Goodness of fit tests on the basis of the kernel quantile estimators in dose-effect relationship

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Abstract: – A simple, nonparametric sample test for equality of a given quantile function is developed which can be applied to a variety of the kernel distribution function estimators for dose-effect relationship data. The test statistic based on a composition of a kernel estimate of the quantile function with a common distribution function estimate. Also test based on a weighted L_2-distance. In the given report we develop theoretical and computer research of this goodness of fit tests for the dose-effect relationship. The asymptotic normality of the corresponding test statistic is established under the null hypothesis. The obtained results can be used for interlaboratory comparison of results of effective dose estimation. A simple simulation study demonstrates that the moderate sample size properties of this procedure are reasonable.

Keywords: – Goodness of fit test, kernel estimators, dose-effect relationship.

1 Introduction

There is a need to test the hypotheses about the coincidence of the observed distribution function of a random variable with a given distribution function or its accessories to a certain kind when alternative distribution is unknown in various problems related to the application of mathematical statistics. These hypotheses can be tested using various statistics. Quite often used tests based on the integrated square error (see [1-5]). They are characterized by a specific choice of measure of the discrepancy between the "true" distribution function and its evaluation. The first tests of this kind were the Cramer-von Mises-Smirnov (CvMS) statistics and the Anderson-Darling (AD) statistics, where CvMS- and AD-statistics belong to the class of quadratic EDF statistics (tests based on the empirical distribution function). If the hypothesized distribution is F_0, and empirical (sample) cumulative distribution function is F_n. then the quadratic EDF-statistics measure the distance between F and F_n by

$$n\int_{-\infty}^{\infty}(F_n(x)-F_0(x))^2\omega(x)\,dF_0(x),$$

where $\omega(x)$ is a weighting function. When the weighting $\omega(x)=1$ and $x_1, x_2, ..., x_n$ be the observed values, increasing order, then the statistic (see [6,7]

$$CS = n\int_{-\infty}^{\infty}(F_n(x)-F_0(x))^2\,dF_0(x) =$$

$$=\frac{1}{12n}+\sum_{j=1}^{n}\left(\frac{2i-1}{2n}-F_0(x_j)\right)^2. \qquad (1)$$

is the CvMS-statistic. The Anderson-Darling test is based on distance (see [2])

$$A = n\int_{-\infty}^{\infty}\frac{(F_n(x)-F_0(x))^2}{F_0(x)(1-F_0(x))}dF_0(x) = \qquad (2)$$

$$=-n-\frac{1}{n}\sum_{j=1}^{n}(2j-1)(\ln F_0(x_j)+\ln(1-F_0(x_{n-j+1})))\,,$$

which is obtained when the weight function is $\omega(x)=(F_0(x)(1-F_0(x)))^{-1}$. For kernel density estimators such test is based on the integrated square error (ISE), which asymptotic normality is established in paper [3]. For dose-effect relationship the most comprehensive study of such tests held Krishtopenko D. S. in [8]

In this paper, we consider the quadratic integrated measure of the deviation of the kernel estimator of the distribution function of the theoretical distribution function. The present work is devoted to the construction and study of goodness of fit test based on estimates of quantile functions in dose-effect relationship (see [9]).

2. Problem statement

We consider the model of binary response which has title *dose-response relationship* [10-18] and which can be described as follows.

Let $\{(X_i,U_i), 1\le i\le n\}$ be a potential repeated sample of an unknown distribution $F(x)Q(y)$, $F(x)=\mathbf{P}(X_i<x)$, $Q(y)=\mathbf{P}(U_i<y)$, $(x,y)\in\mathbf{R}^2$, instead of which one observes the sample $\mathrm{U}^{(n)}=\{(U_i,W_i),\quad 1\le i\le n\}$, where $W_i=I(X_i<U_i)$

are the indicator functions of the event $(X_i < U_i)$. Here U_i are regarded as injected doses, and W_i as an effect of the action of the dose U_i. Let $F(x) = \int_{-\infty}^{x} f(t)\,dt$ and $f(x) > 0$. We shall call this situation the *random plan* of the experiment.

Together with the random plan, we consider *fixed plans* of the experiment. Namely, the injected dose U is supposed to be non-random and we let $U_i = u_i$, $i = 0,1,...,n+1$, where $0 = u_0 < u_1 < ... < u_n < u_{n+1} = 1$.

