visualization of simulation results on the screen, students can better understand the basic features of the processes and systems and develop their intuition. It is also essential that the teaching by means of simulation is much cheaper and faster than the teaching carried by real experiment. In some cases providing the real experiment cannot be feasible.

#### F. Model verification and validation

Verification and validation are important aspects of the process modeling and simulation. They are essential prerequisites to the credible and reliable use of a model and its results [6].

##### 1) Verification

In modeling and simulation, verification is typically defined as the process of determining if executable simulation model is consistent with its specification – e.g. conceptual model. Verification is also concerned with whether the model as designed will satisfy the requirements of the intended application. Verification is concerned with transformational accuracy, i.e., it takes into account simplifying assumptions executable simulation model. Typical questions to be answered during verification are:

- Does the program code of the executable simulation model correctly implement the mathematical model?

- Does the simulation model satisfy the intended uses of the model?
- Does the executable model produce results when it is needed and in the required format?

## 2) Validation

In modeling and simulation, validation is the process of determining the degree to which the model is an accurate representation of the real system / real process. Validation is concerned with representational accuracy, i.e., that of representing the real system / real process in the conceptual model and the results produced by the executable simulation model. The process of validation assesses the accuracy of the models. The accuracy needed should be considered with respect to its intended uses, and differing degrees of required accuracy may be reflected in the methods used for validation. Typical questions to be answered during validation are:

- Is the mathematical model a correct representation of the real system?
- How close are the results produced by the simulation executable model to the behavior of the real system?
- Under what range of inputs are the model's results credible and useful?

Validation and verification are both ultimately activities that compare one thing to another. Validation compares real system / real process and conceptual model. Verification compares conceptual model and executable simulation model. Sometimes validation and verification are done simultaneously in one process.

Validation of the dynamic mathematical model as well as verification of the dynamic simulation model are to be done simultaneously by the comparison of the measured data reached from the real experiment with the dynamic simulation model created and visualized in MS Excel 2D scatter chart.

The whole process of transformation from a real system, the simulation model and its visualization is shown in Fig. 1.

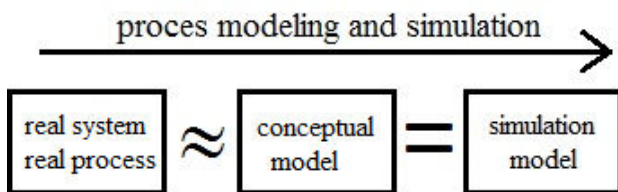


Fig. 1 Process modeling and simulation

Here again let us summarize that the mathematical model that reflects the real system / real process has some limitations and simplifying assumptions (the real system / process and conceptual model are in homomorphic relation).

In contrast, the simulation model is only the computer expression of the conceptual model (the conceptual model and simulation model are in isomorphic relationship).

## III. DYNAMIC SIMULATION AND VISUALIZATION IN MS EXCEL CHART

Before the modeling and dynamic simulation of the mathematical functions will be demonstrated in the case studies, the principle of the dynamic simulation and visualization in MS Excel has to be discussed.

### A. Dynamic simulation model

Dynamic simulation is created based on the dynamic mathematical model. Dynamic mathematical model is model, which represents dynamic real system, e.g. the system which is time dependent.

Time dependent system is based on the system theory such system, which changes its status during the time. In sciences like mathematics, physics, chemistry we are speaking about the time dependent variables, which describe status of the system.

### B. Visualization

Term visualization is close related to the simulation. Visualization is a technique, which shows both input data and output data and its dependencies reached from the simulation model. Visualization helps data analysis and interpretation. Commonly used tools for visualization are 2D and 3D charts, tables, flowcharts, diagram etc.

### C. Animation

Special type of the visualization of the dynamic simulation data is animated diagrams.

Animation generally means a way how a static image can be seemingly set in motion. The principle of an animation is recording of a sequence of images which slightly differ. During a quick viewing of subsequent images an impression of movement occurs, due to the phenomenon of the persistence of vision. The frequency of changing of individual images is proportional to the displacement of the object in consecutive images. (The bigger the displacement of objects in each image is, the greater the frequency of changing these images must be to maintain a sense of motion).

There are many visualizations types of software, which enable animate dynamic simulation data. In the paper we will describe MS Excel as powerful tool for animation of time dependency variables in the scatter chart.

