Krill herd (KH) algorithm applied to the constrained portfolio selection problem

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***Abstract*—**Constrained portfolio selection (optimization) problem extends the classical mean-variance portfolio problem by adding constraints. Such problem becomes computationally intractable which makes the traditional optimization techniques inadequate. Nondeterministic optimization metaheuristics are more appropriate, where swarm intelligence is in the focus of recent research. This paper presents novel krill herd (KH) nature-inspired metaheuristic applied to the constrained portfolio optimization problem. Portfolio selection problem was not much researched by the swarm intelligence algorithms and this is the first application of the krill herd algorithm to this problem. Experimental results show that the krill herd algorithm is a promising technique for portfolio optimization problem since krill herd algorithm results were better compared to other state-of-the-art optimization metaheuristics.

***Keywords*—**Constrained portfolio optimization, krill herd algorithm, nature inspired algorithms, metaheuristic optimization, swarm intelligence.

I. Introduction

PORTFOLIO optimization problem is one of the most studied research topics in the field of finance and economics. In the literature, this task is also known as portfolio selection problem. Financial portfolio is collection of financial instruments (investments), all owned by the same organization or by an individual. It usually includes bonds (investments in debts), stocks (investments in individual businesses), and mutual funds (pools of money from many professional investors). Portfolio structure is generally designed according to the investor's risk sensitivity, objectives of an investment and a time frame.

One of the most important portfolio optimization issues is the risk. Investors are always trying to balance between portfolio's gains and risk. Thus, the goal is to select a portfolio with minimum risk at defined minimal expected returns. This further means reducing non-systematic risks to zero.

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In its basic definition, portfolio optimization problem refers to dealing with the selection of portfolio's assets (or securities) that minimizes the risk subject to the constraint that guarantees a given level of returns. Individual and institutional investors prefer to invest in portfolios rather than in a single asset because by doing this, the risk is mitigated with no negative impact on the expected returns [1]. In other words, portfolio optimization problem seeks for an optimal way to distribute a given budget on a set of available assets [2].

Portfolio optimization problem can also be defined as multi-criteria optimization in which risk has to be minimized, while, on the other hand, return has to be maximized. Unfortunately, this approach to the problem has several drawbacks [3]. Firstly, it might be difficult to gather enough data for accurate estimation of the risk and returns. Secondly, when estimating return and risks using covariance, errors can frequently occur [4]. Thirdly, and finally, this model is considered to be too simplistic for practical purposes because it does not capture essential properties of the real-world trading, such as maximum size of portfolio, transaction costs, preferences over assets, cost management, etc. These properties can be modeled by adding additional constraints to the basic problem definition which transform unconstrained portfolio optimization problem into the constrained one. Constrained problem is more complex, and belong to the class of NP-Complete problems [5].

With the addition of real-world requirements to the basic portfolio optimization problem formulation, the problem is being transformed into constrained, and as such it becomes intractable in a reasonable amount of computational time. In these cases, exact methods can not obtain results, and the use of approximate algorithms, and in particular metaheuristic is necessary. Modern metaheuristics algorithms are typically high level strategies which guide an underlying subordinate heuristic to the desired objective. Metaheuristics methods can find satisfying feasible solution in a reasonable amount of computational time.

Many formulations of portfolio optimization problem were solved using nature-inspired metaheuristisc. One well-known representative is genetic algorithm (GA). GA simulates the process of natural evolution by employing selection, crossover and mutation operator.

GA approach was proposed for solving portfolio optimization problem constrained to cardinality and linear holding weights constraints (HWC) within the risk-aversion formulation [6]. Three additional risk measures, besides the variance, were adopted: mean absolute deviation, semi variance and variance with skew ness. The two latter risk measures are improvement of the variance by taking into account only the returns below the mean for semivariance and by including skewness for variance with skewness. Proposed GA used binary tournament selection and modified replacement strategy and proved to be effective algorithm for tackling portfolio optimization problem.

