The 4-ordered property of some chordal ring networks

Shin-Shin Kao*, Shih-Chin Wey and Hsiu-Chunj Pan

***Abstract*—**A graph G is *k-ordered* if for any sequence of k distinct vertices of G, there exists a cycle in G containing these k vertices in the specified order. Obviously, any cycle in a graph is *1*-ordered, *2*-ordered and *3*-ordered. Thus the study of *k*-ordered graphs always starts with $k = 4$. In this paper, we study the *4*-orderedness of certain chordal rings, denoted by $CR(n;1,q)$ for n being an even integer with $n \geq 6$ and q an odd integer with $3 \leq q \leq n/2$. More specifically, we prove that $CR(n;1,5)$ is *4*-ordered for $n \geq 14$, and $CR(n;1,7)$ is *4*-ordered for $n \geq 18$. The proof is based on computer experimental results by M. Tsai, which can be found in [9], and mathematical induction.

***Keywords*—**4-ordered, chordal ring, cycle embedding, hamiltonian, cycles,

I. INTRODUCTION

We consider finite, undirected and simple graphs only. Let $G = \langle V, E\rangle$ be a graph, where V is the set of vertices of G and $E \subseteq \{(u,v) \mid u, v \in V\}$ is the set of edges of G, respectively. Let u, v be two vertices of G. If $e = (u,v) \in E$, then we say that the vertices u and v are *adjacent* in G. The *degree* of any vertex u is the number of distinct vertices adjacent to u. We use $N(u)$ to denote the set of vertices which are adjacent to u. A path P between two vertices v_0 and v_k is represented by $P=\langle v_0, v_1,\ldots, v_k\rangle$ where each pair of consecutive vertices is connected by an edge. We use P^{-1} to denote the path $\langle v_k, v_{k-1}, v_{k-2}, \ldots, v_0\rangle$. We also write the path $P=\langle v_0, v_1,\ldots, v_k\rangle$ as $\langle v_0, v_1,\ldots, v_i, Q, v_j, v_{j+1},\ldots, v_k\rangle$, where Q denotes the path $\langle v_i, v_{i+1}, \ldots, v_j\rangle$. A *hamiltonian path* between u and v, where u and v are two distinct vertices of G, is a path joining u to v that visits every vertex of G exactly once. A *cycle* is a path of at least three vertices such that the first vertex is the same as the last vertex. A *hamiltonian cycle* of G is a cycle that traverses every vertex of G exactly once. A *hamiltonian graph* is a graph with a hamiltonian cycle. A graph G is *k-ordered* (or *k-ordered hamiltonian*, resp.) if for any sequence of k distinct vertices of G, there exists a cycle (or a hamiltonian cycle, resp.) in G containing these k vertices in the specified order. Obviously, any cycle in a graph is 1-ordered, 2-ordered and 3-ordered. Thus the study of *k*-orderedness (or *k*-ordered hamiltonicity) of any graph always starts with $k = 4$. A graph $G=(V,E)$ is a *k-ordered hamiltonian-connected* graph if for any sequence of k vertices of G, denoted by $\{u = v_0, v_1, \ldots, v = v_{k-1}\}$, there exists a hamiltonian path P between u and v such that P passes these vertices in the specified order. It can be seen that *k*-ordered hamiltonicity and *k*-ordered hamiltonian-connectedness do not imply each other.

The concept of *k*-orderedness and *k*-ordered hamiltonicity has attracted various studies since it was first introduced by Ng and Schultz [8] in 1997. See [2, 5—8]. In [8], the authors posed the question of the existence of 4-ordered 3-regular graphs other than the complete graph K_4 and the complete bipartite graph $K_{3,3}$. In [7], Meszaros answered the question by proving that the Petersen graph and the Heawood graph are non-bipartite, 4-ordered 3-regular graphs. Hsu et al. in [3] provided examples of bipartite non-vertex-transitive 4-ordered 3-regular graphs of order n for any sufficiently large even integer n. In 2013, Hung et al. further gave a complete classification of generalized Petersen graphs, $GP(n,4)$, and showed the following theorems.

Theorem 1.1 [4] Let $n \geq 9$. GP(n,4) is 4-ordered hamiltonian if and only if $n \in \{18, 19\}$ or $n \geq 21$.

Theorem 1.2 [4] Let $n \geq 9$. GP(n,4) is 4-ordered hamiltonian-connected if and only if $n \geq 18$.

Since Petersen graphs have been well-known and often provide examples or counterexamples for interesting graphic properties, the results of [7] and Theorems 1.1—1.2 might leave us an impression that most 4-ordered graphs are 4-ordered hamiltonian, and most 4-ordered hamiltonian graphs are 4-ordered hamiltonian-connected. It might be misleading. Therefore, we intend to study this topic on graphs with real applications. In this paper, we are interested in the 4-orderedness of certain types of chordal rings. The chordal ring family has been adopted as the underlying topology of certain interconnection networks [1] and is studied for the real architecture for parallel and distributed systems due to the advantage of a built-in hamiltonian cycle, symmetry, easy routing and robustness. See [10] and its references. The chordal ring $CR(n;1,q)$, where n is an even integer with $n \geq 6$ and q an odd integer with $3 \leq q \leq n/2$, is defined as follows. Let $G(V,E)=CR(n;1,q)$, where $V= \{a_1, a_2, \ldots, a_n\}$ and $E= \{(a_i, a_{(i+1) mod (n)}: 1 \leq i \leq n\} \cup \{(a_i, a_{(i+q) mod (n)}: i$ is odd and $1 \leq i \leq n\}$. See Figure 1 for an illustration.

The following two lemmas are proved by computer experiments. See [9].

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Lemma 1.1 [9] $CR(n;1,5)$ is 4-ordered for any even integer n with $14 \le n \le 50$.

Lemma 1.2 [9] $CR(n;1,7)$ is 4-ordered for any even integer n with $18 \le n \le 50$.

II. The 4-orderedness of $CR(n;1,5)$

Theorem 2.1 $CR(20+6k;1,5)$ is 4-ordered for $k \ge 0$.

Proof. By Lemma 1.1, $CR(20;1,5)$ is a 4-ordered graph. It is interesting to see whether or not $CR(20+6k;1,5)$ is 4-ordered for $k \ge 1$. We can embed $CR(20+6k;1,5)$ into $CR(26+6k;1,5)$ as follows. Let R be a subset of $V(CR(20+6k;1,5)) \cup E(CR(20+6k;1,5))$. We define a function f, which maps R from $CR(20+6k;1,5)$ into $CR(26+6k;1,5)$ in the following way: (1) If $a_i \in R \cap V(CR(20+6k;1,5))$, where $1 \le i \le 20+6k$, then $f(a_i)=b_i$. (2) If $(a_i, a_j) \in R \cap E(CR(20+6k;1,5))$, where $1 \le i, j \le 20+6k$, then

$$f((a_i, a_j)) = \begin{cases} (b_i, b_{i+1}) & \text{for } 1 \le i \le 19+6k \text{ and } j = i+1; \\ (b_i, b_{i+5}) & \text{for } i = odd \text{ with } 1 \le i \le 15+6k \text{ and } j = i+5; \\ \text{undefined} & \text{otherwise.} \end{cases}$$

Therefore, $CR(26+6k;1,5) - f(CR(20+6k;1,5))$ consists of the vertex set $\{b_{21+6k}, b_{22+6k}, b_{23+6k}, b_{24+6k}, b_{25+6k}, b_{26+6k}\}$ and the edge set $\{(b_{20+6k}, b_{21+6k}), (b_{21+6k}, b_{22+6k}), (b_{22+6k}, b_{23+6k}), (b_{23+6k}, b_{24+6k}), (b_{24+6k}, b_{25+6k}), (b_{25+6k}, b_{26+6k}), (b_{26+6k}, b_1), (b_{17+6k}, b_{22+6k}), (b_{19+6k}, b_{24+6k}), (b_{21+6k}, b_{26+6k}), (b_{23+6k}, b_2), (b_{25+6k}, b_4)\}$. Figure 2.1 gives an illustration, in which f maps R from $CR(20;1,5)$ into $CR(26;1,5)$. We can see that (1) $f(a_i) = b_i$ for $1 \le i \le 20$, denoted by black vertices on both graphs. (2) $f((a_i, a_{i+1})) = (b_i, b_{i+1})$ for $1 \le i \le 19$, denoted by green edges on both graphs. (3) $f((a_i, a_{i+5})) = (b_i, b_{i+5})$ for i is odd with $1 \le i \le 15$, denoted by blue edges on both graphs. (4) $f((a_{20}, a_1)) = \phi$, $f((a_{17}, a_2)) = \phi$ and $f((a_{19}, a_4)) = \phi$, denoted by dashed edges on $CR(20;1,5)$. (5) $CR(26;1,5) - f(CR(20;1,5))$ consists of the vertex set $\{b_{21}, b_{22}, b_{23}, b_{24}, b_{25}, b_{26}\}$ and edge set $\{(b_{20}, b_{21}), (b_{21}, b_{22}), (b_{22}, b_{23}), (b_{23}, b_{24}), (b_{24}, b_{25}), (b_{25}, b_{26}), (b_{26}, b_1), (b_{17}, b_{22}), (b_{19}, b_{24}), (b_{21}, b_{26}), (b_{23}, b_2), (b_{25}, b_4)\}$.

We first present the construction of the required cycle in $CR(26;1,5)$ using the known cycle of $CR(20;1,5)$, denoted by C', as an illustration. There are 20 vertices $a_1, a_2, \ldots, a_{20}$ in $CR(20;1,5)$, and 26 vertices $b_1, b_2, \ldots, b_{26}$ in $CR(26;1,5)$. To prove the theorem, we do case studies by considering different situations. Take $G = CR(26;1,5)$. Let x_1, x_2, x_3 and x_4 be four arbitrary vertices of G. We want to construct a cycle C in G that visits x_i's in the given order. Note that we can always find at least one set of six consecutive vertices, denoted by $S = \{b_i, b_{i+1}, b_{i+2}, \ldots, b_{i+5}\}$, such that $S \cap \{x_1, x_2, x_3, x_4\} = \phi$. Without loss of generality, let $x_1 = b_1$ and $S = \{b_{21}, b_{22}, \ldots, b_{26}\}$. Removing the vertices of S and all edges adjacent to S in G, we obtain a subgraph of $CR(20;1,5)$. Obviously, $S \cap f(CR(20;1,5)) = \phi$. Note that $CR(20;1,5)$ is 4-ordered and hence contains a cycle that visits x_i's in the given order, denoted by C'. We will obtain C by embedding $CR(20;1,5)$ into $CR(26;1,5)$ and rerouting the cycle C'. There are two cases.

Case 1 $(a_{17}, a_2) \in C'$.

Case 1.1 $|\{a_{19}, a_{20}\} \cap C'| = 0$. It means that only the edge $(a_{17}, a_2) \in C'$. Let $C' = \langle a_1, Q_1, a_{17}, a_2, Q_2, a_1 \rangle$, where Q_1 is a path between a_1 and a_{17}, Q_2 is a path between a_2 and a_1, and $Q_1 \cap Q_2 = \phi$. We construct $C = \langle f(a_1), f(Q_1), f(a_{17}), b_{22}, b_{23}, f(a_2), f(Q_2), f(a_1) \rangle = \langle b_1, f(Q_1), b_{17}, b_{22}, b_{23}, b_2, f(Q_2), b_1 \rangle$. See Figure 2.2 and Figure 2.3 for an illustration.

Case 1.2 $|\{a_{19}, a_{20}\} \cap C'| = 1$.

Case 1.2.1 $\langle a_{18}, a_{19}, a_4 \rangle \in C'$. It means that $(a_{17}, a_2) \in C'$ and $\langle a_{18}, a_{19}, a_4 \rangle \in C'$. Let $C' = \langle a_1, Q_1, a_{17}, a_2, Q_2, a_{18}, a_{19}, a_4, Q_3, a_1 \rangle$, where Q_1 is a path between a_1 and a_{17}, Q_2 is a path between a_2 and a_{18}, Q_3 is a path between a_4 and a_1, and $Q_i \cap Q_j = \phi$ for each $i \ne j$ and $\{i, j\} \subseteq \{1,2,3\}$. We construct $C = \langle f(a_1), f(Q_1), f(a_{17}), b_{22}, b_{23}, f(a_2), f(Q_2), f(a_{18}), f(a_{19}), b_{24}, b_{25}, f(a_4), f(Q_3), f(a_1) \rangle = \langle b_1, f(Q_1), b_{17}, b_{22}, b_{23}, b_2, f(Q_2), b_{18}, b_{19}, b_{24}, b_{25}, b_4, f(Q_3), b_1 \rangle$. See Figure 2.4 and Figure 2.5 for an illustration.

Case 1.2.2 $\langle a_{15}, a_{20}, a_1 \rangle \in C'$. It means that $(a_{17}, a_2) \in C'$ and $\langle a_{15}, a_{20}, a_1 \rangle \in C'$. Let $C' = \langle a_1, Q_1, a_{17}, a_2, Q_2, a_{15}, a_{20}, a_1 \rangle$, where Q_1 is a path between a_1 and a_{17}, Q_2 is a path between a_2 and a_{15}, and $Q_1 \cap Q_2 = \phi$. We construct $C = \langle f(a_1), f(Q_1), f(a_{17}), b_{22}, b_{23}, f(a_2), f(Q_2), f(a_{15}), f(a_{20}), b_{21}, b_{26}, f(a_1) \rangle = \langle b_1, f(Q_1), b_{17}, b_{22}, b_{23}, b_2, f(Q_2), b_{15}, b_{20}, b_{21}, b_{26}, b_1 \rangle$. See Figure 2.6 and Figure 2.7 for an illustration.

Case 1.3 $\{a_{19}, a_{20}\} \cap C'| = 2$.

Case 1.3.1 $\langle a_{18}, a_{19}, a_{20}, a_{15} \rangle \in C'$. It means that $(a_{17}, a_2) \in C'$ and $\langle a_{18}, a_{19}, a_{20}, a_{15} \rangle \in C'$. Let $C' = \langle a_1, Q_1, a_{17}, a_2, Q_2, a_{18}, a_{19}, a_{20}, a_{15}, Q_3, a_1 \rangle$, where Q_1 is a path between a_1 and a_{17}, Q_2 is a path between a_2 and a_{18}, Q_3 is a path between a_{15} and a_1, and $Q_i \cap Q_j = \phi$ for each $i \ne j$ and $\{i, j\} \subseteq \{1, 2, 3\}$. We construct $C = \langle f(a_1), f(Q_1), f(a_{17}), b_{22}, b_{23}, f(a_2), f(Q_2), f(a_{18}), f(a_{19}), f(a_{20}), f(a_{15}), f(Q_3), f(a_1) \rangle = \langle b_1, f(Q_1), b_{17}, b_{22}, b_{23}, b_2, f(Q_2), b_{18}, b_{19}, b_{20}, b_{15}, f(Q_3), b_1 \rangle$.

Case 1.3.2 $\langle a_1, a_{20}, a_{19}, a_4 \rangle \in C'$. It means that $\langle a_1, a_{20}, a_{19}, a_4 \rangle \in C'$ and $(a_{17}, a_2) \in C'$. Let $C' = \langle a_1, a_{20}, a_{19}, a_4, Q_1, a_{17}, a_2, Q_2, a_1 \rangle$, where Q_1 is a path between a_4 and a_{17}, Q_2 is a path between a_2 and a_1, and $Q_1 \cap Q_2 = \phi$. We construct $C = \langle f(a_1), b_{26}, b_{21}, f(a_{20}), f(a_{19}), b_{24}, b_{25}, f(a_4), f(Q_1), f(a_{17}), b_{22}, b_{23}, f(a_2), f(Q_2), f(a_1) \rangle = \langle b_1, b_{26}, b_{21}, b_{20}, b_{19}, b_{24}, b_{25}, b_4, f(Q_1), b_{17}, b_{22}, b_{23}, b_2, f(Q_2), b_1 \rangle$.