The principle of the animation of the scatter chart in MS Excel is based on the change of one or more parameters of the charts input data of the dynamic objects or curves. The objects or curves step by step change its position in relation to the static baselines which are time dependence function of one or more parameters. A change of these parameters can be achieved by using so-called *iterative calculation* of values in the cell.

### D. Iteration

In general, iteration means repetition of certain activities - the word comes from the Latin *iteretur* - repeat. The iteration in mathematics and numerical methods is a process of applying a function  $f$  repeatedly, using the output value  $x_{n+1}$  of the function as input argument  $x_n$  of the function in the next step.

This can be written by mathematical formula

$$x_{n+1} = f(x_n) \tag{1}$$

A typical example of the use of iterations is the numerical solution of nonlinear equations by the general iterative method - see e.g. [7]. A typical example of a nonlinear equation is the transcendental equation

$$x = \cos x \tag{2}$$

To find the approximate solution, one has to properly set the initial value of  $x$  (e.g.  $x_0 = 1$ ), to provide the iteration ( $x_{i+1} = \cos x_i$ ) and to see how the value of  $x$  is changing. The termination of the iterations is determined by the required accuracy -  $\varepsilon$ . The calculation will be terminated if the following condition is fulfilled:

$$|x_{i+1} - x_i| < \varepsilon \tag{3}$$

Apart from solving of the nonlinear equations, the iteration calculus can be used e.g. for determination of values of recursively given sequences.

As mentioned e.g. in [4] or [5] the sequences are defined either by explicit formulas for the  $n$ -th terms or recursively which means that the previous values of terms of the sequence are used to generate the value of the next term of the sequence. In other words one needs to know the initial value  $x_0$  and the recurrence construction to be able to find subsequent terms of the sequence.

Let us briefly remind a basic mathematical formulation of recursively defined sequences. Given a real function  $f: N_0 \times R \rightarrow R$  and an initial value  $x_0 \in R$ , consider the sequence of iterates of  $x_0$  under the given function  $f$ :

$$\begin{aligned} &x_0, \\ &x_1 = f(0, x_0), \\ &x_2 = f(0, x_1), \\ &x_3 = f(0, x_2), \\ &\dots \end{aligned} \tag{4}$$

Such a sequence is called a recursively defined sequence. It means that for a given  $x_0$ , we can compute an arbitrary term of sequence  $x_t$ , where  $t = 0, 1, 2, \dots$ , according to the relation:

$$x_{t+1} = f(0, x_t) \tag{5}$$

This relation, together with an initial value  $x_0$  is called a first order recurrence equation. The solution to this equation is any sequence  $x_t$  generated by this equation.

The termination condition in the case of recurrently given sequences is given by number of sequence terms -  $t_{max}$ .

Example of such sequence can be sequence given by following formula:

$$\begin{aligned} &x_0 = 0, \\ &x_{t+1} = x_t + 2 \\ &t_{max} = 5 \end{aligned} \tag{6}$$

with the terms 0, 2, 4, 6, 8.

*E. Iteration in algorithm development and programming*

Iteration in algorithm development and programming means a repeated call of the function whereas the argument of the function is the result of the previous function call. In this sense, the iteration corresponds to the aforementioned case of recursively given sequences. The iteration in algorithm development and programming is therefore called recursion.

An important part of recursion is a terminating condition which determines when the recursion is to be stopped. The recursion in programming is quite often used, as well as e.g. the assignment statement  $A := A + 2$  can be understood as a form of recursion based on (13) with the number of repetition  $t_{max} = 1$ . This expression means that the new variable value equals to the original value increased by 2.

*F. Iteration /recursion in the MS Excel*

The circular reference can occur in the formula of the cell in the Excel sheet. This means that the formula in a cell refers to the value of the same cell, eventually that the formula of the cell refers to the value of another cell, which has a formula linking back to the original cell – see e.g. [8].

The examples of the circular references are shown on the Figure 2.