The GA based technique was used for solving portfolio optimization problem in terms of multi-state continuous optimization over time, where the objective, in addition to increasing the return and decreasing the risk, is to minimize transaction costs [7]. Costs are minimized between each two consecutive time periods. Each GA's chromosome was represented by two arrays, one binary which indicates which asset is present in the portfolio, and second that stores real-valued assets' weights. The authors used an indirect approach for modeling costs as Euclidean distance of the weight vectors of the current position (time $t-1$) and the desired position (time t). The algorithm was tested on real case data sets of monthly historical returns from the NIKKEI and the NASDAQ indexes. The results were satisfying.

GA with RAR operator was implemented for solving Mean-Variance (M-V) portfolio optimization, where cardinality constraints, minimum transaction lots and constraints on sector capitalization are taken account [8]. The sector capitalization constraints suppose that some assets belong to sectors (sets of assets) and state that the capital invested in sector 1 is greater than the one invested in sector 2 and so on. The advantage of these constraints is to let investors invest in some sectors with high-value in a manner to reduce the overall risk. GA was compared with LINGO, an optimization modeling software.

Swarm intelligence employs principles of the collective behavior of social insect colonies and other animal groups in the search process. Swarm intelligence can be classified in the group of population based metaheuristics. These metaheuristics start with initial (usually random) population of candidate problem solutions and iteratively improve them. The key concept of swarm intelligence lies in the effect of emergent behavior of many individuals which exhibit extraordinary collective intelligence without any centralized supervision mechanism. Entire swarm intelligence system is fully adaptive to internal and external changes, and it is established on four basic properties on which self-organization rely: positive feedback, negative feedback, multiple interactions and fluctuations. Positive feedback refers to a situation when one individual directs behavior of the others by some directive. Negative feedback discourages individuals to pursuit bad solution to the problem. Multiple interactions are the basis of the tasks to be carried out by certain rules, while fluctuations refer to the random behaviors of individuals by which the new regions are being explored. Swarm intelligence approach has obvious advantages over other optimization methods and techniques: scalability, adaptation, fault tolerance, parallelism and speed.

Ant colony optimization (ACO) showed satisfying performance in solving many hard optimization problems [9], [10], [11], [12], [13]. This metaheuristics was inspired by the foraging behavior of ants who deposit pheromone trails which help them in finding the shortest path between food sources and their nests. The basic philosophy of the ACO algorithm involves the movement of an ant colony which is directed by two local decision policies: pheromone trails and its attractiveness. ACO adaptations for portfolio optimization problem were found in the literature. The algorithm was tested on real-scenario portfolio benchmark problems [14].

Particle swarm optimization (PSO) is a swarm intelligence algorithm that imitates social behavior of fish schooling or bird flocking. PSO was successfully applied on portfolio optimization problem [15], [16].

Artificial bee colony (ABC) metaheristics is one of the latest simulations of the honey bee swarm. The simulated bee colony consists of employed, onlooker and scout bees. ABC showed outstanding results in global optimization problems [17], [18], [19] and engineering problems [20]. ABC and its modifications were tested on cardinality constrained portfolio optimization [21], [22].

Seeker optimization algorithm (SOA) is based on human search process which uses human reasoning, memory, past experience and human interactions. Seeker operates in the larger environment of candidate solutions called search population. The total population is divided into three equally-sized subpopulations according to the sequence of the seekers. All the agents in the same population form a social unit called neighborhood, and each population performs search in its domain of the search space. SOA was not applied to portfolio optimization problem, but it was adapted for wide variety of other optimization problems [23].

Firefly algorithm (FA) is one of the latest swarm intelligence metaheuristics. It is inspired by the flashing behavior of fireflies. The main algorithm's principle is that each firefly moves towards the brighter firefly. FA was first proposed for unconstrained optimization problems [24], image processing [25] with entropy criteria [26]. FA was also applied on portfolio optimization [27], [28]. Bat algorithm is the latest SI algorithm [29].

In this paper, we present the krill herd (KH) algorithm for portfolio optimization problem. KH was recently proposed by Gandomi and Alavi [30]. The implementation of the KH for portfolio optimization problem was not found in the literature.