Case 1.3.3 $\langle a_{15}, a_{20}, a_{19}, a_4 \rangle \in C'$. It means that $(a_{17}, a_2) \in C'$ and $\langle a_{15}, a_{20}, a_{19}, a_4 \rangle \in C'$. Let $C' = \langle a_1, Q_1, a_{17}, a_2, Q_2, a_{15}, a_{20}, a_{19}, a_4, Q_3, a_1 \rangle$, where Q_1 is a path between a_1 and a_{17}, Q_2 is a path between a_2 and a_{15}, Q_3 is a path between a_4 and a_1, and $Q_i \cap Q_j = \phi$ for each $i \ne j$ and $\{i, j\} \subseteq \{1,2,3\}$. We construct $C = \langle f(a_1), f(Q_1), f(a_{17}), b_{22}, b_{23}, f(a_2), f(Q_2), f(a_{15}), f(a_{20}), f(a_{19}), b_{24}, b_{25}, f(a_4), f(Q_3), f(a_1) \rangle = \langle b_1, f(Q_1), b_{17}, b_{22}, b_{23}, b_2, f(Q_2), b_{15}, b_{20}, b_{19}, b_{24}, b_{25}, b_4, f(Q_3), b_1 \rangle$.

Case 1.3.4 $\langle a_{18}, a_{19}, a_{20}, a_1\rangle \in C'$. It means that $(a_{17}, a_2) \in C'$ and $\langle a_{18}, a_{19}, a_{20}, a_1\rangle \in C'$. Let $C' = \langle a_1, Q_1, a_{17}, a_2, Q_2, a_{18}, a_{19}, a_{20}, a_1\rangle$, where Q_1 is a path between a_1 and a_{17}, Q_2 is a path between a_2 and a_{18}, and $Q_1 \cap Q_2 = \phi$. We construct $C = \langle f(a_1), f(Q_1), f(a_{17}), b_{22}, b_{23}, f(a_2), f(Q_2), f(a_{18}), f(a_{19}), f(a_{20}), b_{21}, b_{26}, f(a_1)\rangle = \langle b_1, f(Q_1), b_{17}, b_{22}, b_{23}, b_2, f(Q_2), b_{18}, b_{19}, b_{20}, b_{21}, b_{26}, b_1\rangle$.

Case 1.3.5 $\langle a_{18}, a_{19}, a_4\rangle \in C'$, $\langle a_{15}, a_{20}, a_1\rangle \in C'$. It means that $(a_{17}, a_2) \in C'$, $\langle a_{18}, a_{19}, a_4\rangle \in C'$ and $\langle a_{15}, a_{20}, a_1\rangle \in C'$. Let $C' = \langle a_1, Q_1, a_{17}, a_2, Q_2, a_{18}, a_{19}, a_4, Q_3, a_{15}, a_{20}, a_1\rangle$, where Q_1 is a path between a_1 and a_{17}, Q_2 is a path between a_2 and a_{18}, Q_3 is a path between a_4 and a_{15}, and $Q_i \cap Q_j = \phi$ for each $i \neq j$ and $\{i, j\} \subseteq \{1,2,3\}$. We construct $C = \langle f(a_1), f(Q_1), f(a_{17}), b_{22}, b_{23}, f(a_2), f(Q_2), f(a_{18}), f(a_{19}), b_{24}, b_{25}, f(a_4), f(Q_3), f(a_{15}), f(a_{20}), b_{21}, b_{26}, f(a_1)\rangle = \langle b_1, f(Q_1), b_{17}, b_{22}, b_{23}, b_2, f(Q_2), b_{18}, b_{19}, b_{24}, b_{25}, b_4, f(Q_3), b_{15}, b_{20}, b_{21}, b_{26}, b_1\rangle$.

Case 2 $(a_{17}, a_2) \notin C'$.

Case 2.1 $|\{a_{19}, a_{20}\} \cap C'| = 0$. It means that $C' = \langle a_1, Q_1, a_1\rangle$, but the edges (a_{17}, a_2), (a_{15}, a_{20}), (a_{19}, a_4), (a_{18}, a_{19}), (a_{19}, a_{20}), $(a_{20}, a_1) \notin C'$.

Case 2.2 $|\{a_{19}, a_{20}\} \cap C'| = 1$.

Case 2.2.1 $\langle a_{18}, a_{19}, a_4\rangle \in C'$. It means that $\langle a_{18}, a_{19}, a_4\rangle \in C'$. Let $C' = \langle a_1, Q_1, a_{18}, a_{19}, a_4, Q_2, a_1\rangle$, where Q_1 is a path between a_1 and a_{18}, Q_2 is a path between a_4 and a_1, and $Q_1 \cap Q_2 = \phi$. We construct $C = \langle f(a_1), f(Q_1), f(a_{18}), f(a_{19}), b_{24}, b_{25}, f(a_4), f(Q_2), f(a_1)\rangle = \langle b_1, f(Q_1), b_{18}, b_{19}, b_{24}, b_{25}, b_4, f(Q_2), b_1\rangle$.

Case 2.2.2 $\langle a_{15}, a_{20}, a_1\rangle \in C'$. It means that $\langle a_{15}, a_{20}, a_1\rangle \in C'$. Let $C' = \langle a_1, Q_1, a_{15}, a_{20}, a_1\rangle$, where Q_1 is a path between a_1 and a_{15}. We construct $C = \langle f(a_1), f(Q_1), f(a_{15}), f(a_{20}), b_{21}, b_{26}, f(a_1)\rangle = \langle b_1, f(Q_1), b_{15}, b_{20}, b_{21}, b_{26}, b_1\rangle$.

Case 2.3 $|\{a_{19}, a_{20}\} \cap C'| = 2$.

Case 2.3.1 $\langle a_{18}, a_{19}, a_{20}, a_{15}\rangle \in C'$. It means that $\langle a_{18}, a_{19}, a_{20}, a_{15}\rangle \in C'$. Let $C' = \langle a_1, Q_1, a_{18}, a_{19}, a_{20}, a_{15}, Q_2, a_1\rangle$, where Q_1 is a path between a_1 and a_{18}, Q_2 is a path between a_{15} and a_1, and $Q_1 \cap Q_2 = \phi$. We construct $C = \langle f(a_1), f(Q_1), f(a_{18}), f(a_{19}), f(a_{20}), f(a_{15}), f(Q_2), f(a_1)\rangle = \langle b_1, f(Q_1), b_{18}, b_{19}, b_{20}, b_{15}, f(Q_2), b_1\rangle$.

Case 2.3.2 $\langle a_1, a_{20}, a_{19}, a_4\rangle \in C'$. It means that $\langle a_1, a_{20}, a_{19}, a_4\rangle \in C'$. Let $C' = \langle a_1, a_{20}, a_{19}, a_4, Q_1, a_1\rangle$, where Q_1 is a path between a_4 and a_1. Let $C = \langle f(a_1), b_{26}, b_{21}, f(a_{20}), f(a_{19}), b_{24}, b_{25}, f(a_4), f(Q_1), f(a_1)\rangle = \langle b_1, b_{26}, b_{21}, b_{20}, b_{19}, b_{24}, b_{25}, b_4, f(Q_1), b_1\rangle$.

Case 2.3.3 $\langle a_{15}, a_{20}, a_{19}, a_4\rangle \in C'$. It means that $\langle a_{15}, a_{20}, a_{19}, a_4\rangle \in C'$. Let $C' = \langle a_1, Q_1, a_{15}, a_{20}, a_{19}, a_4, Q_2, a_1\rangle$, where Q_1 is a path between a_1 and a_{15}, Q_2 is a path between a_4 and a_1, and $Q_1 \cap Q_2 = \phi$. We construct $C = \langle f(a_1), f(Q_1), f(a_{15}), f(a_{20}), f(a_{19}), b_{24}, b_{25}, f(a_4), f(Q_2), f(a_1)\rangle = \langle b_1, f(Q_1), b_{15}, b_{20}, b_{19}, b_{24}, b_{25}, b_4, f(Q_2), b_1\rangle$.

Case 2.3.4 $\langle a_{18}, a_{19}, a_{20}, a_1\rangle \in C'$. It means that $\langle a_{18}, a_{19}, a_{20}, a_1\rangle \in C'$. Let $C' = \langle a_1, Q_1, a_{18}, a_{19}, a_{20}, a_1\rangle$, where Q_1 is a path between a_1 and a_{18}. We construct $C = \langle f(a_1), f(Q_1), f(a_{18}), f(a_{19}), f(a_{20}), b_{21}, b_{26}, f(a_1)\rangle = \langle b_1, f(Q_1), b_{18}, b_{19}, b_{20}, b_{21}, b_{26}, b_1\rangle$.

Case 2.3.5 $\langle a_{18}, a_{19}, a_4\rangle \in C'$, $\langle a_{15}, a_{20}, a_1\rangle \in C'$. It means that $\langle a_{18}, a_{19}, a_4\rangle \in C'$ and $\langle a_{15}, a_{20}, a_1\rangle \in C'$. Let $C' = \langle a_1, Q_1, a_{18}, a_{19}, a_4, Q_2, a_{15}, a_{20}, a_1\rangle$, where Q_1 is a path between a_1 and a_{18}, Q_2 is a path between a_4 and a_{15}, and $Q_1 \cap Q_2 = \phi$. We construct $C = \langle f(a_1), f(Q_1), f(a_{18}), f(a_{19}), b_{24}, b_{25}, f(a_4), f(Q_2), f(a_{15}), f(a_{20}), b_{21}, b_{26}, f(a_1)\rangle = \langle b_1, f(Q_1), b_{18}, b_{19}, b_4, f(Q_2), b_{15}, b_{20}, b_{21}, b_{26}, b_1\rangle$.

Given four arbitrary vertices $\{x_i | 1 \le i \le 4\}$ in $CR(26;1,5)$, we have presented a constructive skill for finding a cycle C in $CR(26;1,5)$ from the known cycle C' in $CR(20;1,5)$ that visits x_i's in the right order. The same technique is applied to derive a cycle $\bar{C}$ in $CR(26+6k;1,5)$ from a cycle $\bar{C}'$ in $CR(20+6k;1,5)$ that passes four arbitrary vertices in the required order. More specifically, using the induction hypothesis, we assume that the statement holds for $CR(20+6k^*;1,5)$ for some integer $k^* \ge 1$. Replacing the vertex label a_i (or b_i, resp.) with a_{i+6k^*} (or b_{i+6k^*} resp.) in the above derivation, we can show that the statement in the theorem holds for $CR(26+6k^*;1,5)$. Hence the theorem is proved by mathematical induction. □

With Lemma 1.1, it is known that $CR(22;1,5)$ and $CR(24;1,5)$ are 4-ordered. It is easy to see that our technique in Theorem 2.1 can be utilized to obtain the following two theorems.

Theorem 2.2. $CR(22+6k;1,5)$ is 4-ordered for $k \ge 0$.

Theorem 2.3. $CR(24+6k;1,5)$ is 4-ordered for $k \ge 0$.

Combining Lemma 1.1 and Theorems 2.1–2.3, we have the following theorem.

Theorem 2.4. $CR(n;1,5)$ is 4-ordered for any even integer n with $n \ge 14$.

III. The 4-orderedness of $CR(n;1,7)$

Theorem 3.1. $CR(26+8k;1,7)$ is 4-ordered for $k \ge 0$.

Proof. By Lemma 1.2, $CR(26; 1, 7)$ is a 4-ordered graph. We can embed $CR(26+8k; 1, 7)$ into $CR(34+8k; 1, 7)$ as follows. Let R be a subset of $V(CR(26 + 8k; 1, 7))$ □ $E(CR(26 + 8k; 1, 7))$. We define a function f, which maps R from $CR(26 + 8k; 1, 7)$ into $CR(34 + 8k; 1, 7)$ in the following way: (1) If $a_i \in R \cap V(CR(26 + 8k; 1, 7))$, where $1 \le i \le 26 + 8k$, then $f(a_i) = b_i$. (2) If $((a_i, a_j)) \in R \cap E(CR(26 + 8k; 1, 7))$, where $1 \le i, j \le 26 + 8k$, then

$$f((a_i\ ,\ a_j)) = \begin{cases} (b_i\ ,\ b_{i+1}) & \text{for } 1 \le i \le 25 + 8k \text{ and } j = i + 1; \\ (b_i\ ,\ b_{i+7}) & \text{for } i = odd \text{ with } 1 \le i \le 19 + 8k \text{ and } j = i + 7; \\ \text{undefined otherwise.} & \end{cases}$$

Therefore, $CR(34 + 8k; 1, 7) - f(CR(26 + 8k; 1, 7))$ consists of the vertex set $\{b_{27+8k}, b_{28+8k}, b_{29+8k}, b_{30+8k}, b_{31+8k}, b_{32+8k}, b_{33+8k}, b_{34+8k}\}$ and the edge set $\{(b_{26+8k}, b_{27+8k}), (b_{27+8k}, b_{28+8k}), (b_{28+8k}, b_{29+8k}), (b_{29+8k}, b_{30+8k}), (b_{30+8k}, b_{31+8k}), (b_{31+8k}, b_{32+8k}), (b_{32+8k}, b_{33+8k}), (b_{33+8k}, b_{34+8k}), (b_{34+8k}, b_1), (b_{21+8k}, b_{28+8k}), (b_{23+8k}, b_{30+8k}), (b_{25+8k}, b_{32+8k}), (b_{27+8k}, b_{34+8k}), (b_{29+8k}, b_2), (b_{31+8k}, b_4), (b_{33+8k}, b_6), (b_{35+8k}, b_8)\}$. Figure 3.1 gives an illustration, in which f maps R from $CR(26; 1, 7)$ into $CR(34; 1, 7)$. We can see that (1) $f(a_i) = b_i$ for $1 \le i \le 26$, denoted by black vertices on both graphs. (2) $f((a_i, a_{i+1})) = \langle b_i, b_{i+1}\rangle$ for $1 \le i \le 25$, denoted by brown edges on both graphs. (3) $f((a_i, a_{i+7})) = \langle b_i, b_{i+7}\rangle$ for i is odd with $1 \le i \le 19$, denoted by blue edges on both graphs. (4) $f((a_{26}, a_1)) = \varphi$, $f((a_{21}, a_2)) = \varphi$, $f((a_{23}, a_4)) = \varphi$ and $f((a_{25}, a_6)) = \varphi$, denoted by dashed edges on $CR(26; 1, 7)$. (5) $CR(34; 1, 7) - f(CR(26; 1, 7))$ consists of the vertex set $\{b_{27}, b_{28}, b_{29}, b_{30}, b_{31}, b_{32}, b_{33}, b_{34}\}$ and edge set $\{(b_{26}, b_{27}), (b_{27}, b_{28}), (b_{28}, b_{29}), (b_{29}, b_{30}), (b_{30}, b_{31}), (b_{31}, b_{32}), (b_{32}, b_{33}), (b_{33}, b_{34}), (b_{34}, b_1), (b_{21}, b_{28}), (b_{23}, b_{30}), (b_{25}, b_{32}), (b_{27}, b_{34}), (b_{29}, b_2), (b_{31}, b_4), (b_{33}, b_6)\}$.