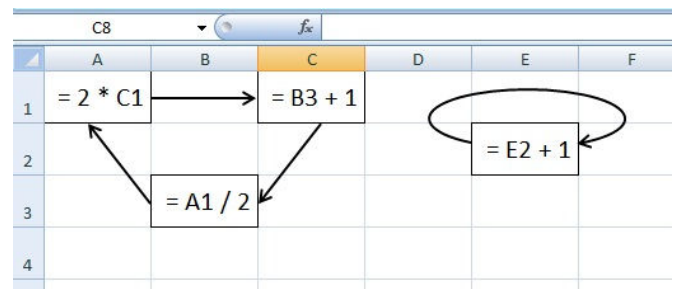


Fig. 2 Examples of circular reference in MS Excel

The value of such cells can be determined only by the iterative calculation. If such an iterative calculation is not possible in MS Excel option tools (default is disabled), the calculation will be executed and an error message will be shown.

The iterative calculation in version of MS Excel 2007 can be enabled in the "Excel Options" dialog box (which is reached by clicking to the Excel option button in the main menu) in the folder "Formulas" – see Figure 3. The "Enable iterative calculation" checkbox has to be checked.

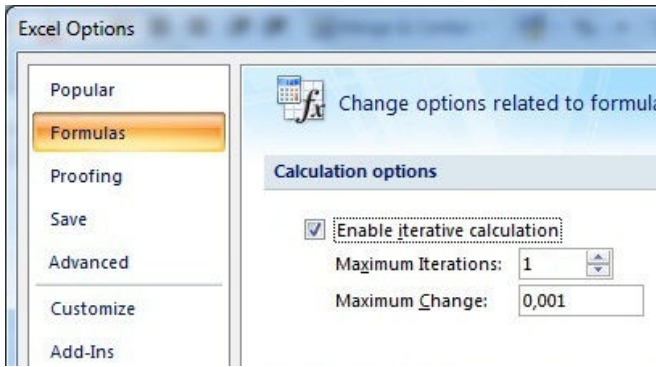


Fig. 3 Setting of iterative calculation parameters in MS Excel

The item "Maximum Iterations" specifies the maximum number of iterations, repetitions, which Excel provides in an iterative calculation. The item "Maximum Change" sets the accuracy of the iterative calculation - the value of the cell before and after the iteration is compared. If the absolute difference of the both values is smaller than the accuracy of the calculation, the iteration is stopped. The default value of "Maximum Iterations" is set to 100 and the default value of "Maximum Change" is set to 0.001.

The iterative calculation and its termination can be expressed in the form of the algorithm flowchart in Figure 4, where the following variables are used:

- NI - the number of the already done iterations;
- Xold - the value of the cell before the iteration;
- Xnew - the value of the cell after the iteration;
- MI - the maximum iterations;
- MC - the maximum change.

It is clear that the terminating condition is given either by the maximum of iterations (13) or by condition (10).

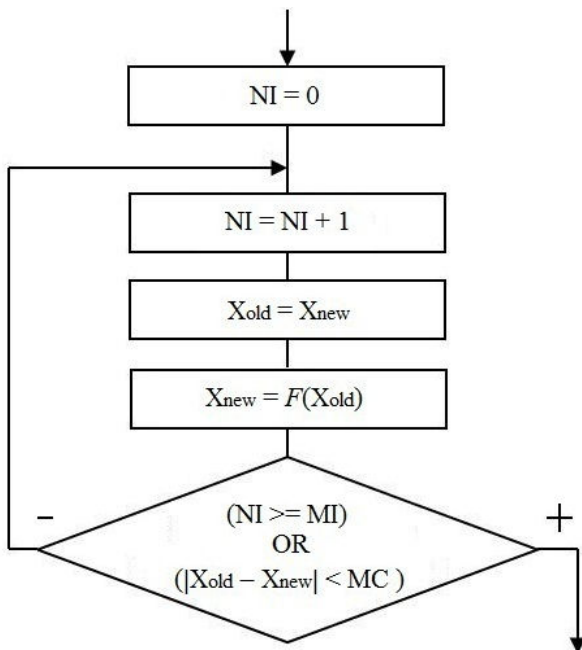


Fig. 4 Flowchart of iterative calculation in MS Excel

IV. CASE STUDY 1  
ANIMATION OF QUADRATIC FUNCTION

Case study demonstrating animation of the XY - scatter chart will be introduced in the following text – see [8].

Animation will be realized in quadratic function:

$$f(x) = a(x + b)^2 + c, \tag{7}$$

where the one of the parameters  $a, b, c$  can be iteratively changed and this change will change the source data for MS Excel chart.