The paper is organized as follows. After Introduction, in Section 2, this paper presents mathematical formulation of the portfolio optimization models. In this section, we show different problem formulations that can be found in the literature. Section 3 introduces KH metaheuristic and explains its search process. Experimental data, problem establishments and experimental results are presented in Section 4. In this section, we also show comparative analysis between KH and GA and FA. Final conclusion is given in Section 5.

II. PORTFOLIO OPTIMIZATION DEFINITIONS

The main principle when making financial investments decisions is diversification where investors invest into different types of assets. Portfolio diversification minimizes investors' exposure to the risks, while maximizing returns on portfolios.

Many methods can be applied to solving multiobjective optimization problems such as portfolio optimization. One essential method is to transform the multi-objective optimization problem into a single objective optimization problem. This method can be further divided into two sub-types. In the first approach, one important objective function is selected for optimization, while the rest of objective functions are defined as constrained conditions. Alternatively, only one evaluation function is created by weighting the multiple objective functions.

The first method is defined by Markowitz and is called the standard mean-variance model [31]. It was first introduced more than 50 years ago and its basic assumptions are a rational investor with either ultivariate normally distributed asset returns, or, in the case of arbitrary returns, a quadratic utility function [1]. If those assumptions hold, then the optimal portfolio for the investor lies on the mean-variance efficient frontier.

In this model, the selection of risky portfolio is considered as one objective function and the mean return on an asset is considered to be one of the constraints [16]. It can be formulated as follows:

$$min\,\sigma^2_{R_P} = \sigma^2_p = \sum_{i=1}^{N}\sum_{j=1}^{N}\omega_i\omega_j Cov(\overline{R_i}\,\overline{R_j}) \tag{1}$$

Subject to

$$\overline{R_p} = E(R_p) = \sum_{i=1}\omega_i \overline{R_i} \geq R \tag{2}$$

$$\sum_{i=1}^{N}\omega_i = 1 \tag{3}$$

$$\omega_i \geq 0,\ \forall i \in (1,2,...,N) \tag{4}$$

where N is the number of available assets, $\overline{R_i}$ is the mean return on an asset i and $Cov(\overline{R_i}\,\overline{R_j})$ is covariance of returns of assets i and j respectively. Weight variable ω_i controls the proportion of the capital that is invested in asset i, and constraint in (3) ensures that the whole available capital is invested. In this model, the goal is to minimize the portfolio risk σ^2_p, for a given value of portfolio expected return $\overline{R_p}$.

In the presented standard mean-variance model, variables are real and they range between zero and one, as they represent the fraction of available money to invest in an asset. This choice is quite straightforward, and has the advantage of being independent of the actual budget.

The second method refers to the construction of only one evaluation function that models portfolio selection problem. This method comprises two distinct models: efficient frontier and Sharpe ratio model [14].

In efficient frontier model, the goal is to find the different objective function values by varying desired mean return R. The best practice is to introduce new parameter $\lambda \in [0,1]$ which is called risk aversion indicator. In this case, the model is approximated to only one objective function:

$$min\,\lambda[\sum_{i=1}^{N}\sum_{j=1}^{N}\omega_i\omega_j Cov(\overline{R_i}\,\overline{R_j})] - (1-\lambda)[\sum_{i=1}^{N}\omega_i \overline{R_i}] \tag{5}$$

Subject to

$$\sum_{i=1}^{N}\omega_i = 1 \tag{6}$$

$$\omega_i \geq 0, \forall i \in (1,2,...N) \tag{7}$$

λ controls the relative importance of the mean return to the risk of the investor. When λ is zero, mean return of the portfolio is maximized regardless of the risk. Contrary, when λ equals 1, risk of the portfolio is being minimized regardless of the mean return. Thus, with the increase of λ, the relative importance of the risk to the investor increases, and importance of the mean return decreases, and vice-versa.

With the change of the value of λ, objective function value changes also. The reason for this change is that the objective function is composed of the mean return value and the variance (risk). The dependencies between changes of λ and the mean return and variance intersections are shown on a continuous curve which is called efficient frontier in the Markowitz theory. Since each point on this curve indicates an optimum, portfolio optimization problem is considered as multi-objective, but λ transforms it into single-objective optimization task.