We first present the construction of the required cycle in $CR(34; 1, 7)$ using the known cycle of $CR(26; 1, 7)$ as an illustration. There are twenty-six vertices $a_1, a_2, \ldots, a_{26}$ in $CR(26; 1, 7)$, and thirty-four vertices $b_1, b_2, \ldots, b_{34}$ in $CR(34; 1, 7)$. To prove the theorem, we do case studies by considering different situations. Take $G = CR(34; 1, 7)$. Let x_1, x_2, x_3 and x_4 be four arbitrary vertices of G. We want to construct a cycle C in G that visits x_i's in the given order. Note that we can always find at least one set of eight consecutive vertices, denoted by $S = \{b_i, b_{i+1}, b_{i+2}, \ldots, b_{i+7}\}$, such that $S \cap \{x_1, x_2, x_3, x_4\} = \varphi$. Without loss of generality, let $x_1 = b_1$ and $S = \{b_{27}, b_{28}, \ldots, b_{34}\}$. Removing the vertices of S and all edges adjacent to S in G, we obtain a subgraph of $CR(26; 1, 7)$. Obviously, $S \cap f(CR(26; 1, 7)) = \varphi$. Note that $CR(26; 1, 7)$ is 4-ordered and hence contains a cycle that visits x_i's in the given order, denoted by C'. We will obtain C by embedding $CR(26; 1, 7)$ into $CR(34; 1, 7)$ and rerouting the cycle C'. There are two cases.

Case 1 $(a_{21}, a_2) \in C'$.

Case 1.1 $|\{a_{23}, a_{24}, a_{25}, a_{26}\} \cap C'| = 0$.
It means that only the edge $(a_{21}, a_2) \in C'$. See Figure 3.2 for an illustration. Let $C' = \langle a_1, Q_1, a_{21}, a_2, Q_2, a_1\rangle$, where Q_1 is a path between a_1 and a_{21}, Q_2 is a path between a_2 and a_1, and $Q_1 \cap Q_2 = \varphi$. We construct $C = \langle f(a_1), f(Q_1), f(a_{21}), b_{28}, b_{29}, f(a_2), f(Q_2), f(a_1)\rangle = \langle b_1, f(Q_1), b_{21}, b_{28}, b_{29}, b_2, f(Q_2), b_1\rangle$.

Case 1.2 $|\{a_{23}, a_{24}, a_{25}, a_{26}\} \cap C'| = 1$.

Case 1.2.1 $\langle a_{19}, a_{26}, a_1\rangle \in C'$.

It means that $(a_{21}, a_2) \in C'$ and $\langle a_{19}, a_{26}, a_1\rangle \in C'$. Let $C' = \langle a_1, Q_1, a_{21}, a_2, Q_2, a_{19}, a_{26}, a_1\rangle$, where Q_1 is a path between a_1 and a_{21}, Q_2 is a path between a_2 and a_{19}, and $Q_1 \cap Q_2 = \varphi$. We construct $C = \langle f(a_1), f(Q_1), f(a_{21}), b_{28}, b_{29}, f(a_2), f(Q_2), f(a_{19}), f(a_{26}), b_{27}, b_{34}, f(a_1)\rangle = \langle b_1, f(Q_1), b_{21}, b_{28}, b_{29}, b_2, f(Q_2), b_{19}, b_{26}, b_{27}, b_{34}, b_1\rangle$.

Case 1.2.2 $\langle a_{22}, a_{23}, a_4\rangle \in C'$.

It means that $(a_{21}, a_2) \in C'$ and $\langle a_{22}, a_{23}, a_4\rangle \in C'$. Let $C' = \langle a_1, Q_1, a_{21}, a_2, Q_2, a_{22}, a_{23}, a_4, Q_3, a_1\rangle$, where Q_1 is a path between a_1 and a_{21}, Q_2 is a path between a_2 and a_{22}, Q_3 is a path between a_4 and a_1, and $Q_i \cap Q_j = \varphi$ for each $i \ne j$ and $\{i,j\} \subseteq \{1, 2, 3\}$. We construct $C = \langle f(a_1), f(Q_1), f(a_{21}), b_{28}, b_{29}, f(a_2), f(Q_2), f(a_{22}), f(a_{23}), b_{30}, b_{31}, f(a_4), f(Q_3), f(a_1)\rangle = \langle b_1, f(Q_1), b_{21}, b_{28}, b_{29}, b_2, f(Q_2), b_{22}, b_{23}, b_{30}, b_{31}, b_4, f(Q_3), b_1\rangle$. See Figure 3.3 for an illustration.

Case 1.3 $|\{a_{23}, a_{24}, a_{25}, a_{26}\} \cap C'| = 2$.

Case 1.3.1 $\langle a_{17}, a_{24}, a_{23}, a_{22}\rangle \in C'$.

It means that $(a_{21}, a_2) \in C'$ and $\langle a_{17}, a_{24}, a_{23}, a_{22}\rangle \in C'$. Let $C' = \langle a_1, Q_1, a_{21}, a_2, Q_2, a_{17}, a_{24}, a_{23}, a_{22}, Q_3, a_1\rangle$, where Q_1 is a path between a_1 and a_{21}, Q_2 is a path between a_2 and a_{17}, Q_3 is a path between a_{22} and a_1, and $Q_i \cap Q_j = \varphi$ for each $i \ne j$ and $\{i,j\} \subseteq \{1, 2, 3\}$. We construct $C = \langle f(a_1), f(Q_1), f(a_{21}), b_{28}, b_{29}, f(a_2), f(Q_2), f(a_{17}), f(a_{24}), f(a_{23}), f(a_{22}), f(Q_3), f(a_1)\rangle = \langle b_1, f(Q_1), b_{21}, b_{28}, b_{29}, b_2, f(Q_2), b_{17}, b_{24}, b_{23}, b_{22}, f(Q_3), b_1\rangle$.

Case 1.3.2 $\langle a_6, a_{25}, a_{26}, a_1\rangle \in C'$.

It means that $(a_{21}, a_2) \in C'$ and $\langle a_6, a_{25}, a_{26}, a_1\rangle \in C'$. Let $C' = \langle a_1, Q_1, a_{21}, a_2, Q_2, a_6, a_{25}, a_{26}, a_1\rangle$, where Q_1 is a path between a_1 and a_{21}, Q_2 is a path between a_2 and a_6, and $Q_1 \cap Q_2 = \varphi$. We construct $C = \langle f(a_1), f(Q_1), f(a_{21}), b_{28}, b_{29}, f(a_2), f(Q_2), f(a_6), b_{33}, b_{32}, f(a_{25}), f(a_{26}), b_{27}, b_{34}, f(a_1)\rangle = \langle b_1, f(Q_1), b_{21}, b_{28}, b_{29}, b_2, f(Q_2), b_6, b_{33}, b_{32}, b_{25}, b_{26}, b_{27}, b_{34}, b_1\rangle$. See Figure 3.4 for an illustration.

Case 1.3.3 $\langle a_{17}, a_{24}, a_{25}, a_6\rangle \in C'$.

It means that $(a_{21}, a_2) \in C'$ and $\langle a_{17}, a_{24}, a_{25}, a_6\rangle \in C'$. Let $C' = \langle a_1, Q_1, a_{21}, a_2, Q_2, a_{17}, a_{24}, a_{25}, a_6, Q_3, a_1\rangle$, where Q_1 is a path between a_1 and a_{21}, Q_2 is a path between a_2 and a_{17}, Q_3 is a path between a_6 and a_1, and $Q_i \cap Q_j = \varphi$ for each $i \ne j$ and $\{i,j\} \subseteq \{1, 2, 3\}$. We construct $C = \langle f(a_1), f(Q_1), f(a_{21}), b_{28}, b_{29}, f(a_2), f(Q_2), f(a_{17}), f(a_{24}), f(a_{25}), b_{32}, b_{33}, f(a_6), f(Q_3), f(a_1)\rangle = \langle b_1, f(Q_1), b_{21}, b_{28}, b_{29}, b_2, f(Q_2), b_{17}, b_{24}, b_{25}, b_{32}, b_{33}, b_6, f(Q_3), b_1\rangle$.

Case 1.3.4 $\langle a_{17}, a_{24}, a_{23}, a_4\rangle \in C'$.

It means that $(a_{21}, a_2) \in C'$ and $\langle a_{17}, a_{24}, a_{23}, a_4\rangle \in C'$. Let $C' = \langle a_1, Q_1, a_{21}, a_2, Q_2, a_{17}, a_{24}, a_{23}, a_4, Q_3, a_1\rangle$, where Q_1 is a path between a_1 and a_{21}, Q_2 is a path between a_2 and a_{17}, Q_3 is a path between a_4 and a_1, and $Q_i \cap Q_j = \varphi$ for each $i \ne j$ and $\{i,j\} \subseteq \{1, 2,$

3}. We construct $C = \langle f(a_1), f(Q_1), f(a_{21}), b_{28}, b_{29}, f(a_2), f(Q_2), f(a_{17}), f(a_{24}), f(a_{23}), b_{30}, b_{31}, f(a_4), f(Q_3), f(a_1)\rangle = \langle b_1, f(Q_1), b_{21}, b_{28}, b_{29}, b_2, f(Q_2), b_{17}, b_{24}, b_{23}, b_{30}, b_{31}, b_4, f(Q_3), b_1\rangle$.

Case 1.3.5 $\langle a_{19}, a_{26}, a_{25}, a_6\rangle \in C'$.

It means that $(a_{21}, a_2) \in C'$ and $\langle a_{19}, a_{26}, a_{25}, a_6\rangle \in C'$. Let $C' = \langle a_1, Q_1, a_{21}, a_2, Q_2, a_{19}, a_{26}, a_{25}, a_6, Q_3, a_1\rangle$, where Q_1 is a path between a_1 and a_{21}, Q_2 is a path between a_2 and a_{19}, Q_3 is a path between a_6 and a_1, and $Q_i \cap Q_j = \varphi$ for each $i \neq j$ and $\{i,j\} \subseteq \{1, 2, 3\}$. We construct $C = \langle f(a_1), f(Q_1), f(a_{21}), b_{28}, b_{29}, f(a_2), f(Q_2), f(a_{19}), f(a_{26}), f(a_{25}), b_{32}, b_{33}, f(a_6), f(Q_3), f(a_1)\rangle = \langle b_1, f(Q_1), b_{21}, b_{28}, b_{29}, b_2, f(Q_2), b_{19}, b_{26}, b_{25}, b_{32}, b_{33}, b_6, f(Q_3), b_1\rangle$.

Case 1.3.6 $\langle a_{22}, a_{23}, a_4\rangle \in C'$, $\langle a_{19}, a_{26}, a_1\rangle \in C'$.

It means that $(a_{21}, a_2) \in C'$, $\langle a_{22}, a_{23}, a_4\rangle \in C'$ and $\langle a_{19}, a_{26}, a_1\rangle \in C'$. Let $C' = \langle a_1, Q_1, a_{21}, a_2, Q_2, a_{22}, a_{23}, a_4, Q_3, a_{19}, a_{26}, a_1\rangle$, where Q_1 is a path between a_1 and a_{21}, Q_2 is a path between a_2 and a_{22}, Q_3 is a path between a_4 and a_{19}, and $Q_i \cap Q_j = \varphi$ for each $i \neq j$ and $\{i,j\} \subseteq \{1,2,3\}$. We construct $C = \langle f(a_1), f(Q_1), f(a_{21}), b_{28}, b_{29}, f(a_2), f(Q_2), f(a_{22}), f(a_{23}), b_{30}, b_{31}, f(a_4), f(Q_3), f(a_{19}), f(a_{26}), b_{27}, b_{34}, f(a_1)\rangle = \langle b_1, f(Q_1), b_{21}, b_{28}, b_{29}, b_2, f(Q_2), b_{22}, b_{23}, b_{30}, b_{31}, b_4, f(Q_3), b_{19}, b_{26}, b_{27}, b_{34}, b_1\rangle$.

Case 1.4 $|\{a_{23}, a_{24}, a_{25}, a_{26}\} \cap C'| = 3$.

Case 1.4.1 $\langle a_{17}, a_{24}, a_{25}, a_{26}, a_1\rangle \in C'$.

It means that $(a_{21}, a_2) \in C'$ and $\langle a_{17}, a_{24}, a_{25}, a_{26}, a_1\rangle \in C'$. Let $C' = \langle a_1, Q_1, a_{21}, a_2, Q_2, a_{17}, a_{24}, a_{25}, a_{26}, a_1\rangle$, where Q_1 is a path between a_1 and a_{21}, Q_2 is a path between a_2 and a_{17}, and $Q_1 \cap Q_2 = \varphi$. We construct $C = \langle f(a_1), f(Q_1), f(a_{21}), b_{28}, b_{29}, f(a_2), f(Q_2), f(a_{17}), f(a_{24}), f(a_{25}), f(a_{26}), b_{27}, b_{34}, f(a_1)\rangle = \langle b_1, f(Q_1), b_{21}, b_{28}, b_{29}, b_2, f(Q_2), b_{17}, b_{24}, b_{25}, b_{26}, b_{27}, b_{34}, b_1\rangle$.

Case 1.4.2 $\langle a_{22}, a_{23}, a_{24}, a_{25}, a_6\rangle \in C'$.

It means that $(a_{21}, a_2) \in C'$ and $\langle a_{22}, a_{23}, a_{24}, a_{25}, a_6\rangle \in C'$. Let $C' = \langle a_1, Q_1, a_{21}, a_2, Q_2, a_{22}, a_{23}, a_{24}, a_{25}, a_6, Q_3, a_1\rangle$, where Q_1 is a path between a_1 and a_{21}, Q_2 is a path between a_2 and a_{22}, Q_3 is a path between a_6 and a_1, and $Q_i \cap Q_j = \varphi$ for each $i \neq j$ and $\{i,j\} \subseteq \{1, 2, 3\}$. We construct $C = \langle f(a_1), f(Q_1), f(a_{21}), b_{28}, b_{29}, f(a_2), f(Q_2), f(a_{22}), f(a_{23}), f(a_{24}), f(a_{25}), b_{32}, b_{33}, f(a_6), f(Q_3), f(a_1)\rangle = \langle b_1, f(Q_1), b_{21}, b_{28}, b_{29}, b_2, f(Q_2), b_{22}, b_{23}, b_{24}, b_{25}, b_{32}, b_{33}, b_6, f(Q_3), b_1\rangle$. See Figure 3.5 for an illustration.

Case 1.4.3 $\langle a_{17}, a_{24}, a_{25}, a_{26}, a_{19}\rangle \in C'$.

It means that $(a_{21}, a_2) \in C'$ and $\langle a_{17}, a_{24}, a_{25}, a_{26}, a_{19}\rangle \in C'$. Let $C' = \langle a_1, Q_1, a_{21}, a_2, Q_2, a_{17}, a_{24}, a_{25}, a_{26}, a_{19}, Q_3, a_1\rangle$, where Q_1 is a path between a_1 and a_{21}, Q_2 is a path between a_2 and a_{17}, Q_3 is a path between a_{19} and a_1, and $Q_i \cap Q_j = \varphi$ for each $i \neq j$ and $\{i,j\} \subseteq \{1, 2, 3\}$. We construct $C = \langle f(a_1), f(Q_1), f(a_{21}), b_{28}, b_{29}, f(a_2), f(Q_2), f(a_{17}), f(a_{24}), f(a_{25}), f(a_{26}), f(a_{19}), f(Q_3), f(a_1)\rangle = \langle b_1, f(Q_1), b_{21}, b_{28}, b_{29}, b_2, f(Q_2), b_{17}, b_{24}, b_{25}, b_{26}, b_{19}, f(Q_3), b_1\rangle$.