If the parameter  $c$  of the quadratic equation (7) changes (with constant parameters  $a$  and  $b$ ), the graph will move in the direction of the  $y$ -axis. Preparation of data for the XY – scatter graph quadratic functions and inserting the chart will be discussed later.

A. Setting of parameters and formulas for iterative recalculation

Now the setting of the MS Excel iterative recalculation of the parameter  $c$  and calculation of other data needed for drawing of the quadratic function in the XY – scatter chart is to be presented.

The value of the parameter  $c$  is saved in cell B4. Simultaneously the circular reference formula

$$= B4 + E2,$$

is entered into cell B4.

This formula expresses that the value of cell B2 after one iterative recalculation step increases by the increment value entered in cell E2 - iteration (animation) step.

Iterative recalculation starts by pressing of F9 key. The number of the iteration steps is set by the value of "Maximum Iterations". To make only one step of the iteration, defined by formula in cell B4, the value of "Maximum Iterations" has to be equal to 1, as is evident from the Figure 3.

By repeated pressings, eventually by holding down the key F9, the value of cell B4 will step by step increase by the value of the animation step – value in cell E2.

If only the formula

$$= B4 + E2$$

was entered to the cell B4, the value of B4 would after each iteration step still increase. It is therefore necessary to determine the *terminating condition*, i.e. *minimum* and *maximum* limits, given by the values in the cells C2, D2 respectively, so that only within this interval the value of cell B4 will increase.

This can be done by adding the following condition to formula in cell B4:

$$= IF (B4 >= D2, C2, B4 + E2).$$

B4		fx = IF(B4>=D2;C2;B4+E2)			
A	B	C	D	E	F
1	$f(x) = a(x+b)^2 + c$	Min	Max	Step	
2	$a = 2,00$	-30	30	0,1	
3	$b = 2,00$				
4	$c = 3,60$				
5	$x$	$f(x)$			
6	-10	131,6			
7	-9,9	128,42			
8	-9,8	125,28			
9	-9,7	122,18			
10	-9,6	119,12			

Fig. 5 Solution of the iterative calculation of the parameter  $c$ 

This condition compares the value in cell B4 with the value in cell D2. If the condition is true ( $B4 \geq D2$ ) the value of B4 changes to the minimum given by value in cell C2. Otherwise, the iterative calculation is carried out and the value of B4 cells increases by the value of the iteration step (value of the cell E2). The situation is shown on the Figure 5.

The iterative recalculation of the parameter  $c$  of quadratic function is changed within the range from a minimum value (value of the cell C2) to a maximum value (value of the cell D2) with the iteration step (the value of the cell E2) by repeated pressings or by holding of key F9. Within this interval, the centre of parabola will "move up" from lowest (minimum value of  $c$ ) to highest (maximum value of  $c$ ) within the XY scatter chart.

### B. Creation of the XY scatter chart

The chart of the quadratic function (7) will be drawn in the interval  $x \in \langle 10, 10 \rangle$ .

The iterative calculation will be used to change the parameter  $c$ , whose value is saved in cell B4 (see above), the coefficients  $a$  and  $b$  are constant and are saved in the cells B2 and B3.

Into one of the columns (e.g. column A) the  $x$  coordinates of all displayed points of quadratic function are saved.

With regard to the accuracy of the displayed chart it is necessary to choose the coordinates relatively "thick". In our case study we take difference between  $x$  coordinate:  $dx = 0.1$ .

The auto copy of series of numbers by dragging the handle of the cells is used to create the set of  $x$  coordinates. Number of such created points is 201.