Sharpe ratio (SR) model combines the information from the mean and variance of an asset. This simple model is risk-adjusted measure of mean return and can be described with the following equation [32]:

$$SR = \frac{R_p - R_f}{StdDev(p)}, \tag{8}$$

where p denotes portfolio, R_p is the mean return of the portfolio p, and R_f is a test available rate of return on a risk-free asset. *StdDev(p)* is a measure of the risk in portfolio (standard deviation of R_p). By adjusting the portfolio weights ω_i, portfolio's Sharpe ratio can be maximized.

The models we talked about so far refer only to the basic problem definitions. Those definitions do not seem realistic because they do not consider several aspects, such as [33]:

- the existence of frictional aspects like the transaction costs, sectors with high capitalization and taxation;

- the existence of specific impositions arising from the legal,economic, etc. environment;
- the finite divisibility of the assets to select.

Taking into account all above mentioned additional portfolio optimization constraints, new portfolio optimization problem can be established [16]. This model is called extended mean-variance model and it is classified as a quadratic mixed-integer programming model necessitating the use of efficient heuristics to find the solution. It can be formulated as follows:

$$min\,\sigma_{R_P}^2 = \sigma_p^2 = \sum_{i=1}^{N}\sum_{j=1}^{N}\omega_i\omega_j Cov(\overline{R_i}\,\overline{R_j}) \tag{9}$$

where

$$\omega_i = \frac{x_i c_i z_i}{\sum_{j=1}^{N} x_j c_j z_j},\ i = 1,...,N \tag{10}$$

$$\sum_{i=1}^{N} z_i = M \le N, M, N \in N, \forall i = 1,...N, z_i \in \{0,1\} \tag{11}$$

Subject to

$$\sum_{i=1}^{N} x_i c_i z_i \overline{R_i} \ge BR \tag{12}$$

$$\sum_{i=1}^{N} x_i c_i z_i \le B \tag{13}$$

$$0 \le B_{low_i} \le x_i c_i \le B_{up_i} \le B, i = 1,...N \tag{14}$$

$$\sum_{i_S} W_{i_S} \ge \sum_{i_{S'}} W_{i_{S'}}, \forall y_s y_{s'} \ne 0, s, s' \in \{1,...S\}, s < s' \tag{15}$$

where

$$y_s = \begin{cases} 1, if \sum_{i_S} z_i > 0 \\ 0, if \sum_{i_{S'}} z_i = 0 \end{cases} \tag{16}$$

where M represents the number of selected assets among possible N assets. B is the total available budget, while B_{low_i} and B_{up_i} are lower and upper limits respectively of the budget that can be invested in asset i. S is the total number of sectors. c_i represents the minimum transaction lot for asset i, and x_i denotes the number of c_i that is purchased. According to this, $x_i c_i$ are integer values that show the units of asset i in the portfolio.

Decision variable z_i is defined for modeling cardinality constraint. It is equal to 1 if an asset i is present in the portfolio, otherwise it is equal to 0. Equation (11) represents the cardinality constraint and inequality (12) is the same as (2). In order to make the search process easier, budget constraint in (13) is converted to inequality. Equation (14) shows lower and upper bounds of budget constraint.

Sector capitalization constraint improves portfolio's structure decisions by preferring investments in assets that belong to the sector with higher capitalization value. The assets which belong to the sector with more capitalization should have more shares in the final portfolio. This constraint is held only if securities from the corresponding sectors are selected Equation (15) introduces sector capitalization constraint into extended mean-variance model. Despite the fact that a certain sector has high capitalization, security from this sector that has low return and/or high risk must be excluded from final portfolio's structure. To make such exclusion, variable y_s is defined and it has a value of 1 if the corresponding sector has at least one selected asset, and 0 otherwise. In (15) i_s is a set of assets which can be found in sector S. Sectors are sorted in descending order by their capitalization value. Sector 1 has the highest capitalization value, while sector S has the lowest value.

III. KH ALGORITHM OVERVIEW

Zooplankton aggregation occurs as the result of biological and physical processes [34]. In the ocean environment, the density distribution of plankton depends on the irculation patterns, such as musicale vertices and fronts [34]. The behavior of individuals responding to their environment also plays a significant role in the generation of dense, quasi horizontal patches of zooplankton commonly called swarms. With pysical and chemical cues, collective movements and the formation of groups in the population can be triggered, and this is referred to as social behavior.