Case 1.4.4 $\langle a_6, a_{25}, a_{24}, a_{23}, a_4\rangle \in C'$.

It means that $(a_{21}, a_2) \in C'$ and $\langle a_6, a_{25}, a_{24}, a_{23}, a_4\rangle \in C'$. Let $C' = \langle a_1, Q_1, a_{21}, a_2, Q_2, a_6, a_{25}, a_{24}, a_{23}, a_4, Q_3, a_1\rangle$, where Q_1 is a path between a_1 and a_{21}, Q_2 is a path between a_2 and a_6, Q_3 is a path between a_4 and a_1, and $Q_i \cap Q_j = \varphi$ for each $i \neq j$ and $\{i,j\} \subseteq \{1, 2, 3\}$. We construct $C = \langle f(a_1), f(Q_1), f(a_{21}), b_{28}, b_{29}, f(a_2), f(Q_2), f(a_6), b_{33}, b_{32}, f(a_{25}), f(a_{24}), f(a_{23}), b_{30}, b_{31}, f(a_4), f(Q_3), f(a_1)\rangle = \langle b_1, f(Q_1), b_{21}, b_{28}, b_{29}, b_2, f(Q_2), b_6, b_{33}, b_{32}, b_{25}, b_{24}, b_{23}, b_{30}, b_{31}, b_4, f(Q_3), b_1\rangle$.

Case 1.4.5 $\langle a_{17}, a_{24}, a_{23}, a_{22}\rangle \in C'$, $\langle a_{19}, a_{26}, a_1\rangle \in C'$.

It means that $(a_{21}, a_2) \in C'$, $\langle a_{17}, a_{24}, a_{23}, a_{22}\rangle \in C'$ and $\langle a_{19}, a_{26}, a_1\rangle \in C'$. Let $C' = \langle a_1, Q_1, a_{21}, a_2, Q_2, a_{17}, a_{24}, a_{23}, a_{22}, Q_3, a_{19}, a_{26}, a_1\rangle$, where Q_1 is a path between a_1 and a_{21}, Q_2 is a path between a_2 and a_{17}, Q_3 is a path between a_{22} and a_{19}, and $Q_i \cap Q_j = \varphi$ for each $i \neq j$ and $\{i,j\} \subseteq \{1, 2, 3\}$. We construct $C = \langle f(a_1), f(Q_1), f(a_{21}), b_{28}, b_{29}, f(a_2), f(Q_2), f(a_{17}), f(a_{24}), f(a_{23}), f(a_{22}), f(Q_3), f(a_{19}), f(a_{26}), b_{27}, b_{34}, f(a_1)\rangle = \langle b_1, f(Q_1), b_{21}, b_{28}, b_{29}, b_2, f(Q_2), b_{17}, b_{24}, b_{23}, b_{22}, f(Q_3), b_{19}, b_{26}, b_{27}, b_{34}, b_1\rangle$.

Case 1.4.6 $\langle a_{22}, a_{23}, a_4\rangle \in C'$, $\langle a_6, a_{25}, a_{26}, a_1\rangle \in C'$.

It means that $(a_{21}, a_2) \in C'$, $\langle a_{22}, a_{23}, a_4\rangle \in C'$ and $\langle a_6, a_{25}, a_{26}, a_1\rangle \in C'$. Let $C' = \langle a_1, Q_1, a_{21}, a_2, Q_2, a_{22}, a_{23}, a_4, Q_3, a_6, a_{25}, a_{26}, a_1\rangle$, where Q_1 is a path between a_1 and a_{21}, Q_2 is a path between a_2 and a_{22}, Q_3 is a path between a_4 and a_6, and $Q_i \cap Q_j = \varphi$ for each $i \neq j$ and $\{i,j\} \subseteq \{1, 2, 3\}$. We construct $C = \langle f(a_1), f(Q_1), f(a_{21}), b_{28}, b_{29}, f(a_2), f(Q_2), f(a_{22}), f(a_{23}), b_{30}, b_{31}, f(a_4), f(Q_3), f(a_6), b_{33}, b_{32}, f(a_{25}), f(a_{26}), b_{27}, b_{34}, f(a_1)\rangle = \langle b_1, f(Q_1), b_{21}, b_{28}, b_{29}, b_2, f(Q_2), b_{22}, b_{23}, b_{30}, b_{31}, b_4, f(Q_3), b_6, b_{33}, b_{32}, b_{25}, b_{26}, b_{27}, b_{34}, b_1\rangle$.

Case 1.4.7 $\langle a_{17}, a_{24}, a_{23}, a_4\rangle \in C'$, $\langle a_{19}, a_{26}, a_1\rangle \in C'$.

It means that $(a_{21}, a_2) \in C'$, $\langle a_{17}, a_{24}, a_{23}, a_4\rangle \in C'$ and $\langle a_{19}, a_{26}, a_1\rangle \in C'$. Let $C' = \langle a_1, Q_1, a_{21}, a_2, Q_2, a_{17}, a_{24}, a_{23}, a_4, Q_3, a_{19}, a_{26}, a_1\rangle$, where Q_1 is a path between a_1 and a_{21}, Q_2 is a path between a_2 and a_{17}, Q_3 is a path between a_4 and a_{19}, and $Q_i \cap Q_j = \varphi$ for each $i \neq j$ and $\{i,j\} \subseteq \{1, 2, 3\}$. We construct $C = \langle f(a_1), f(Q_1), f(a_{21}), b_{28}, b_{29}, f(a_2), f(Q_2), f(a_{17}), f(a_{24}), f(a_{23}), b_{30}, b_{31}, f(a_4),$

$f(Q_3), f(a_{19}), f(a_{26}), b_{27}, b_{34}, f(a_1)\rangle = \langle b_1, f(Q_1), b_{21}, b_{28}, b_{29}, b_2, f(Q_2), b_{17}, b_{24}, b_{23}, b_{30}, b_{31}, b_4, f(Q_3), b_{19}, b_{26}, b_{27}, b_{34}, b_1\rangle$.

Case 1.4.8 $\langle a_{19}, a_{26}, a_{25}, a_6\rangle \in C'$, $\langle a_{22}, a_{23}, a_4\rangle \in C'$.
It means that $(a_{21}, a_2) \in C'$, $\langle a_{19}, a_{26}, a_{25}, a_6\rangle \in C'$ and $\langle a_{22}, a_{23}, a_4\rangle \in C'$. Let $C' = \langle a_1, Q_1, a_{21}, a_2, Q_2, a_{19}, a_{26}, a_{25}, a_6, Q_3, a_{22}, a_{23}, a_4, Q_4, a_1\rangle$, where Q_1 is a path between a_1 and a_{21}, Q_2 is a path between a_2 and a_{19}, Q_3 is a path between a_6 and a_{22}, Q_4 is a path between a_4 and a_1, and $Q_i \cap Q_j = \varphi$ for each $i \neq j$ and $\{i, j\} \subseteq \{1, 2, 3, 4\}$. We construct $C = \langle f(a_1), f(Q_1), f(a_{21}), b_{28}, b_{29}, f(a_2), f(Q_2), f(a_{19}), f(a_{26}), f(a_{25}), b_{32}, b_{33}, f(a_6), f(Q_3), f(a_{22}), f(a_{23}), b_{30}, b_{31}, f(a_4), f(Q_4), f(a_1)\rangle = \langle b_1, f(Q_1), b_{21}, b_{28}, b_{29}, b_2, f(Q_2), b_{19}, b_{26}, b_{25}, b_{32}, b_{33}, b_6, f(Q_3), b_{22}, b_{23}, b_{30}, b_{31}, b_4, f(Q_4), b_1\rangle$.

Case 1.4.9 $\langle a_{17}, a_{24}, a_{25}, a_6\rangle \in C'$, $\langle a_{19}, a_{26}, a_1\rangle \in C'$.
It means that $(a_{21}, a_2) \in C'$, $\langle a_{17}, a_{24}, a_{25}, a_6\rangle \in C'$ and $\langle a_{19}, a_{26}, a_1\rangle \in C'$. Let $C' = \langle a_1, Q_1, a_{21}, a_2, Q_2, a_{17}, a_{24}, a_{25}, a_6, Q_3, a_{19}, a_{26}, a_1\rangle$, where Q_1 is a path between a_1 and a_{21}, Q_2 is a path between a_2 and a_{17}, Q_3 is a path between a_6 and a_{19}, and $Q_i \cap Q_j = \varphi$ for each $i \neq j$ and $\{i,j\} \subseteq \{1, 2, 3\}$. We construct $C = \langle f(a_1), f(Q_1), f(a_{21}), b_{28}, b_{29}, f(a_2), f(Q_2), f(a_{17}), f(a_{24}), f(a_{25}), b_{32}, b_{33}, f(a_6), f(Q_3), f(a_{19}), f(a_{26}), b_{27}, b_{34}, f(a_1)\rangle = \langle b_1, f(Q_1), b_{21}, b_{28}, b_{29}, b_2, f(Q_2), b_{17}, b_{24}, b_{25}, b_{32}, b_{33}, b_6, f(Q_3), b_{19}, b_{26}, b_{27}, b_{34}, b_1\rangle$.

Case 1.4.10 $\langle a_{17}, a_{24}, a_{25}, a_6\rangle \in C'$, $\langle a_{22}, a_{23}, a_4\rangle \in C'$.
It means that $(a_{21}, a_2) \in C'$, $\langle a_{17}, a_{24}, a_{25}, a_6\rangle \in C'$ and $\langle a_{22}, a_{23}, a_4\rangle \in C'$. Let $C' = \langle a_1, Q_1, a_{21}, a_2, Q_2, a_{17}, a_{24}, a_{25}, a_6, Q_3, a_{22}, a_{23}, a_4, Q_4, a_1\rangle$, where Q_1 is a path between a_1 and a_{21}, Q_2 is a path between a_2 and a_{17}, Q_3 is a path between a_6 and a_{22}, Q_4 is a path between a_4 and a_1, and $Q_i \cap Q_j = \varphi$ for each $i \neq j$ and $\{i, j\} \subseteq \{1, 2, 3, 4\}$. We construct $C = \langle f(a_1), f(Q_1), f(a_{21}), b_{28}, b_{29}, f(a_2), f(Q_2), f(a_{17}), f(a_{24}), f(a_{25}), b_{32}, b_{33}, f(a_6), f(Q_3), f(a_{22}), f(a_{23}), b_{30}, b_{31}, f(a_4), f(Q_4), f(a_1)\rangle = \langle b_1, f(Q_1), b_{21}, b_{28}, b_{29}, b_2, f(Q_2), b_{17}, b_{24}, b_{25}, b_{32}, b_{33}, b_6, f(Q_3), b_{22}, b_{23}, b_{30}, b_{31}, b_4, f(Q_4), b_1\rangle$.

Case 1.5 $|\{a_{23}, a_{24}, a_{25}, a_{26}\} \cap C'| = 4$.

Case 1.5.1 $\langle a_{19}, a_{26}, a_{25}, a_{24}, a_{23}, a_{22}\rangle \in C'$.
It means that $(a_{21}, a_2) \in C'$, $\langle a_{19}, a_{26}, a_{25}, a_{24}, a_{23}, a_{22}\rangle \in C'$. Let $C' = \langle a_1, Q_1, a_{21}, a_2, Q_2, a_{19}, a_{26}, a_{25}, a_{24}, a_{23}, a_{22}, Q_3, a_1\rangle$, where Q_1 is a path between a_1 and a_{21}, Q_2 is a path between a_2 and a_{19}, Q_3 is a path between a_{22} and a_1, and $Q_i \cap Q_j = \varphi$ for each $i \neq j$ and $\{i,j\} \subseteq \{1, 2, 3\}$. We construct $C = \langle f(a_1), f(Q_1), f(a_{21}), b_{28}, b_{29}, f(a_2), f(Q_2), f(a_{19}), f(a_{26}), f(a_{25}), f(a_{24}), f(a_{23}), f(a_{22}), f(Q_3), f(a_1)\rangle = \langle b_1, f(Q_1), b_{21}, b_{28}, b_{29}, b_2, f(Q_2), b_{19}, b_{26}, b_{25}, b_{24}, b_{23}, b_{22}, f(Q_3), b_1\rangle$.

Case 1.5.2 $\langle a_4, a_{23}, a_{24}, a_{25}, a_{26}, a_1\rangle \in C'$.
It means that $(a_{21}, a_2) \in C'$, $\langle a_4, a_{23}, a_{24}, a_{25}, a_{26}, a_1\rangle \in C'$. Let $C' = \langle a_1, Q_1, a_{21}, a_2, Q_2, a_4, a_{23}, a_{24}, a_{25}, a_{26}, a_1\rangle$, where Q_1 is a path between a_1 and a_{21}, Q_2 is a path between a_2 and a_4, and $Q_1 \cap Q_2 = \varphi$. We construct $C = \langle f(a_1), f(Q_1), f(a_{21}), b_{28}, b_{29}, f(a_2), f(Q_2), f(a_4), b_{31}, b_{30}, f(a_{23}), f(a_{24}), f(a_{25}), f(a_{26}), b_{27}, b_{34}, f(a_1)\rangle = \langle b_1, f(Q_1), b_{21}, b_{28}, b_{29}, b_2, f(Q_2), b_4, b_{31}, b_{30}, b_{23}, b_{24}, b_{25}, b_{26}, b_{27}, b_{34}, b_1\rangle$. See Figure 3.6 for an illustration.

Case 1.5.3 $\langle a_{19}, a_{26}, a_{25}, a_{24}, a_{23}, a_4\rangle \in C'$.
It means that $(a_{21}, a_2) \in C'$, $\langle a_{19}, a_{26}, a_{25}, a_{24}, a_{23}, a_4\rangle \in C'$. Let $C' = \langle a_1, Q_1, a_{21}, a_2, Q_2, a_{19}, a_{26}, a_{25}, a_{24}, a_{23}, a_4, Q_3, a_1\rangle$, where Q_1 is a path between a_1 and a_{21}, Q_2 is a path between a_2 and a_{19}, Q_3 is a path between a_4 and a_1, and $Q_i \cap Q_j = \varphi$ for each $i \neq j$ and $\{i,j\} \subseteq \{1, 2, 3\}$. We construct $C = \langle f(a_1), f(Q_1), f(a_{21}), b_{28}, b_{29}, f(a_2), f(Q_2), f(a_{19}), f(a_{26}), f(a_{25}), f(a_{24}), f(a_{23}), b_{30}, b_{31}, f(a_4), f(Q_3), f(a_1)\rangle = \langle b_1, f(Q_1), b_{21}, b_{28}, b_{29}, b_2, f(Q_2), b_{19}, b_{26}, b_{25}, b_{24}, b_{23}, b_{30}, b_{31}, b_4, f(Q_3), b_1\rangle$.

Case 1.5.4 $\langle a_{22}, a_{23}, a_{24}, a_{25}, a_{26}, a_1\rangle \in C'$.
It means that $(a_{21}, a_2) \in C'$, $\langle a_{22}, a_{23}, a_{24}, a_{25}, a_{26}, a_1\rangle \in C'$. Let $C' = \langle a_1, Q_1, a_{21}, a_2, Q_2, a_{22}, a_{23}, a_{24}, a_{25}, a_{26}, a_1\rangle$, where Q_1 is a path between a_1 and a_{21}, Q_2 is a path between a_2 and a_{22}, and $Q_1 \cap Q_2 = \varphi$. We construct $C = \langle f(a_1), f(Q_1), f(a_{21}), b_{28}, b_{29}, f(a_2), f(Q_2), f(a_{22}), f(a_{23}), f(a_{24}), f(a_{25}), f(a_{26}), b_{27}, b_{34}, f(a_1)\rangle = \langle b_1, f(Q_1), b_{21}, b_{28}, b_{29}, b_2, f(Q_2), b_{22}, b_{23}, b_{24}, b_{25}, b_{26}, b_{27}, b_{34}, b_1\rangle$.