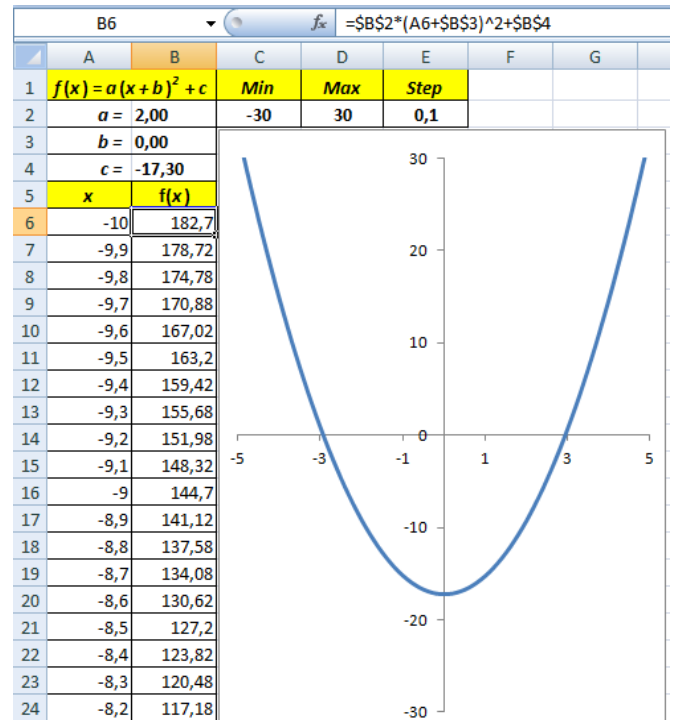


Fig. 6 Data settings for animated XY scatter chart

In column B, the values of the function  $f(x)$  is calculated. Formula saved in cell B6 is shown on Figure 6. Links of the coefficients  $a$ ,  $b$ ,  $c$  to cells B2, B3 and B4 are absolute. Links to the function argument – values  $x$  in column A are relative. Formula can be copied by auto copy of the cells.

Source data for the chart are saved in the range A6:B206. These data are to be recalculated after each iterative recalculation of the coefficient  $c$  of quadratic function (7) within the limits and step as mentioned in the above text. Simultaneously the curve of quadratic function is shifted in the direction of the  $y$  axis. Suitable choice of the size of the iteration step (value in cell E2),

The graph appears as an animated if value of iteration step is set appropriately – see Figure 6.

## V. CASE STUDY 2

### NUMERIC CALCULATION AND ANIMATION OF DAMPED HARMONIC OSCILLATOR

In this case study the animation of time dependence of displacement of damped harmonic oscillator will be demonstrated.

#### A. Mathematical analysis of the damped harmonic oscillator

The equation of the motion in the form

$$m\ddot{x} + b\dot{x} + kx = 0. \quad (8)$$

can be understood as mathematical model of the damped harmonic oscillator – see e.g. [9].

The constant  $m$  is mass of the oscillator,  $k$  is spring constant and  $b$  is damping constant of the oscillator.

The variable  $x(t)$  is time dependent displacement from the equilibrium position.

The mathematical model does not require any restrictions on constant  $m$ ,  $b$  and  $k$  (except  $m = 0$  so that the equation is genuinely second order). But, since this mathematical model is model of real physical phenomenon the following restriction on constant  $m$ ,  $b$  and  $k$  are required:

$$\begin{aligned} m &> 0 \\ b &\geq 0 \\ k &> 0 \end{aligned} \quad (9)$$

### 1) Solution for undamped oscillator

The undamped system has the damping constant  $b = 0$ . The equation (8) can be written in the form:

$$m\ddot{x} + kx = 0. \quad (10)$$

The solution of the (10) is:

$$\begin{aligned} x(t) &= c_1 \cos(\omega t) + c_2 \sin(\omega t) = \\ x(t) &= A \cos(\omega t - \varphi) \end{aligned} \quad (11)$$

The motion is periodic, repeating itself in a time dependence of displacement  $x(t)$  is sinusoidal with constant amplitude  $A$ . Moreover the motion of undamped harmonic oscillator is characterized by its angular frequency  $\omega = (k/m)^{1/2}$  (the period and frequency are determined by the size of the mass  $m$  and the force constant  $k$ ) and also depends on the initial phase,  $\varphi$ .

### 2) Solution for damped oscillator

When the damping is added ( $b > 0$ ) to the system described by (8) the system is called a damped harmonic oscillator.

Damped oscillators are real oscillators because friction or damping slows the motion of the system at any time. Due to frictional force, the velocity decreases proportional to the acting frictional force. Whereas undamped harmonic motion oscillates with only the restoring force acting on the system, damped harmonic motion experiences friction.