Antarctic krill is one of the best-studied species of marine animal. Krill herds exist on a space scales of 10 to 100 meters. In the last three decades, many studies have been conducted for the sake of understanding the ecology and distribution of the krill.

Although there are yet notable uncertainties about the forces determining the distribution of the krill herd, conceptual models have been proposed to explain the observed formation of the krill herds.

When predators attack krill, they remove only individual krill, and the krill density is reduced. The formation of the krill after the attack depends on several parameters. The herding of the krill individuals is a multi-objective process including two main goals: (1) increasing krill density, and (2) seaching food.

The position of an individual krill that is timedependent in 2D surface is govern by the following factors: movement induced by other krill individuals, foraging activity and random diffusion.

The Lagrangian model is generalized to an n dimensional decision space [30]:

$$\frac{dX_i}{d_t} = N_i + F_i + D_i, \tag{17}$$

where N_i is the motion induced by other krill individuals, F_i is foraging motion, and D_i is the physical diffusion of the i-th krill.

The movement of krill individual is defined by:

$$N_i^{new} = N^{\max}\alpha_i + \omega_n N_i^{old}, \quad (18)$$

where

$$\alpha_i = \alpha_i^{local} + \alpha_i^{\text{target}}, \quad (19)$$

where $N^{\max}$ is the maximum induce speed, ω_n is the inertia weight of the motion in the range [0, 1], N_i^{old} is the last motion induced, α_i^{local} is the local effect provided by the neighbors, α_i^{target} is the target direction provided by the best krill individual.

The sensing distance for each krill individual can be determined using different heuristic methods. Here, it is determined using the following formula for each iteration [30]:

$$d_{i,s} = \frac{1}{5N}\sum_{j=1}^{N} \| X_i - X_j \|, \quad (20)$$

where $d_{i,s}$ is the sensing distance for the i-th krill individual in the population, and N is the number of krill individuals. The factor 5 in the denominator is empirically calculated [30]. If the distance between two krill individuals is less than the defined sensing distance, they are considered to be neighbors (Eq. (20)).

The known target vector of each krill individual is the lowest fitness of an individual krill. The global optimum is defined as followed:

$$\alpha_i^{\text{target}} = C^{best} K_{i,best} x_{i,bset}, \quad (21)$$

where C^{best} is the effective coefficient of the krill individual with the best fitness of the i-th krill individual. The value of C^{best} can be defined as follows:

$$C^{best} = 2(rand + \frac{1}{l_{\max}}), \quad (22)$$

where $rand$ is a random number between 0 and 1, l is the current iteration number, and $l_{\max}$ is the maximum number of iterations.

As mentioned above, the krill motion consists of foraging motion, motion influenced by other individuals, and the physical diffusion. The foraging motion formulation is based on two main effective parameters: the food location, and the previous experience about the food location. Foraging motion of the *i*-th krill individual is formulated as follows:

$$F_i = V_f \beta_i + \omega_f F_i^{old} \quad (23)$$

where

$$\beta_i = \beta_i^{food} + \beta_i^{best}, \quad (24)$$

where V_f is the foraging speed, ω_f is the inertia weight of the foraging motion, and it is defined in range [0,1], β_i^{food} is the food attractiveness, and β_i^{best} is the effect of the best krill found in the population so far. According to empirical test, the best value for the V_f is 0.02 ms-1.