Case 1.5.5 $\langle a_{17}, a_{24}, a_{23}, a_{22}\rangle \in C'$, $\langle a_{19}, a_{26}, a_{25}, a_6\rangle \in C'$.
It means that $(a_{21}, a_2) \in C'$, $\langle a_{17}, a_{24}, a_{23}, a_{22}\rangle \in C'$ and $\langle a_{19}, a_{26}, a_{25}, a_6\rangle \in C'$. Let $C' = \langle a_1, Q_1, a_{21}, a_2, Q_2, a_{17}, a_{24}, a_{23}, a_{22}, Q_3, a_{19}, a_{26}, a_{25}, a_6, Q_4, a_1\rangle$, where Q_1 is a path between a_1 and a_{21}, Q_2 is a path between a_2 and a_{17}, Q_3 is a path between a_{22} and a_{19}, Q_4 is a path between a_6 and a_1, and $Q_i \cap Q_j = \varphi$ for each $i \neq j$ and $\{i,j\} \subseteq \{1, 2, 3, 4\}$. We construct $C = \langle f(a_1), f(Q_1), f(a_{21}), b_{28}, b_{29}, f(a_2), f(Q_2), f(a_{17}), f(a_{24}), f(a_{23}), f(a_{22}), f(Q_3), f(a_{19}), f(a_{26}), f(a_{25}), b_{32}, b_{33}, f(a_6), f(Q_4), f(a_1)\rangle = \langle b_1, f(Q_1), b_{21}, b_{28}, b_{29}, b_2, f(Q_2), b_{17}, b_{24}, b_{23}, b_{22}, f(Q_3), b_{19}, b_{26}, b_{25}, b_{32}, b_{33}, b_6, f(Q_4), b_1\rangle$.

Case 1.5.6 $\langle a_{17}, a_{24}, a_{23}, a_4\rangle \in C'$, $\langle a_6, a_{25}, a_{26}, a_1\rangle \in C'$.
It means that $(a_{21}, a_2) \in C'$, $\langle a_{19}, a_{26}, a_{25}, a_{24}, a_{23}, a_4\rangle \in C'$. Let $C' = \langle a_1, Q_1, a_{21}, a_2, Q_2, a_{19}, a_{26}, a_{25}, a_{24}, a_{23}, a_4, Q_3, a_6, a_{25}, a_{26}, a_1\rangle$, where Q_1 is a path between a_1 and a_{21}, Q_2 is a path between a_2 and a_{19}, Q_3 is a path between a_4 and a_6, and $Q_i \cap Q_j = \varphi$ for each $i \neq j$ and $\{i,j\} \subseteq \{1,2,3\}$. We construct $C = \langle f(a_1), f(Q_1), f(a_{21}), b_{28}, b_{29},$

$f(a_2), f(Q_2), f(a_{19}), f(a_{26}), f(a_{25}), f(a_{24}), f(a_{23}), b_{30}, b_{31}, f(a_4), f(Q_3), f(a_6), b_{33}, b_{32}, f(a_{25}), f(a_{26}), b_{27}, b_{34}, f(a_1)\rangle = \langle b_1, f(Q_1), b_{21}, b_{28}, b_{29}, b_2, f(Q_2), b_{19}, b_{26}, b_{25}, b_{24}, b_{23}, b_{30}, b_{31}, b_4, f(Q_3), b_6, b_{33}, b_{32}, b_{25}, b_{26}, b_{27}, b_{34}, b_1\rangle$.

Case 1.5.7 $\langle a_{17}, a_{24}, a_{23}, a_{22}\rangle \in C'$, $\langle a_6, a_{25}, a_{26}, a_1\rangle \in C'$. It means that $(a_{21}, a_2) \in C'$, $\langle a_{17}, a_{24}, a_{23}, a_{22}\rangle \in C'$ and $\langle a_6, a_{25}, a_{26}, a_1\rangle \in C'$. Let $C' = \langle a_1, Q_1, a_{21}, a_2, Q_2, a_{17}, a_{24}, a_{23}, a_{22}, Q_3, a_6, a_{25}, a_{26}, a_1\rangle$, where Q_1 is a path between a_1 and a_{21}, Q_2 is a path between a_2 and a_{17}, Q_3 is a path between a_{22} and a_6, and $Q_i \cap Q_j = \varphi$ for each $i \neq j$ and $\{i,j\} \subseteq \{1, 2, 3\}$. We construct $C = \langle f(a_1), f(Q_1), f(a_{21}), b_{28}, b_{29}, f(a_2), f(Q_2), f(a_{17}), f(a_{24}), f(a_{23}), f(a_{22}), f(Q_3), f(a_6), b_{33}, b_{32}, f(a_{25}), f(a_{26}), b_{27}, b_{34}, f(a_1)\rangle = \langle b_1, f(Q_1), b_{21}, b_{28}, b_{29}, b_2, f(Q_2), b_{17}, b_{24}, b_{23}, b_{22}, f(Q_3), b_6, b_{33}, b_{32}, b_{25}, b_{26}, b_{27}, b_{34}, b_1\rangle$.

Case 1.5.8 $\langle a_{17}, a_{24}, a_{25}, a_{26}, a_{19}\rangle \in C'$, $\langle a_{22}, a_{23}, a_4\rangle \in C'$. It means that $(a_{21}, a_2) \in C'$, $\langle a_{17}, a_{24}, a_{25}, a_{26}, a_{19}\rangle \in C'$ and $\langle a_{22}, a_{23}, a_4\rangle \in C'$. Let $C' = \langle a_1, Q_1, a_{21}, a_2, Q_2, a_{17}, a_{24}, a_{25}, a_{26}, a_{19}, Q_3, a_{22}, a_{23}, a_4, Q_4, a_1\rangle$, where Q_1 is a path between a_1 and a_{21}, Q_2 is a path between a_2 and a_{17}, Q_3 is a path between a_{19} and a_{22}, Q_4 is a path between a_4 and a_1 and $Q_i \cap Q_j = \varphi$ for each $i \neq j$ and $\{i, j\} \subseteq \{1, 2, 3, 4\}$. We construct $C = \langle f(a_1), f(Q_1), f(a_{21}), b_{28}, b_{29}, f(a_2), f(Q_2), f(a_{17}), f(a_{24}), f(a_{25}), f(a_{26}), f(a_{19}), f(Q_3), f(a_{22}), f(a_{23}), b_{30}, b_{31}, f(a_4), f(Q_4), f(a_1)\rangle = \langle b_1, f(Q_1), b_{21}, b_{28}, b_{29}, b_2, f(Q_2), b_{17}, b_{24}, b_{25}, b_{26}, b_{19}, f(Q_3), b_{22}, b_{23}, b_{30}, b_{31}, b_4, f(Q_4), b_1\rangle$.

Case 1.5.9 $\langle a_6, a_{25}, a_{24}, a_{23}, a_4\rangle \in C'$, $\langle a_{19}, a_{26}, a_1\rangle \in C'$. It means that $(a_{21}, a_2) \in C'$, $\langle a_6, a_{25}, a_{24}, a_{23}, a_4\rangle \in C'$ and $\langle a_{19}, a_{26}, a_1\rangle \in C'$. Let $C' = \langle a_1, Q_1, a_{21}, a_2, Q_2, a_6, a_{25}, a_{24}, a_{23}, a_4, Q_3, a_{19}, a_{26}, a_1\rangle$, where Q_1 is a path between a_1 and a_{21}, Q_2 is a path between a_2 and a_6, Q_3 is a path between a_4 and a_{19}, and $Q_i \cap Q_j = \varphi$ for each $i \neq j$ and $\{i,j\} \subseteq \{1, 2, 3\}$. We construct $C = \langle f(a_1), f(Q_1), f(a_{21}), b_{28}, b_{29}, f(a_2), f(Q_2), f(a_6), b_{33}, b_{32}, f(a_{25}), f(a_{24}), f(a_{23}), b_{30}, b_{31}, f(a_4), f(Q_3), f(a_{19}), f(a_{26}), b_{27}, b_{34}, f(a_1)\rangle = \langle b_1, f(Q_1), b_{21}, b_{28}, b_{29}, b_2, f(Q_2), b_6, b_{33}, b_{32}, b_{25}, b_{24}, b_{23}, b_{30}, b_{31}, b_4, f(Q_3), b_{19}, b_{26}, b_{27}, b_{34}, b_1\rangle$.

Case 1.5.10 $\langle a_{22}, a_{23}, a_4\rangle \in C'$, $\langle a_{17}, a_{24}, a_{25}, a_{26}, a_1\rangle \in C'$ It means that $(a_{21}, a_2) \in C'$, $\langle a_{22}, a_{23}, a_4\rangle \in C'$ and $\langle a_{17}, a_{24}, a_{25}, a_{26}, a_1\rangle \in C'$. Let $C' = \langle a_1, Q_1, a_{21}, a_2, Q_2, a_{22}, a_{23}, a_4, Q_3, a_{17}, a_{24}, a_{25}, a_{26}, a_1\rangle$, where Q_1 is a path between a_1 and a_{21}, Q_2 is a path between a_2 and a_{22}, Q_3 is a path between a_4 and a_{17}, and $Q_i \cap Q_j = \varphi$ for each $i \neq j$ and $\{i,j\} \subseteq \{1, 2, 3\}$. We construct $C = \langle f(a_1), f(Q_1), f(a_{21}), b_{28}, b_{29}, f(a_2), f(Q_2), f(a_{22}), f(a_{23}), b_{30}, b_{31}, f(a_4), f(Q_3), f(a_{17}), f(a_{24}), f(a_{25}), f(a_{26}), b_{27}, b_{34}, f(a_1)\rangle = \langle b_1, f(Q_1), b_{21}, b_{28}, b_{29}, b_2, f(Q_2), b_{22}, b_{23}, b_{30}, b_{31}, b_4, f(Q_3), b_{17}, b_{24}, b_{25}, b_{26}, b_{27}, b_{34}, b_1\rangle$.

Case 1.5.11 $\langle a_{22}, a_{23}, a_{24}, a_{25}, a_6\rangle \in C'$, $\langle a_{19}, a_{26}, a_1\rangle \in C'$ It means that $(a_{21}, a_2) \in C'$, $\langle a_{22}, a_{23}, a_{24}, a_{25}, a_6\rangle \in C'$ and $\langle a_{19}, a_{26}, a_1\rangle \in C'$. Let $C' = \langle a_1, Q_1, a_{21}, a_2, Q_2, a_{22}, a_{23}, a_{24}, a_{25}, a_6, Q_3, a_{19}, a_{26}, a_1\rangle$, where Q_1 is a path between a_1 and a_{21}, Q_2 is a path between a_2 and a_{22}, Q_3 is a path between a_6 and a_{19}, and $Q_i \cap Q_j = \varphi$ for each $i \neq j$ and $\{i, j\} \subseteq \{1, 2, 3\}$. We construct $C = \langle f(a_1), f(Q_1), f(a_{21}), b_{28}, b_{29}, f(a_2), f(Q_2), f(a_{22}), f(a_{23}), f(a_{24}), f(a_{25}), b_{32}, b_{33}, f(a_6), f(Q_3), f(a_{19}), f(a_{26}), b_{27}, b_{34}, f(a_1)\rangle = \langle b_1, f(Q_1), b_{21}, b_{28}, b_{29}, b_2, f(Q_2), b_{22}, b_{23}, b_{24}, b_{25}, b_{32}, b_{33}, b_6, f(Q_3), b_{19}, b_{26}, b_{27}, b_{34}, b_1\rangle$.

Case 1.5.12 $\langle a_{17}, a_{24}, a_{23}, a_4\rangle \in C'$, $\langle a_{19}, a_{26}, a_{25}, a_6\rangle \in C'$. It means that $(a_{21}, a_2) \in C'$, $\langle a_{17}, a_{24}, a_{23}, a_4\rangle \in C'$ and $\langle a_{19}, a_{26}, a_{25}, a_6\rangle \in C'$. Let $C' = \langle a_1, Q_1, a_{21}, a_2, Q_2, a_{17}, a_{24}, a_{23}, a_4, Q_3, a_{19}, a_{26}, a_{25}, a_6, Q_4, a_1\rangle$, where Q_1 is a path between a_1 and a_{21}, Q_2 is a path between a_2 and a_{17}, Q_3 is a path between a_4 and a_{19}, Q_4 is a path between a_6 and a_1,and $Q_i \cap Q_j = \varphi$ for each $i \neq j$ and $\{i,j\} \subseteq \{1, 2, 3, 4\}$. We construct $C = \langle f(a_1), f(Q_1), f(a_{21}), b_{28}, b_{29}, f(a_2), f(Q_2), f(a_{17}), f(a_{24}), f(a_{23}), b_{30}, b_{31}, f(a_4), f(Q_3), f(a_{19}), f(a_{26}), f(a_{25}), b_{32}, b_{33}, f(a_6), f(Q_4), f(a_1)\rangle = \langle b_1, f(Q_1), b_{21}, b_{28}, b_{29}, b_2, f(Q_2), b_{17}, b_{24}, b_{23}, b_{30}, b_{31}, b_4, f(Q_3), b_{19}, b_{26}, b_{25}, b_{32}, b_{33}, b_6, f(Q_3), b_1\rangle$.

Case 1.5.13 $\langle a_{17}, a_{24}, a_{25}, a_6\rangle \in C'$, $\langle a_{22}, a_{23}, a_4\rangle \in C'$, $\langle a_{19}, a_{26}, a_1\rangle \in C'$.

It means that $(a_{21}, a_2) \in C'$, $\langle a_{17}, a_{24}, a_{25}, a_6\rangle \in C'$, $\langle a_{22}, a_{23}, a_4\rangle \in C'$ and $\langle a_{19}, a_{26}, a_1\rangle \in C'$. Let $C' = \langle a_1, Q_1, a_{21}, a_2, Q_2, a_{17}, a_{24}, a_{25}, a_6, Q_3, a_{22}, a_{23}, a_4, Q_4, a_{19}, a_{26}, a_1\rangle$, where Q_1 is a path between a_1 and a_{21}, Q_2 is a path between a_2 and a_{17}, Q_3 is a path between a_6 and a_{22}, Q_4 is a path between a_4 and a_{19},and $Q_i \cap Q_j = \varphi$ for each ij=jand $\{i,j\} \subseteq \{1, 2, 3, 4\}$. We construct $C = \langle f(a_1), f(Q_1), f(a_{21}), b_{28}, b_{29}, f(a_2), f(Q_2), f(a_{17}), f(a_{24}), f(a_{25}), b_{32}, b_{33}, f(a_6), f(Q_3), f(a_{22}), f(a_{23}), b_{30}, b_{31}, f(a_4), f(Q_4), f(a_{19}), f(a_{26}), b_{27}, b_{34}, f(a_1)\rangle = \langle b_1, f(Q_1), b_{21}, b_{28}, b_{29}, b_2, f(Q_2), b_{17}, b_{24}, b_{25}, b_{32}, b_{33}, b_6, f(Q_3), b_{22}, b_{23}, b_{30}, b_{31}, b_4, f(Q_4), b_{19}, b_{26}, b_{27}, b_{34}, b_1\rangle$.

Case 2 $(a_{21}, a_2) \notin C'$.

Case 2.1 $|\{a_{23}, a_{24}, a_{25}, a_{26}\} \cap C'| = 0$.