The solution of the (8) is given by solution of its characteristic equation:

$$ms^2 + bs + kx = 0 \quad (12)$$

with characteristic roots:

$$s_{1,2} = \frac{-b \pm \sqrt{b^2 - 4km}}{2m} \quad (13)$$

There are three cases of the damped oscillating systems depending on the sign of the expression under the square root:

- $b^2 < 4km$ : (*underdamping*,  $b$  is small relative to  $m$  and  $k$ );
- $b^2 > 4km$ : (*overdamping*,  $b$  is large relative to  $m$  and  $k$ );
- $b^2 = 4km$ : (*critical damping*).

Mathematically, while characteristic roots of (13) of underdamping are complex, the characteristic roots of overdamping are real.

### 3) Underdamping

If  $b^2 < 4km$  then the term under the square root is negative and the characteristic roots are complex.

In order for  $b^2 < 4km$  the damping constant  $b$  must be relatively small.

First we use the roots (13) to solve (12). Under consideration

$$\omega_d = \frac{\sqrt{b^2 - 4km}}{2m} \quad (14)$$

The characteristic complex roots are:

$$s_{1,2} = -\frac{b}{2m} \pm i\omega_d \quad (15)$$

and complex partial exponential solutions are:

$$\begin{aligned} x_1 &= e^{(-b/2m + i\omega_d)t} \\ x_2 &= e^{(-b/2m - i\omega_d)t} \end{aligned} \quad (16)$$

The general real solution is found by taking linear combinations of the two basic solutions (16):

$$\begin{aligned} x(t) &= e^{-bt/2m} (c_1 \cos(\omega_d t) + c_2 \sin(\omega_d t)) \\ x(t) &= Ae^{-bt/2m} \cos(\omega_d t + \varphi) \end{aligned} \quad (17)$$

The mathematical model (17) expects the system still oscillating with angular frequency  $\omega_d$ , and with decreasing initial amplitude  $A$  and initial phase  $\varphi$ . Amplitude  $A$  and phase  $\varphi$  are given by initial conditions.

### 4) Overdamping

If  $b^2 > 4km$  then the term under the square root is positive and the characteristic roots are real.

In order for  $b^2 > 4km$  the damping constant  $b$  must be relatively large.

The characteristic real roots are:

$$s_{1,2} = \frac{-b \pm \sqrt{b^2 - 4km}}{2m} \quad (18)$$

and real partial exponential solutions are:

$$\begin{aligned} x_1 &= e^{(-b + \sqrt{b^2 - 4km}/2m)t} \\ x_2 &= e^{(-b - \sqrt{b^2 - 4km}/2m)t} \end{aligned} \quad (19)$$

The general real solution is found by taking linear combinations of the two basic solutions (19):

$$x(t) = c_1 e^{(-b + \sqrt{b^2 - 4km}/2m)t} + c_2 e^{(-b - \sqrt{b^2 - 4km}/2m)t} \quad (20)$$

The mathematical model (20) describes does not oscillating system. Regarding the fact that the roots  $x_1$  and  $x_2$  are both negative every solution in this case goes asymptotically to the equilibrium  $x = 0$ . Constant  $c_1$  and  $c_2$  are calculated from initial conditions.

### 5) Critical damping

If  $b^2 = 4km$  then the term under the square root is zero and

the solution has one real root (or two repeated real roots):

$$s_{1,2} = \frac{-b}{2m} \tag{21}$$

Real partial exponential solutions are:

$$\begin{aligned} x_1 &= e^{(-b/2m)t} \\ x_2 &= te^{(-b/2m)t} \end{aligned} \tag{22}$$

The general real solution is found by taking linear combinations of the two basic solutions (19):

$$x(t) = e^{(-b/2m)t} (c_1 + c_2 t) \tag{23}$$

The model (23) as well as the model (20) describes does not oscillating system.

*B. Numerical solution of the damped harmonic oscillator created in MS Excel*

The equation (8) can be solved not only analytically (see previous paragraph) but also numerically. Numeric solution can be provided not only by specialized software tool but also in MS Excel spreadsheet.

*1) Analysis of the numerical solution*

The numerical solution of the (8) is based on numerical solution of nonlinear equations by the general iterative method, which was already described above.

The equation (8) can be rewrite

$$x(t) = -(m\ddot{x}(t) + b\dot{x}(t)) / k . \tag{24}$$

To solve this equation numerically means finding equality between left and right side of (24) for set of  $N$  time points in time interval  $t \in \langle 0, t_{max} \rangle$ .