The effect of the food depends on its location. The center of the food is discovered first and it is used for formulation of food attraction. This can only be estimated. In [30], the virtual center of food concentration is estimated according to the fitness distribution of the krill individuals, which is inspired from the “center of mass”. This center of food in each iteration is defined as:

$$X^{food} = \frac{\sum_{i=1}^{N}\frac{1}{k_i}X_i}{\sum_{i=1}^{N}\frac{1}{k_i}} \quad (25)$$

The food attraction of the *i*-th krill individual is defined as:

$$\beta_i^{food} = C^{food} K_{i,food} X_{i,food} \quad (26)$$

where C^{food} is the food coefficient, and it is defined as follows:

$$C^{food} = 2(1 - \frac{l}{l_{\max}}) \quad (27)$$

Physical diffusion of the krill individuals is a random process, and it is used for exploration of the search space. It is formulated using maximum diffusion speed and a random directional vector:

$$D_i = D^{\max}\delta, \quad (28)$$

where $D^{\max}$ is the maximum diffusion speed, δ is a random directional vector. Empirically calculated maximum diffusion speed is in the range [0.002, 0.010] ms^{-1} [30].

Above defined motions frequently change the position of a krill individual towards the best fitness. Motions contain two global and two local strategies, which make KH very powerful algorithm [30]. The position of a krill individual in the time interval $[t, t + \Delta t]$ is given below:

$$X_i(t + \Delta t) = X_i(t) + \Delta t \frac{dX_i}{dt} \quad (29)$$

Pseudo-code of the KH algorithm is given below [30]:

```
 I Definition: defining algorithm parameters, bounds
of the problem, etc.
 II Initialization: creation of the initial
population of solutions
 III Fitness evaluation: evaluate all krill based on
its current position
 IV Motion calculation:
    based on the position of other individuals
    foraging motion
    physical diffusion
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 V Updating: update the krill individual position in
the population
VI Repeating: go to step III until termination
criteria is met
VII End
```

We should note that in implementation presented in [30], genetic operators were used. In our demonstration, no genetic operators were employed.

IV. Practical application and experimental results

In this section, we present mathematical model of portfolio optimization problem used in testing KH approach, data used in the experiments and experimental results. We used the same problem formulation and data set as in [35].

A. Problem formulation

The main objective of the applied portfolio optimization mathematical model used to test KH algorithm is to select weights of the each asset in the portfolio in order to maximize the portfolio's return and to minimize the portfolio's risk. We transformed multi-objective problem into single one with constraints.

The expected return of each individual security i is presented as follows:

$$E(\omega_i) = w_i \mu_i \,, \tag{30}$$

where ω_i denotes the weight of individual asset i, and μ_i is the expected return of i. Total expected return of the portfolio P can be formulated as follows:

$$E(P) = \sum_{i=1}^{n} E(\omega_i)\,, \tag{31}$$

where n is the number of securities in the portfolio P.

In our problem formulation, first goal is to maximize portfolio's expected return, and thus, the expression shown in (31) is objective function for the portfolio's return and it should be maximized.

The objective function of the portfolio variance (risk) is presented as a polynomial of second degree:

$$\begin{aligned}\sigma^2(P) = \sigma^2(\omega_i) = \sum_{i=1}^{n}(\omega_i^2 \sigma^2(\mu_i)) + \\ \sum_{i=1}^{n}\sum_{j=i+1}^{n} 2\omega_i \omega_j Cov(\mu_i, \mu_j)\,,\end{aligned} \tag{32}$$

where $\sigma^2(\omega_i)$ is variance of asset i, and $Cov(\mu_i, \mu_j)$ is covariance between securities i and j.

According to (31) and (32), the multi-objective function to be minimized is transformed into single-objective in the following form:

$$H(P) = E(P) - \sigma^2(P) \tag{33}$$

Alternatively, considering individual asset i, not the whole portfolio P, (33) can be formulated as:

$$H(\omega_i) = E(\omega_i) - \sigma^2(\omega_i) \tag{34}$$

Subject to:

$$\sum_{i=1}^{n} \omega_i = 1 \tag{35}$$

$$\omega_i^{\min} \le \omega_i \le \omega_i^{\max} \tag{36}$$

To make sure that the portfolio's return is positive in all test cases, we used the following constraint:

$$\sum_{i=1}^{n} \mu_i \omega_i \ge 0\,, \tag{37}$$

where $\omega_i^{\min}$ and $\omega_i^{\max}$ are minimum and maximum weights of asset i respectively.

B. Data used in the experiments

For testing purposes, we used simple historical data set from [35]. Data set is shown in Table 1. The mean return on each asset and covariance matrix are given in Tables 2 and 3 respectively.