It means that $C' = \langle a_1, Q_1, a_1\rangle$, but the edges (a_{22}, a_{23}), (a_{23}, a_{24}), (a_{24}, a_{25}), (a_{25}, a_{26}), (a_{26}, a_1), (a_{17}, a_{24}), (a_{19}, a_{26}), (a_{21}, a_2), (a_{23}, a_4), $(a_{25}, a_6) \notin C'$. If the cycle C' that departs from a_1 and pass through a_2 has to pass a_8 for returning to a_1. Hence $C' = \langle a_1, a_2, Q_1, a_8, a_1\rangle$ We construct $C = \langle f(a_1), f(a_2), f(Q_1), f(a_8), f(a_1)\rangle = \langle b_1, b_2, f(Q_1), b_8, b_1\rangle$. On the other hand, if the cycle

C' that departs from a_1 and pass through a_8 has to pass a_2 for returning to a_1. Hence $C' = \langle a_1, a_8, Q_1, a_2, a_1\rangle$. We construct $C = \langle f(a_1), f(a_8), f(Q_1), f(a_2), f(a_1)\rangle = \langle b_1, b_8, f(Q_1), b_2, b_1\rangle$.

Case 2.2 $|\{a_{23}, a_{24}, a_{25}, a_{26}\} \cap C'| = 1$.

Case 2.2.1 $\langle a_{19}, a_{26}, a_1\rangle \in C'$.

It means that $\langle a_{19}, a_{26}, a_1\rangle \in C'$. Let $C' = \langle a_1, Q_1, a_{19}, a_{26}, a_1\rangle$, where Q_1 is a path between a_1 and a_{21}. We construct $C = \langle f(a_1), f(Q_1), f(a_{19}), f(a_{26}), b_{27}, b_{34}, f(a_1)\rangle = \langle b_1, f(Q_1), b_{19}, b_{26}, b_{27}, b_{34}, b_1\rangle$.

Case 2.2.2 $\langle a_{22}, a_{23}, a_4\rangle \in C'$.

It means that $\langle a_{22}, a_{23}, a_4\rangle \in C'$. Let $C' = \langle a_1, Q_1, a_{22}, a_{23}, a_4, Q_2, a_1\rangle$, where Q_1 is a path between a_1 and a_{22}, Q_2 is a path between a_4 and a_1, and $Q_1 \cap Q_2 = \varphi$. We construct $C = \langle f(a_1), f(Q_1), f(Q_2), f(a_{22}), f(a_{23}), b_{30}, b_{31}, f(a_4), f(Q_2), f(a_1)\rangle = \langle b_1, f(Q_1), b_{22}, b_{23}, b_{30}, b_{31}, b_4, f(Q_2), b_1\rangle$. See Figure 3.7 for an illustration.

Case 2.3 $|\{a_{23}, a_{24}, a_{25}, a_{26}\} \cap C'| = 2$.

Case 2.3.1 $\langle a_{17}, a_{24}, a_{23}, a_{22}\rangle \in C'$.

It means that $\langle a_{17}, a_{24}, a_{23}, a_{22}\rangle \in C'$. Let $C' = \langle a_1, Q_1, a_{17}, a_{24}, a_{23}, a_{22}, Q_2, a_1\rangle$, where Q_1 is a path between a_1 and a_{17}, Q_2 is a path between a_{22} and a_1, and $Q_1 \cap Q_2 = \varphi$. We construct $C = \langle f(a_1), f(Q_1), f(a_{17}), f(a_{24}), f(a_{23}), f(a_{22}), f(Q_2), f(a_1)\rangle = \langle b_1, f(Q_1), b_{17}, b_{24}, b_{23}, b_{22}, f(Q_2), b_1\rangle$.

Case 2.3.2 $\langle a_6, a_{25}, a_{26}, a_1\rangle \in C'$.

It means that $\langle a_6, a_{25}, a_{26}, a_1\rangle \in C'$. Let $C' = \langle a_1, Q_1, a_6, a_{25}, a_{26}, a_1\rangle$, where Q_1 is a path between a_1 and a_6. We construct $C = \langle f(a_1), f(Q_1), f(a_6), b_{33}, b_{32}, f(a_{25}), f(a_{26}), b_{27}, b_{34}, f(a_1)\rangle = \langle b_1, f(Q_1), b_6, b_{33}, b_{32}, b_{25}, b_{26}, b_{27}, b_{34}, b_1\rangle$.

Case 2.3.3 $\langle a_{17}, a_{24}, a_{25}, a_6\rangle \in C'$.

It means that $\langle a_{17}, a_{24}, a_{25}, a_6\rangle \in C'$. Let $C' = \langle a_1, Q_1, a_{17}, a_{24}, a_{25}, a_6, Q_2, a_1\rangle$, where Q_1 is a path between a_1 and a_{17}, Q_2 is a path between a_6 and a_1, and $Q_1 \cap Q_2 = \varphi$. We construct $C = \langle f(a_1), f(Q_1), f(a_{17}), f(a_{24}), f(a_{25}), b_{32}, b_{33}, f(a_6), f(Q_2), f(a_1)\rangle = \langle b_1, f(Q_1), b_{17}, b_{24}, b_{25}, b_{32}, b_{33}, b_6, f(Q_2), b_1\rangle$.

Case 2.3.4 $\langle a_{17}, a_{24}, a_{23}, a_4\rangle \in C'$.

It means that $\langle a_{17}, a_{24}, a_{23}, a_4\rangle \in C'$. Let $C' = \langle a_1, Q_1, a_{17}, a_{24}, a_{23}, a_4, Q_2, a_1\rangle$, where Q_1 is a path between a_1 and a_{17}, Q_2 is a path between a_4 and a_1, and $Q_1 \cap Q_2 = \varphi$. We construct $C = \langle f(a_1), f(Q_1), f(a_{17}), f(a_{24}), f(a_{23}), b_{30}, b_{31}, f(a_4), f(Q_2), f(a_1)\rangle = \langle b_1, f(Q_1), b_{17}, b_{24}, b_{23}, b_{30}, b_{31}, b_4, f(Q_2), b_1\rangle$.

Case 2.3.5 $\langle a_{19}, a_{26}, a_{25}, a_6\rangle \in C'$.

It means that $\langle a_{19}, a_{26}, a_{25}, a_6\rangle \in C'$. Let $C' = \langle a_1, Q_1, a_{19}, a_{26}, a_{25}, a_6, Q_2, a_1\rangle$, where Q_1 is a path between a_1 and a_{19}, Q_2 is a path between a_6 and a_1, and $Q_1 \cap Q_2 = \varphi$. We construct $C = \langle f(a_1), f(Q_1), f(a_{19}), f(a_{26}), f(a_{25}), b_{32}, b_{33}, f(a_6), f(Q_2), f(a_1)\rangle = \langle b_1, f(Q_1), b_{19}, b_{26}, b_{25}, b_{32}, b_{33}, b_6, f(Q_2), b_1\rangle$.

Case 2.3.6 $\langle a_{22}, a_{23}, a_4\rangle \in C'$, $\langle a_{19}, a_{26}, a_1\rangle \in C'$.

It means that $\langle a_{22}, a_{23}, a_4\rangle \in C'$ and $\langle a_{19}, a_{26}, a_1\rangle \in C'$. Let $C' = \langle a_1, Q_1, a_{22}, a_{23}, a_4, Q_2, a_{19}, a_{26}, a_1\rangle$, where Q_1 is a path between a_1 and a_{22}, Q_2 is a path between a_4 and a_{19}, and $Q_1 \cap Q_2 = \varphi$. We construct $C = \langle f(a_1), f(Q_1), f(a_{22}), f(a_{23}), b_{30}, b_{31}, f(a_4), f(Q_2), f(a_{19}), f(a_{26}), b_{27}, b_{34}, f(a_1)\rangle = \langle b_1, f(Q_1), b_{22}, b_{23}, b_{30}, b_{31}, b_4, f(Q_2), b_{19}, b_{26}, b_{27}, b_{34}, b_1\rangle$.

Case 2.4 $|\{a_{23}, a_{24}, a_{25}, a_{26}\} \cap C'| = 3$.

Case 2.4.1 $\langle a_{17}, a_{24}, a_{25}, a_{26}, a_1\rangle \in C'$.

It means that $\langle a_{17}, a_{24}, a_{25}, a_{26}, a_1\rangle \in C'$. Let $C' = \langle a_1, Q_1, a_{17}, a_{24}, a_{25}, a_{26}, a_1\rangle$, where Q_1 is a path between a_1 and a_{17}. We construct $C = \langle f(a_1), f(Q_1), f(a_{17}), f(a_{24}), f(a_{25}), f(a_{26}), b_{27}, b_{34}, f(a_1)\rangle = \langle b_1, f(Q_1), b_{17}, b_{24}, b_{25}, b_{26}, b_{27}, b_{34}, b_1\rangle$.

Case 2.4.2 $\langle a_{22}, a_{23}, a_{24}, a_{25}, a_6\rangle \in C'$.

It means that $\langle a_{22}, a_{23}, a_{24}, a_{25}, a_6\rangle \in C'$. Let $C' = \langle a_1, Q_1, a_{22}, a_{23}, a_{24}, a_{25}, a_6, Q_2, a_1\rangle$, where Q_1 is a path between a_1 and a_{22}, Q_2 is a path between a_6 and a_1, and $Q_i \cap Q_2 = \varphi$. We construct $C = \langle f(a_1), f(Q_1), f(a_{22}), f(a_{23}), f(a_{24}), f(a_{25}), b_{32}, b_{33}, f(a_6), f(Q_2), f(a_1)\rangle = \langle b_1, f(Q_1), b_{22}, b_{23}, b_{24}, b_{25}, b_{32}, b_{33}, b_6, f(Q_2), b_1\rangle$.

Case 2.4.3 $\langle a_{17}, a_{24}, a_{25}, a_{26}, a_{19}\rangle \in C'$.

It means that $\langle a_{17}, a_{24}, a_{25}, a_{26}, a_{19}\rangle \in C'$. Let $C' = \langle a_1, Q_1, a_{17}, a_{24}, a_{25}, a_{26}, a_{19}, Q_2, a_1\rangle$, where Q_1 is a path between a_1 and a_{17}, Q_2 is a path between a_{19} and a_1, and $Q_1 \cap Q_2 = \varphi$. We construct $C = \langle f(a_1), f(Q_1), f(a_{17}), f(a_{24}), f(a_{25}), f(a_{26}), f(a_{19}), f(Q_2), f(a_1)\rangle = \langle b_1, f(Q_1), b_{17}, b_{24}, b_{25}, b_{26}, b_{19}, f(Q_2), b_1\rangle$.

Case 2.4.4 $\langle a_6, a_{25}, a_{24}, a_{23}, a_4\rangle \in C'$.

It means that $\langle a_6, a_{25}, a_{24}, a_{23}, a_4\rangle \in C'$. Let $C' = \langle a_1, Q_1, a_6, a_{25}, a_{24}, a_{23}, a_4, Q_2, a_1\rangle$, where Q_1 is a path between a_1 and a_6, Q_2 is a path between a_4 and a_1, and $Q_1 \cap Q_2 = \varphi$. We construct $C = \langle f(a_1), f(Q_1), f(a_6), b_{33}, b_{32}, f(a_{25}), f(a_{24}), f(a_{23}), b_{30}, b_{31}, f(a_4),$

$f(Q_2), f(a_1)\rangle = \langle b_1, f(Q_1), b_6, b_{33}, b_{32}, b_{25}, b_{24}, b_{23}, b_{30}, b_{31}, b_4, f(Q_2), b_1\rangle$.

Case 2.4.5 $\langle a_{17}, a_{24}, a_{23}, a_{22}\rangle \in C'$, $\langle a_{19}, a_{26}, a_1\rangle \in C'$.
It means that $\langle a_{17}, a_{24}, a_{23}, a_{22}\rangle \in C'$ and $\langle a_{19}, a_{26}, a_1\rangle \in C'$. Let $C' = \langle a_1, Q_1, a_{17}, a_{24}, a_{23}, a_{22}, Q_2, a_{19}, a_{26}, a_1\rangle$, where Q_1 is a path between a_1 and a_{17}, Q_2 is a path between a_{22} and a_{19}, and $Q_1 \cap Q_2 = \varphi$. We construct $C = \langle f(a_1), f(Q_1), f(a_{17}), f(a_{24}), f(a_{23}), f(a_{22}), f(Q_2), f(a_{19}), f(a_{26}), b_{27}, b_{34}, f(a_1)\rangle = \langle b_1, f(Q_1), b_{17}, b_{24}, b_{23}, b_{22}, f(Q_2), b_{19}, b_{26}, b_{27}, b_{34}, b_1\rangle$.

Case 2.4.6 $\langle a_{22}, a_{23}, a_4\rangle \in C'$, $\langle a_6, a_{25}, a_{26}, a_1\rangle \in C'$.
It means that $\langle a_{22}, a_{23}, a_4\rangle \in C'$ and $\langle a_6, a_{25}, a_{26}, a_1\rangle \in C'$. Let $C' = \langle a_1, Q_1, a_{22}, a_{23}, a_4, Q_2, a_6, a_{25}, a_{26}, a_1\rangle$, where Q_1 is a path between a_1 and a_{22}, Q_2 is a path between a_4 and a_6, and $Q_1 \cap Q_2 = \varphi$. We construct $C = \langle f(a_1), f(Q_1), f(a_{22}), f(a_{23}), b_{30}, b_{31}, f(a_4), f(Q_2), f(a_6), b_{33}, b_{32}, f(a_{25}), f(a_{26}), b_{27}, b_{34}, f(a_1)\rangle = \langle b_1, f(Q_1), b_{22}, b_{23}, b_{30}, b_{31}, b_4, f(Q_2), b_6, b_{33}, b_{32}, b_{25}, b_{26}, b_{27}, b_{34}, b_1\rangle$.

Case 2.4.7 $\langle a_{17}, a_{24}, a_{23}, a_4\rangle \in C'$, $\langle a_{19}, a_{26}, a_1\rangle \in C'$.
It means that $\langle a_{17}, a_{24}, a_{23}, a_4\rangle \in C'$ and $\langle a_{19}, a_{26}, a_1\rangle \in C'$. Let $C' = \langle a_1, Q_1, a_{17}, a_{24}, a_{23}, a_4, Q_2, a_{19}, a_{26}, a_1\rangle$, where Q_1 is a path between a_1 and a_{17}, Q_2 is a path between a_4 and a_{19}, and $Q_1 \cap Q_2 = \varphi$. We construct $C = \langle f(a_1), f(Q_1), f(a_{17}), f(a_{24}), f(a_{23}), b_{30}, b_{31}, f(a_4), f(Q_2), f(a_{19}), f(a_{26}), b_{27}, b_{34}, f(a_1)\rangle = \langle b_1, f(Q_1), b_{17}, b_{24}, b_{23}, b_{30}, b_{31}, b_4, f(Q_2), b_{19}, b_{26}, b_{27}, b_{34}, b_1\rangle$.