The time interval  $\langle 0, t_{max} \rangle$  can be split to  $N - 1$  equidistant subintervals:

$$\langle t, t+dt \rangle = \langle t, t+t_{max}/(N - 1) \rangle. \tag{25}$$

In order to find numeric solution of (24) it is necessary numerically calculate the first and the second derivative at each interval (25).

The first derivative at time  $t$  can be numerically calculated based on formulas:

$$\dot{x}(t) = \frac{x(t) - x(t - dt)}{dt} . \tag{26}$$

The second derivative at time  $t$  can be numerically calculated based on formulas:

$$\ddot{x}(t) = \frac{\dot{x}(t) - \dot{x}(t - dt)}{dt} , \tag{27}$$

*2) Numeric solution in MS Excel spreadsheet*

The input data of the numeric solution model are parameters of the damped harmonic motion  $m, b, k$  and initial condition of the motion  $x(0)$  and  $\dot{x}(0)$ .

In our particular case the time interval  $\langle 0, 10 \rangle$  is split to  $N = 101$  equidistant subintervals with  $dt = 0.1$ .

The specific solution of the numerical calculation in MS Excel spreadsheet is shown on the Figure 7.

E15		fx =(D15-D14)/B15			
	A	B	C	D	E
1	Input data:				
2	$m$	1,0			
3	$b$	4,0			
4	$k$	3,0			
6	$x(0)$	1,000			
7	$x'(0)$	0,000			
9	$dt$	0,100			
10	$t_{max}$	10,000			
12	$t$	$dt$	$x(t)$	$x'(t)$	$x''(t)$
13	0,000		1,000	0,000	
14	0,100	0,100	1,000	0,000	0,000
15	0,200	0,100	0,199	-8,013	-80,133
16	0,300	0,100	0,296	0,969	89,818

Fig. 7 Numerical solution in MS Excel spreadsheet

The column A includes set of the  $N$  values of time  $t$ . In the column B there are corresponding values of  $dt$ .

Columns C, D and E includes formulas for calculation of function  $x(t)$  and derivatives  $\dot{x}(t)$  and  $\ddot{x}(t)$ .

For example the cell D15 contains formula

$$=(C15-C14) / B15$$

for numeric calculation of  $\dot{x}(t)$  at time point  $t = 0.2$  – see (26).

Cell E15 contains formula

$$=(D15-D14) / B15$$

for numeric calculation of  $\ddot{x}(t)$  at time point  $t = 0.2$  – see (27).

Numerical solution of (24) is calculated based on iterative calculation of the cell. Cells in column C contains circular reference formula

$$=- (\$B\$2 * E15 + \$B\$3 * D15) / \$B\$4.$$

If iterative calculation of the cell in MS Excel is enabled, “Maximum Iterations” is set to 100 and the default value of “Maximum Change” is set to 0.001, this circular formula provides numerical solution of (24) for every  $N$  time point of the interval  $\langle 0, t_{max} \rangle$  - see Figure 7.

*C. Animation of time dependence of displacement of the damped harmonic oscillator*

*1) Analysis of the dynamic simulation model*

Visualization of time dependence of displacement  $x(t)$  of damped harmonic oscillator can be realized in similar way as in Case study 1 – by means of iterative recalculation of the cells. The solution of the animation is shown on the Figure 8.



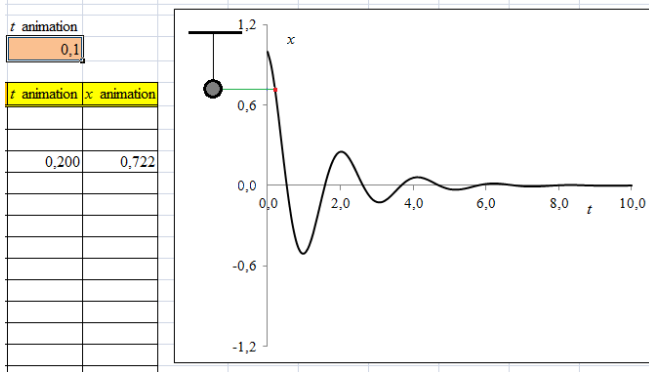


Fig. 8 Animation of damped harmonic oscillator

Right part represents static XY scatter chart – time dependence of displacement  $x(t)$ . Source data of this static chart are values from column A and column C – see Figure 7.