TABLE I
Historical data set for the experiments

Year	Stock 1	Stock 2	Stock 3	Stock 4	Stock 5
2007	-0.15	0.29	0.38	0.18	-0.10
2008	0.05	0.18	0.63	-0.12	0.15
2009	-0.43	0.24	0.46	0.42	0.15
2010	0.79	0.25	0.36	0.24	0.10
2011	0.32	0.17	-0.57	0.30	0.25

TABLE II
Mean returns for each asset

Stock 1	0.116
Stock 2	0.226
Stock 3	0.252
Stock 4	0.204
Stock 5	0.11

TABLE III
Covariance matrix

	Stock 1	Stock 2	Stock 3	Stock 4	Stock 5
Stock 1	0.21728	-0.003376	-0.053492	-0.009264	0.01064
Stock 2	-0.003376	0.00253	0.008468	0.002376	-0.00456
Stock 3	-0.053492	0.008468	0.22247	-0.31128	-0.02392
Stock 4	-0.009264	0.002376	-0.031128	0.04068	0.00276
Stock 5	0.01064	-0.00456	-0.02392	0.00276	0.01675

C. Algorithm settings

In this subsection, we show experimental results for testing KH for constrained portfolio optimization problem (see Subsection A for problem formulation). Tests were performed on Intel Core 3770K processor @4.2GHz with 8GB of RAM memory, Windows 8 x64 operating system and Visual Studio 2012 with .NET 4.5 Framework. Number of krills in the population KN was set to 40, while maximum iteration number IN was set to 6000, yielding total of 240,000 objective function evaluations (40*6000). The same number of objective function evaluations was used in [35]. The algorithm was tested on 30 independent runs. Each run starts with a different random number seed.

We also ran additional test where we wanted to see whether our algorithm could perform better if it used more function evaluations. For this additional test we set maximum iteration number *IN* to 8000 while keeping solution number *KN* on the previous value. In this way, we employed 320.000 function evaluations (40*8000) which is 33.3% more than in the first experiment.

Since the data set used in the experiment consists of five portfolio's assets, dimension *D* of a problem is 5. Each krill in the population is a 5-dimensional vector. In initialization phase, krill *x* is created using the following expression:

$$x_i = \omega_i^{\min} + rand(0,1)(\omega_i^{\max} - \omega_i^{\min}), \tag{38}$$

where *rand(0, 1)* is a random number uniformly distributed between 0 and 1.

To guarantee the feasibility of solutions, we used the following pseudo code:

while(true)

$\chi = \sum_{i=1}^{D} x_i$, $x_i = x_i / \chi, \eta = \sum_{i=1}^{D} \max(0, x_i - \omega_i^{\max}), \phi = \sum_{i=1}^{D} \max(0, \omega_i^{\min} - x_i)$

if ($\eta = 0$ **and** $\phi = 0$) then exit the pseudo-code

if ($x_i > \omega_i^{\max}$) **then** ($x_i = \omega_i^{\max}$)

if ($x_i < \omega_i^{\min}$) **then** ($x_i = \omega_i^{\min}$)

end while

We set the foraging speed V_f to 0.02 like in [30], and diffusion speed $D_{\max}$ to 0.006 (the aritmetic average of the range of recommended paramters, see Section 3).

Moreover, we also used constraint handling techniques to direct the search process towards the feasible region of the search space. Equality constraints decrease efficiency of the search process by making the feasible space very small compared to the entire search space. For improving the search process, the equality constraints can be replaced by inequality constraints using the following expression [36]:

$$|h(x)| - \varepsilon \le 0, \tag{39}$$

where $\varepsilon > 0$ is very small violation tolerance. The ε was dynamically adjusted according to the current algorithm's iteration:

$$\varepsilon(t+1) = \frac{\varepsilon(t)}{dec}, \tag{40}$$

where *t* is the current iteration, and *dec* is a value slightly larger than 1. When the value of ε reaches the predetermined threshold value, (33) is no longer applied.

Summary of all parameters is given in Table IV.