Case 2.4.8 $\langle a_{19}, a_{26}, a_{25}, a_6\rangle \in C'$, $\langle a_{22}, a_{23}, a_4\rangle \in C'$.
It means that $\langle a_{19}, a_{26}, a_{25}, a_6\rangle \in C'$ and $\langle a_{22}, a_{23}, a_4\rangle \in C'$. Let $C' = \langle a_1, Q_1, a_{19}, a_{26}, a_{25}, a_6, Q_2, a_{22}, a_{23}, a_4, Q_3, a_1\rangle$, where Q_1 is a path between a_1 and a_{19}, Q_2 is a path between a_6 and a_{22}, Q_3 is a path between a_4 and a_1, and $Q_i \cap Q_j = \varphi$ for each $i \neq j$ and $\{i, j\} \subseteq \{1, 2, 3\}$. We construct $C = \langle f(a_1), f(Q_1), f(a_{19}), f(a_{26}), f(a_{25}), b_{32}, b_{33}, f(a_6), f(Q_2), f(a_{22}), f(a_{23}), b_{30}, b_{31}, f(a_4), f(Q_3), f(a_1)\rangle = \langle b_1, f(Q_1), b_{19}, b_{26}, b_{25}, b_{32}, b_{33}, b_6, f(Q_2), b_{22}, b_{23}, b_{30}, b_{31}, b_3, f(Q_4), b_1\rangle$.

Case 2.4.9 $\langle a_{17}, a_{24}, a_{25}, a_6\rangle \in C'$, $\langle a_{19}, a_{26}, a_1\rangle \in C'$.
It means that $\langle a_{17}, a_{24}, a_{25}, a_6\rangle \in C'$ and $\langle a_{19}, a_{26}, a_1\rangle \in C'$. Let $C' = \langle a_1, Q_1, a_{17}, a_{24}, a_{25}, a_6, Q_2, a_{19}, a_{26}, a_1\rangle$, where Q_1 is a path between a_1 and a_{17}, Q_2 is a path between a_6 and a_{19} , and $Q_1 \cap Q_2 = \varphi$. We construct $C = \langle f(a_1), f(Q_1), f(a_{17}), f(a_{24}), f(a_{25}), b_{32}, b_{33}, f(a_6), f(Q_2), f(a_{19}), f(a_{26}), b_{27}, b_{34}, f(a_1)\rangle = \langle b_1, f(Q_1), b_{17}, b_{24}, b_{25}, b_{32}, b_{33}, b_6, f(Q_2), b_{19}, b_{26}, b_{27}, b_{34}, b_1\rangle$.

Case 2.4.10 $\langle a_{17}, a_{24}, a_{25}, a_6\rangle \in C'$, $\langle a_{22}, a_{23}, a_4\rangle \in C'$.
It means that $\langle a_{17}, a_{24}, a_{25}, a_6\rangle \in C'$ and $\langle a_{22}, a_{23}, a_4\rangle \in C'$. Let $C' = \langle a_1, Q_1, a_{17}, a_{24}, a_{25}, a_6, Q_2, a_{22}, a_{23}, a_4, Q_3, a_1\rangle$, where Q_1 is a path between a_1 and a_{17}, Q_2 is a path between a_6 and a_{22}, Q_3 is a path between a_4 and a_1, Q_4 is a path between a_4 and a_1, and $Q_i \cap Q_j = \varphi$ for each $i \neq j$ and $\{i,j\} \subseteq \{1,2,3\}$. We construct $C = \langle f(a_1), f(Q_1), f(a_{17}), f(a_{24}), f(a_{25}), b_{32}, b_{33}, f(a_6), f(Q_2), f(a_{22}), f(a_{23}), b_{30}, b_{31}, f(a_4), f(Q_3), f(a_1)\rangle = \langle b_1, f(Q_1), b_{17}, b_{24}, b_{25}, b_{32}, b_{33}, b_6, f(Q_2), b_{22}, b_{23}, b_{30}, b_{31}, b_4, f(Q_3), b_1\rangle$.

Case 2.5 $|\{a_{23}, a_{24}, a_{25}, a_{26}\} \cap C'| = 4$.

Case 2.5.1 $\langle a_{19}, a_{26}, a_{25}, a_{24}, a_{23}, a_{22}\rangle \in C'$.
It means that $\langle a_{19}, a_{26}, a_{25}, a_{24}, a_{23}, a_{22}\rangle \in C'$. Let $C' = \langle a_1, Q_1, a_{19}, a_{26}, a_{25}, a_{24}, a_{23}, a_{22}, Q_2, a_1\rangle$, where Q_1 is a path between a_1 and a_{19}, Q_2 is a path between a_{22} and a_1, and $Q_1 \cap Q_2 = \varphi$. We construct $C = \langle f(a_1), f(Q_1), f(a_{19}), f(a_{26}), f(a_{25}), f(a_{24}), f(a_{23}), f(a_{22}), f(Q_2), f(a_1)\rangle = \langle b_1, f(Q_1), b_{19}, b_{26}, b_{25}, b_{24}, b_{23}, b_{22}, f(Q_2), b_1\rangle$.

Case 2.5.2 $\langle a_4, a_{23}, a_{24}, a_{25}, a_{26}, a_1\rangle \in C'$.
It means that $\langle a_4, a_{23}, a_{24}, a_{25}, a_{26}, a_1\rangle \in C'$. Let $C' = \langle a_1, Q_1, a_4, a_{23}, a_{24}, a_{25}, a_{26}, a_1\rangle$, where Q_1 is a path between a_1 and a_4. We construct $C = \langle f(a_1), f(Q_1), f(a_4), b_{31}, b_{30}, f(a_{23}), f(a_{24}), f(a_{25}), f(a_{26}), b_{27}, b_{34}, f(a_1)\rangle = \langle b_1, f(Q_1), b_4, b_{31}, b_{30}, b_{23}, b_{24}, b_{25}, b_{26}, b_{27}, b_{34}, b_1\rangle$.

Case 2.5.3 $\langle a_{19}, a_{26}, a_{25}, a_{24}, a_{23}, a_{22}\rangle \in C'$.
It means that $\langle a_{19}, a_{26}, a_{25}, a_{24}, a_{23}, a_4\rangle \in C'$. Let $C' = \langle a_1, Q_1, a_{19}, a_{26}, a_{25}, a_{24}, a_{23}, a_4, Q_2, a_1\rangle$, where Q_1 is a path between a_1 and a_{19}, Q_2 is a path between a_4 and a_1, and $Q_1 \cap Q_2 = \varphi$. We construct $C = \langle f(a_1), f(Q_1), f(a_{19}), f(a_{26}), f(a_{25}), f(a_{24}), f(a_{23}), f(a_4), f(Q_2), f(a_1)\rangle = \langle b_1, f(Q_1), b_{19}, b_{26}, b_{25}, b_{24}, b_{23}, b_4, f(Q_2), b_1\rangle$.

Case 2.5.4 $\langle a_{22}, a_{23}, a_{24}, a_{25}, a_{26}, a_1\rangle \in C'$.
It means that $\langle a_{22}, a_{23}, a_{24}, a_{25}, a_{26}, a_1\rangle \in C'$. Let $C' = \langle a_1, Q_1, a_{22}, a_{23}, a_{24}, a_{25}, a_{26}, a_1\rangle$, where Q_1 is a path between a_1 and a_{22}. We construct $C = \langle f(a_1), f(Q_1), f(a_{22}), f(a_{23}), f(a_{24}), f(a_{25}), f(a_{26}), b_{27}, b_{34}, f(a_1)\rangle = \langle b_1, f(Q_1), b_{22}, b_{23}, b_{24}, b_{25}, b_{26}, b_{27}, b_{34}, b_1\rangle$.

Case 2.5.5 $\langle a_{17}, a_{24}, a_{23}, a_{22}\rangle \in C'$, $\langle a_{19}, a_{26}, a_{25}, a_6\rangle \in C'$.
It means that $\langle a_{17}, a_{24}, a_{23}, a_{22}\rangle \in C'$ and $\langle a_{19}, a_{26}, a_{25}, a_6\rangle \in C'$. Let $C' = \langle a_1, Q_1, a_{17}, a_{24}, a_{23}, a_{22}, Q_2, a_{19}, a_{26}, a_{25}, a_6, Q_3, a_1\rangle$, where Q_1 is a path between a_1 and a_{17}, Q_2 is a path between a_{22} and a_{19}, Q_3 is a path between a_6 and a_1, and $Q_i \cap Q_j = \varphi$ for each $i \neq j$ and $\{i,j\} \subseteq \{1,2,3\}$. We construct $C = \langle f(a_1), f(Q_1), f(a_{17}), f(a_{24}),$

$f(a_{23}), f(a_{22}), f(Q_3), f(a_{19}), f(a_{26}), f(a_{25}), b_{32}, b_{33}, f(a_6), f(Q_4), f(a_1)\rangle = \langle b_1, f(Q_1), b_{17}, b_{24}, b_{23}, b_{22}, f(Q_3), b_{19}, b_{26}, b_{25}, b_{32}, b_{33}, b_6, f(Q_4), b_1\rangle$.

Case 2.5.6 $\langle a_{17}, a_{24}, a_{23}, a_4\rangle \in C'$, $\langle a_6, a_{25}, a_{26}, a_1\rangle \in C'$. It means that $\langle a_{19}, a_{26}, a_{25}, a_{24}, a_{23}, a_4\rangle \in C'$. Let $C' = \langle a_1, Q_1, a_{19}, a_{26}, a_{25}, a_{24}, a_{23}, a_4, Q_2, a_6, a_{25}, a_{26}, a_1\rangle$, where Q_1 is a path between a_1 and a_{19} , Q_2 is a path between a_4 and a_6 , and $Q_1 \cap Q_2 = \varphi$. Weconstruct $C = \langle f(a_1), f(Q_1), f(a_{19}), f(a_{26}), f(a_{25}), f(a_{24}), f(a_{23}), b_{30}, b_{31}, f(a_4), f(Q_2), f(a_6), b_{33}, b_{32}, f(a_{25}), f(a_{26}), b_{27}, b_{34}, f(a_1)\rangle = \langle b_1, f(Q_1), b_{19}, b_{26}, b_{25}, b_{24}, b_{23}, b_{30}, b_{31}, b_4, f(Q_2), b_6, b_{33}, b_{32}, b_{25}, b_{26}, b_{27}, b_{34}, b_1\rangle$.

Case 2.5.7 $\langle a_{17}, a_{24}, a_{23}, a_{22}\rangle \in C'$, $\langle a_6, a_{25}, a_{26}, a_1\rangle \in C'$. It means that $\langle a_{17}, a_{24}, a_{23}, a_{22}\rangle \in C'$ and $\langle a_6, a_{25}, a_{26}, a_1\rangle \in C'$. Let $C' = \langle a_1, Q_1, a_{17}, a_{24}, a_{23}, a_{22}, Q_2, a_6, a_{25}, a_{26}, a_1\rangle$, where Q_1 is a path between a_1 and a_{17}, Q_2 is a path between a_{22} and a_6, and $Q_1 \cap Q_2 = \varphi$. We construct $C = \langle f(a_1), f(Q_1), f(a_{17}), f(a_{24}), f(a_{23}), f(a_{22}), f(Q_2), f(a_6), b_{33}, b_{32}, f(a_{25}), f(a_{26}), b_{27}, b_{34}, f(a_1)\rangle = \langle b_1, f(Q_1), b_{17}, b_{24}, b_{23}, b_{22}, f(Q_2), b_6, b_{33}, b_{32}, b_{25}, b_{26}, b_{27}, b_{34}, b_1\rangle$.

Case 2.5.8 $\langle a_{17}, a_{24}, a_{25}, a_{26}, a_{19}\rangle \in C'$, $\langle a_{22}, a_{23}, a_4\rangle \in C'$. It means that $\langle a_{17}, a_{24}, a_{25}, a_{26}, a_{19}\rangle \in C'$ and $\langle a_{22}, a_{23}, a_4\rangle \in C'$. Let $C' = \langle a_1, Q_1, a_{17}, a_{24}, a_{25}, a_{26}, a_{19}, Q_2, a_{22}, a_{23}, a_4, Q_3, a_1\rangle$, where Q_1 is a path between a_1 and a_{17}, Q_2 is a path between a_{19} and a_{22}, Q_3 is a path between a_4 and a_1 and $Q_i \cap Q_j = \varphi$ for each $i \neq j$ and $\{i,j\} \subseteq \{1,2,3\}$. Weconstruct $C = \langle f(a_1), f(Q_1), f(a_{17}), f(a_{24}), f(a_{25}), f(a_{26}), f(a_{19}), f(Q_2), f(a_{22}), f(a_{23}), b_{30}, b_{31}, f(a_4), f(Q_3), f(a_1)\rangle = \langle b_1, f(Q_1), b_{17}, b_{24}, b_{25}, b_{26}, b_{19}, f(Q_2), b_{22}, b_{23}, b_{30}, b_{31}, b_4, f(Q_3), b_1\rangle$.

Case 2.5.9 $\langle a_6, a_{25}, a_{24}, a_{23}, a_4\rangle \in C'$, $\langle a_{19}, a_{26}, a_1\rangle \in C$. It means that $\langle a_6, a_{25}, a_{24}, a_{23}, a_4\rangle \in C'$ and $\langle a_{19}, a_{26}, a_1\rangle \in C'$. Let $C' = \langle a_1, Q_1, a_6, a_{25}, a_{24}, a_{23}, a_4, Q_2, a_{19}, a_{26}, a_1\rangle$, where Q_1 is a path between a_1 and a_6, Q_2 is a path between a_4 and a_{19}, and $Q_1 \cap Q_2 = \varphi$. We construct $C = \langle f(a_1), f(Q_1), f(a_6), b_{33}, b_{32}, f(a_{25}), f(a_{24}), f(a_{23}), b_{30}, b_{31}, f(a_4), f(Q_2), f(a_{19}), f(a_{26}), b_{27}, b_{34}, f(a_1)\rangle = \langle b_1, f(Q_1), b_6, b_{33}, b_{32}, b_{25}, b_{24}, b_{23}, b_{30}, b_{31}, b_4, f(Q_2), b_{19}, b_{26}, b_{27}, b_{34}, b_1\rangle$.

Case 2.5.10 $\langle a_{22}, a_{23}, a_4\rangle \in C'$, $\langle a_{17}, a_{24}, a_{25}, a_{26}, a_1\rangle \in C'$. It means that $\langle a_{22}, a_{23}, a_4\rangle \in C'$ and $\langle a_{17}, a_{24}, a_{25}, a_{26}, a_1\rangle \in C'$. Let $C' = \langle a_1, Q_1, a_{22}, a_{23}, a_4, Q_2, a_{17}, a_{24}, a_{25}, a_{26}, a_1\rangle$, where Q_1 is a path between a_1 and a_{22}, Q_2 is a path between a_4 and a_{17}, and $Q_1 \cap Q_2 = \varphi$. We construct $C = \langle f(a_1), f(Q_1), f(a_{22}), f(a_{23}), b_{30}, b_{31}, f(a_4), f(Q_2), f(a_{17}), f(a_{24}), f(a_{25}), f(a_{26}), b_{27}, b_{34}, f(a_1)\rangle = \langle b_1, f(Q_1), b_{22}, b_{23}, b_{30}, b_{31}, b_4, f(Q_2), b_{17}, b_{24}, b_{25}, b_{26}, b_{27}, b_{34}, b_1\rangle$.

Case 2.5.11 $\langle a_{22}, a_{23}, a_{24}, a_{25}, a_6\rangle \in C'$, $\langle a_{19}, a_{26}, a_1\rangle \in C'$. It means that $\langle a_{22}, a_{23}, a_{24}, a_{25}, a_6\rangle \in C'$ and $\langle a_{19}, a_{26}, a_1\rangle \in C'$. Let $C' = \langle a_1, Q_1, a_{22}, a_{23}, a_{24}, a_{25}, a_6, Q_2, a_{19}, a_{26}, a_1\rangle$, where Q_1 is a path between a_1 and a_{22}, Q_2 is a path between a_6 and a_{19}, and $Q_1 \cap Q_2 = \varphi$. We construct $C = \langle f(a_1), f(Q_1), f(a_{22}), f(a_{23}), f(a_{24}), f(a_{25}), b_{32}, b_{33}, f(a_6), f(Q_2), f(a_{19}), f(a_{26}), b_{27}, b_{34}, f(a_1)\rangle = \langle b_1, f(Q_1), b_{22}, b_{23}, b_{24}, b_{25}, b_{32}, b_{33}, b_6, f(Q_2), b_{19}, b_{26}, b_{27}, b_{34}, b_1\rangle$.