Left part of the diagram represents visualization of the motion (animation) of the damped harmonic oscillator in form of moving ball.

Main parameter of animation recalculation is time  $t$ . The parameter is changed within the range from  $t = 0$  to  $t = 10$ . with the iteration step  $dt = 0.1$  by repeated pressings or by holding of key F9. The formula providing iterative recalculation of the time is stored in cell F10

$$=IF(F10>B10;0;F10+B9/100).$$

In contrast to the Case study 1, here the value of "Maximum Iterations" is set to 100 (see previous paragraph) and therefore iteration step in the formula is divided by 100.

Source data – XY coordinates of the ball are stored in columns F and G. Cells e.g. F15 in columns G contains formula

$$=IF(AND($F$10>=A15;$F$10<A16);A15;"")$$

that shows value of cell A15 ( $t$ ) in the cell F15 only if value in cell F10 is closed to value in cell A15 (in the column F there is shown only one actual value of the time  $t$  from column A) – see Figure 8.

Similarly, in the column G there is shown only one actual value of  $x(t)$  from column C by using formula:

$$=IF(F15="";"";C15).$$

### 2) Examples of animations

The example of animation of underdamping harmonic oscillator:

$$3\ddot{x} + 4\dot{x} + 1x = 0. \tag{28}$$

is shown on Figure 9.

The example of animation of overdamping harmonic oscillator:

$$1\ddot{x} + 4\dot{x} + 3x = 0. \tag{29}$$

is shown on Figure 10.

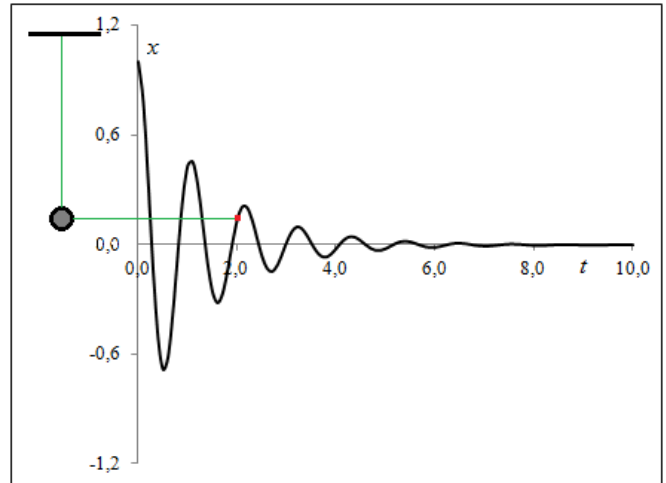


Fig. 9 Animation of underdamping harmonic oscillator

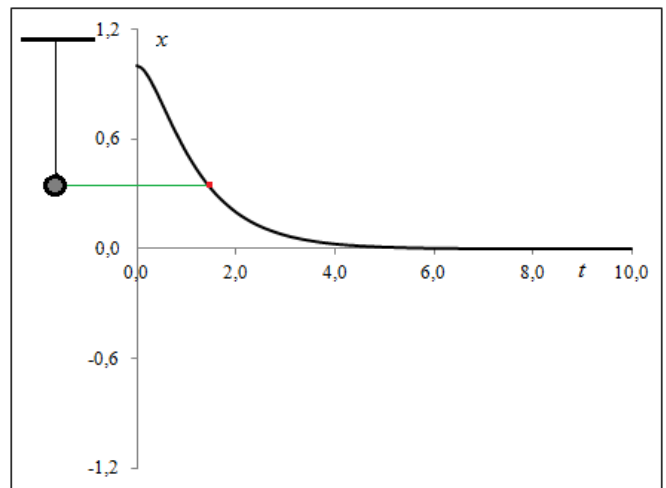


Fig. 10 Animation of overdamping harmonic oscillator

## VI. CONCLUSION

There are various approaches how to provide training and development of theoretically acquired knowledge in mathematics.

A new way of the teaching / learning strategy by using the system approach was presented in the paper.

The processes of creation of numerical solution, dynamic computer simulation model, visualization and animation have been presented step by step in the paper.

The system approach can be set as the default paradigm for a wide integration of the principles of the algorithm development into education.

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