TABLE IV
FA PARAMETER SET

Parameter	Value
Number of krills in the population (*KN*)	40
Number of iterations (*IN*)	6000
Foraging speed V_f	0.02
Diffusion speed D_{max}	0.006
Initial violation tolerance (ε)	1.0
Decrement (*dec*)	1.002
ω^{min}	0
ω^{max}	1

D. Experimental results and comparative analysis

In experimental results, we show best, mean and worst results for objective function value, variance (risk) and average return of portfolios (Table 5). In Table 6, we show portfolio weights for the best and worst results.

TABLE V
EXPERIMENTAL RESULTS WITH 240.000 EVALUATIONS

	Best	Worst	Mean
Objective function	4.582	4.702	4.682
Variance	0.039	0.069	0.075
Return	0.209	0.183	0.195

TABLE VI
PORTFOLIO WEIGHTS FOR BEST AND WORST RESULTS IN 240.000 EVALUATIONS TEST

	ω_1	ω_2	ω_3	ω_4	ω_5
Best	0.072	0.415	0.287	0.226	0.079
Worst	0.035	0.188	0.259	0.232	0.286

According to the experiment results presented in Tables V and VI, KH for portfolio optimization performs similar like GA approach in [35]. In [35], three variants of GA were shown: single-point, two-point and arithmetic. Arithmetic variant performed significantly better than other two variants, and also better than the KH presented in this paper. But, at the other hand, KH showed better performance than singlepoint and two-point variants of the GA presented in [35]. GA results are shown in Table VII.

TABLE VII
GA EXPERIMENTAL RESULTS

Objective function	Variance	Return
Single-point variant		
4.900	0.019	0.204
Two-point variant		
4.598	0.080	0.221
Arithmetic variant		
4.532	0.0325	0.222

FA presented in [27] was tested on the same data set as KH shown in this paper. Moreover, FA also employed 240.000 function evaluations. FA results are shown in Table VIII.

TABLE VIII
FA EXPERIMENTAL RESULTS WITH 240,000 EVALUATIONS

	Best	Worst	Mean
Objective function	4.542	4.698	4.615
Variance	0.036	0.072	0.059
Return	0.218	0.198	0.205

Comparative analysis of KH and FA results (Table V vs. Table VIII) showed that the FA algorithm performs significantly better in all – best, worst and mean categories. FA's bests, worsts and means results are better than KH's results for 0.9%, 0.8% and 1.4% respectively.

We also wanted to see how KH algorithm performs when the number of function evaluations is slightly greater. So, we ran additional test, but this time, we set the number of iterations (*IN*) to 8000, while the number of krill agents (*KN*) remained the same as in the first experiment. This parameter set gives 320.000 (40*8000) function evaluations which is 33.3 % higher than in the first experiment. The results are shown in the tables below.

TABLE IX
EXPERIMENTAL RESULTS WITH 320,000 EVALUATIONS

	Best	Worst	Mean
Objective function	4.563	4.690	4.672
Variance	0.036	0.058	0.045
Return	0.238	0.215	0.221

TABLE X
PORTFOLIO WEIGHTS FOR BEST AND WORST RESULTS IN 320,000 EVALUATIONS TEST

	ω_1	ω_2	ω_3	ω_4	ω_5
Best	0.039	0.368	0.391	0.067	0.135
Worst	0.064	0.341	0.363	0.187	0.045

As can be seen from Table IX, with higher number of function evaluations, our KH algorithm still performs worse than arithmetic variant of GA [35] and FA [27] with 240.000 function evaluations.

V. CONCLUSION

In this paper, KH for constrained portfolio optimization problem was presented. The algorithm was tested on a standard benchmark set of five assets.

Two experiments were conducted with different number of function evaluations. In the first experiment (240,000 evaluations), KH performed better than single-point and two-point variant of GA, while the arithmetic variant of GA outperformed our KH approach. FA also outscored KH algorithm.

In the second experiment (320,000 evaluations), KH generated better results than in 240,000 evaluations tests, but still worse than the arithmetic variant of GA and FA.

KH was applied only to the basic portfolio optimization problem definition. It has potential with some modification. There is a large potential for applying metaheuristic techniques to this class of problems, because they appear not to be investigated enough.

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