Case 2.5.12 $\langle a_{17}, a_{24}, a_{23}, a_4\rangle \in C'$, $\langle a_{19}, a_{26}, a_{25}, a_6\rangle \in C'$. It means that $\langle a_{17}, a_{24}, a_{23}, a_4\rangle \in C'$ and $\langle a_{19}, a_{26}, a_{25}, a_6\rangle \in C'$. Let $C' = \langle a_1, Q_1, a_{17}, a_{24}, a_{23}, a_4, Q_2, a_{19}, a_{26}, a_{25}, a_6, Q_3, a_1\rangle$, where Q_1 is a path between a_1 and a_{17}, Q_2 is a path between a_4 and a_{19}, Q_3 is a path between a_6 and a_1, and $Q_i \cap Q_j = \varphi$ for each $i \neq j$ and $\{i,j\} \subseteq \{1, 2, 3\}$. We construct $C = \langle f(a_1), f(Q_1), f(a_{17}), f(a_{24}), f(a_{23}), b_{30}, b_{31}, f(a_4), f(Q_2), f(a_{19}), f(a_{26}), f(a_{25}), b_{32}, b_{33}, f(a_6), f(Q_3), f(a_1)\rangle = \langle b_1, f(Q_1), b_{17}, b_{24}, b_{23}, b_{30}, b_{31}, b_4, f(Q_2), b_{19}, b_{26}, b_{25}, b_{32}, b_{33}, b_6, f(Q_3), b_1\rangle$.

Case 2.5.13 $\langle a_{17}, a_{24}, a_{25}, a_6\rangle \in C'$, $\langle a_{22}, a_{23}, a_4\rangle \in C'$, $\langle a_{19}, a_{26}, a_1\rangle \in C'$.

It means that $\langle a_{17}, a_{24}, a_{25}, a_6\rangle \in C'$, $\langle a_{22}, a_{23}, a_4\rangle \in C'$ and $\langle a_{19}, a_{26}, a_1\rangle \in C'$. Let $C' = \langle a_1, Q_1, a_{17}, a_{24}, a_{25}, a_6, Q_2, a_{22}, a_{23}, a_4, Q_3, a_{19}, a_{26}, a_1\rangle$, where Q_1 is a path between a_1 and a_{17}, Q_2 is a path between a_6 and a_{22}, Q_3 is a path between a_4 and a_{19}, and $Q_i \cap Q_j = \varphi$ foreach $i \neq j$ and $\{i,j\} \subseteq \{1, 2, 3\}$. We construct $C = \langle f(a_1), f(Q_1), f(a_{17}), f(a_{24}), f(a_{25}), b_{32}, b_{33}, f(a_6), f(Q_2), f(a_{22}), f(a_{23}), b_{30}, b_{31}, f(a_4), f(Q_3), f(a_{19}), f(a_{26}), b_{27}, b_{34}, f(a_1)\rangle = \langle b_1, f(Q_1), b_{17}, b_{24}, b_{25}, b_{32}, b_{33}, b_6, f(Q_2), b_{22}, b_{23}, b_{30}, b_{31}, b_4, f(Q_3), b_{19}, b_{26}, b_{27}, b_{34}, b_1\rangle$.

We want to construct the required cycle C of $CR(34+8k; 1, 7)$ by rerouting the cycle C' of $CR(26 + 8k; 1, 7)$ in each of the above cases. Here we omit the lengthy path description in each case since it is tedious and indeed very similar to what we've done in Section II. □

With Lemma 1.2, it is known that $CR(28;1,7)$, $CR(30;1,7)$ and $CR(32;1,7)$ are 4-ordered. It is easy to see that our technique in Theorem 3.1 can be utilized to obtain the following three theorems.

Theorem 3.2. $CR(28+8k;1,7)$ is 4-ordered for $k \geq 0$.

Theorem 3.3. $CR(30+8k;1,7)$ is 4-ordered for $k \geq 0$.

Theorem 3.4. $CR(32+8k;1,7)$ is 4-ordered for $k \geq 0$.

Combining Lemma 1.2 and Theorem 3.1- 3.4, we have the following theorem.

Theorem 3.5. *CR(n;1,7)* is 4-ordered for any even integer *n* with $n \geq 18$.

IV. 4. CONCLUSION

Let $n \geq 6$ be an even integer. In this paper, we show the 4-orderedness of certain chordal rings, which are widely applied in real applications. More precisely, we prove that $CR(n;1,5)$ for $n \geq 14$, and $CR(n;1,7)$ for $n \geq 18$, are 4-ordered. Our derivation combines computer experimental results for small *n*, and mathematical induction for general *n*'s. An natural question to be explored is the 4-ordered hamiltonicity of the chordal rings. In particular, the 4-ordered hamiltonicity for the graphs in $CR(n;1,5)$ and $CR(n;1,7)$. Currently, computer experiments already shows that the 4-ordered hamiltonicity only exists on $CR(n;1,5)$, or $CR(n;1,7)$, and some other chordal rings for specific *n*'s. We have the following conjecture.

Conjecture 4.1 *CR(n,1,5)* is a 4-ordered hamiltonian graph if *n=14*, *n=12k+2* or *n=12k+10* with .

Furthermore, the 4-ordered hamiltonian-connectedness of the chordal ring family remains an open problem.

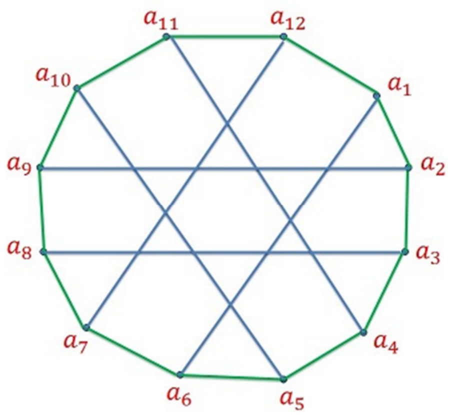

Figure 1: CR(12; 1, 5)

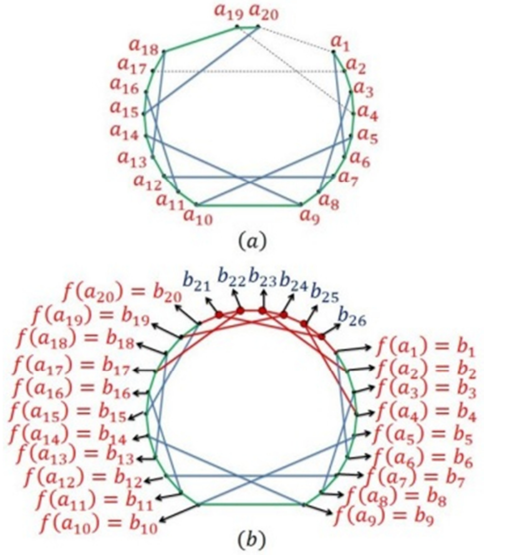

Figure 2.1: (a) CR(20; 1, 5); (b) CR(26; 1, 5) and the function *f*.

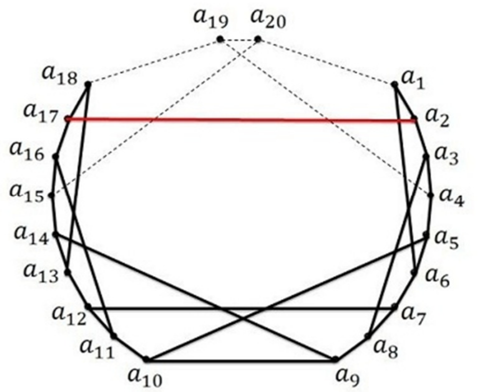

Figure 2.2: Case 1.1 in Theorem 2.1., where (,) ∈ .

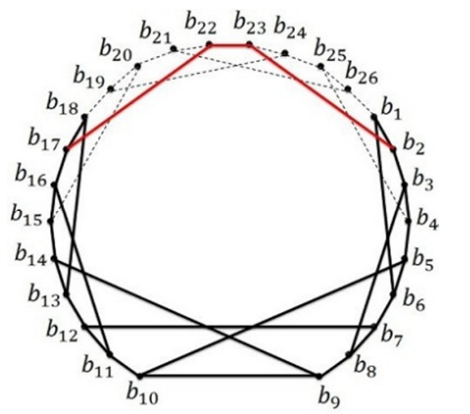

Figure 2.3: The cycle C constructed in Case 1.1 in Theorem 2.1.

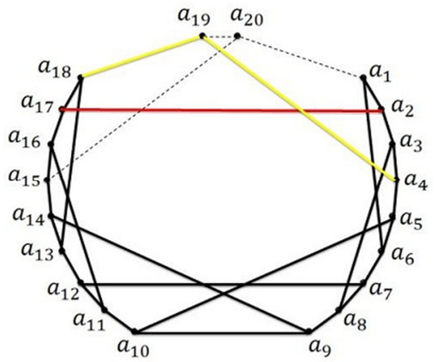

Figure 2.4: Case 1.2.1 in Theorem 2.1., where (,) ∈ and (, , ∈ .

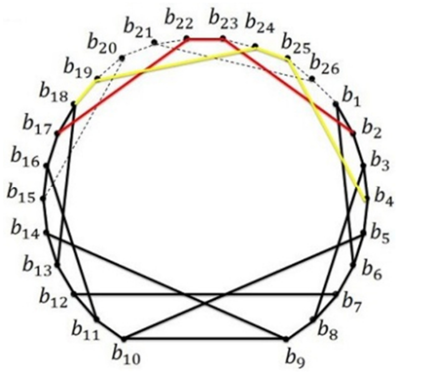

Figure 2.5: The cycle C constructed in Case 1.2.1 in Theorem 2.1

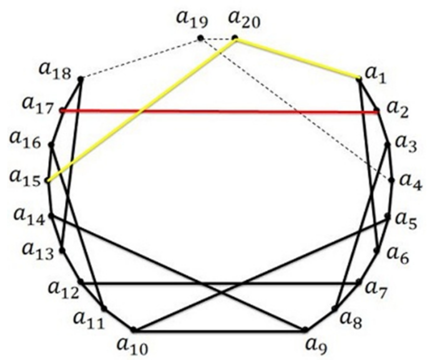

Figure 2.6: Case 1.2.2 in Theorem 2.1., where (,) ∈ and (, , ∈ .

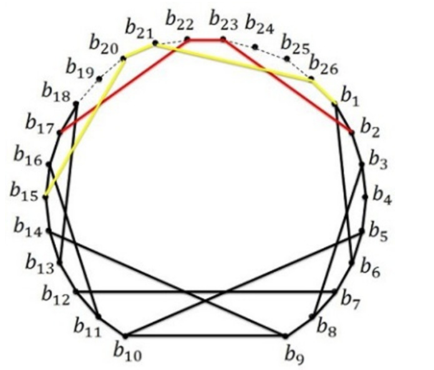

Figure 2.7: The cycle C constructed in Case 1.2.2 in Theorem 2.1.

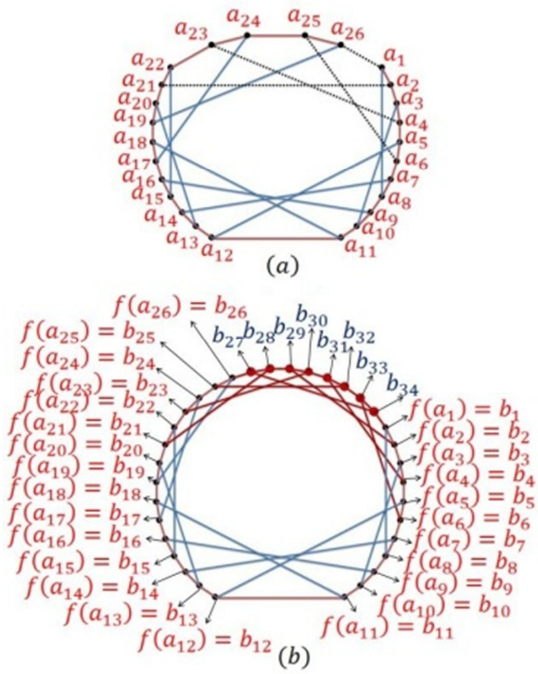

Figure 3.1: (a) *CR*(26; 1, 7); (b) *CR*(34; 1, 7) and the function *f*.

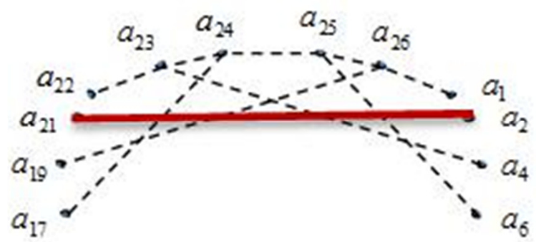

Figure 3.2: Case 1.1 of Theorem 3.1.

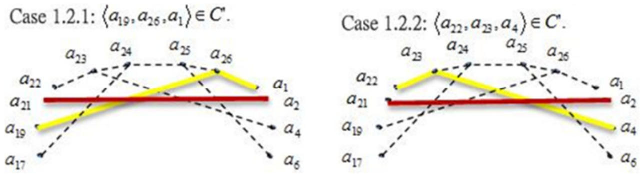

Figure 3.3: Case 1.2.1 – Case 1.2.2 of Theorem 3.1

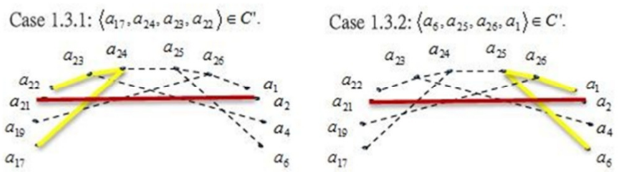

Figure 3.4: Case 1.3.1 – Case 1.3.2 of Theorem 3.1

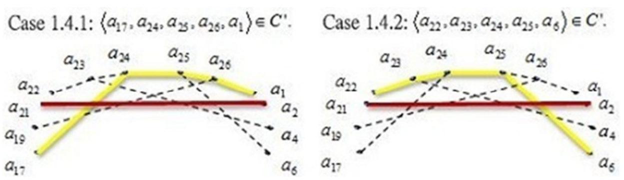

Figure 3.5: Case 1.4.1 – Case 1.4.2 of Theorem 3.1

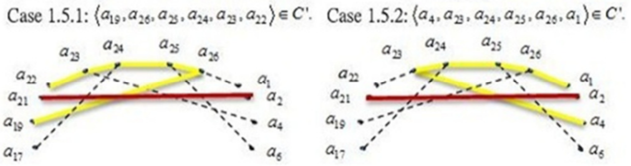

Figure 3.6: Case 1.5.1 – Case 1.5.2 of Theorem 3.1.

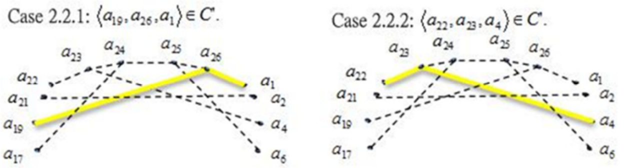

Figure 3.7: Case 2.2.1 – Case 2.2.2 of Theorem 3.1.

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