On the main problem of the dose-response relationship is to estimate the *effective doses* $ED_{100\lambda} = F^{-1}(\lambda) = x_\lambda$, $0 < \lambda < 1$, by the sample $\mathsf{U}^{(n)}$. For fixed plans of an experiment, we shall consider several nonparametric estimator and we shall find their asymptotic (as $n \to \infty$) distributions.

The nonparametric approach to the estimating supposes the presence of kernel functions $K_r(x)$, $K_d(x)$, being in fact symmetric densities of distributions with the support, say $[-1,1]$, and bandwidth h_r, h_d, which are smoothing non-random parameters depending on the sampling size n and converging to zero as $n \to \infty$, but $nh_r \to \infty$, $nh_d \to \infty$ as $n \to \infty$. We also let $H_d(u) = \int_{-\infty}^{u} K_d(x)\,dx.$

If there is evidence that the distribution function is (strictly) increasing we define

$$\hat{x}_n(\lambda) = \frac{1}{n}\sum_{i=1}^{n} H_d\left(\frac{\lambda - F_{nh_r}(i/n)}{h_d}\right), \qquad (3)$$

as an estimate of $x_\lambda = F^{-1}(\lambda)$, where

$$F_{nh_r}(x) = \frac{1}{nh_r}\sum_{i=1}^{n} K_r\left(\frac{x-u_i}{h_r}\right)W_i = \qquad (4)$$

$$= \frac{1}{n}\sum_{i=1}^{n} K_{r,h_r}(x-u_i)W_i .$$

is classical Nadaraya-Watson estimate (see [19,20] or [15]).

Let

$$\rho_n^2 = \int_0^1 (\hat{x}_{1,n}(\lambda) - x_\lambda)^2 \omega(\lambda)\,d\lambda , \qquad (5)$$

where $\omega(\lambda) \geq 0, (\int \omega(\lambda)\,d\lambda = 1)$ is the weight function.

Then

$$\rho_n^2 = 2I_{1,n} + I_{2,n} + I_{3,n}, \qquad (6)$$

where

$$I_{1,n} = \int_0^1 (\hat{x}_{1,n}(\lambda) - \mathbf{E}(\hat{x}_{1,n}(\lambda))(\mathbf{E}(\hat{x}_{1,n}(\lambda) - x_\lambda)\omega(\lambda)\,d\lambda,$$

$$I_{2,n} = \int_0^1 (\mathbf{E}(\hat{x}_{1,n}(\lambda)) - x_\lambda)^2\,\omega(\lambda)\,d\lambda,$$

$$I_{3,n} = \int_0^1 (\hat{x}_{1,n}(\lambda) - \mathbf{E}(\hat{x}_{1,n}(\lambda)))^2\,\omega(\lambda)\,d\lambda .$$

The terms $I_{1,n}, I_{2,n}$ and $I_{3,n}$ will be studied in detail.

Last integral we will present in the form: $I_{3,n} = J_{3,n} + J_{4,n}$, where

$$J_{n3} = \frac{2}{nh_r^{1/2}}\sum_{1\leq i<j\leq n}(W_i - F(u_i))(W_j - F(u_j)) \times$$
$$\times\int_0^1 \frac{K_{r,h_r}(x_\lambda - u_i)K_{r,h_r}(x_\lambda - u_j)}{f^2(x_\lambda)}\omega(\lambda)\,d\lambda , \qquad (7)$$

$$J_{n4} = \frac{1}{nh_r^{1/2}}\sum_{i=1}^{n}(W_i - F(u_i))^2\int_0^1 \frac{K_{r,h_r}^2(x_\lambda - u_i)}{f^2(x_\lambda)}\omega(\lambda)\,d\lambda . \qquad (8)$$

Let

$$\varsigma_{nj} = \sum_{i=j+1}^{n}(W_i - F(u_i))(W_j - F(u_j)) \times$$
$$\times\int_0^1 \frac{K_{r,h_r}(x_\lambda - u_i)K_{r,h_r}(x_\lambda - u_j)}{f^2(x_\lambda)}\omega(\lambda)\,d\lambda . \qquad (9)$$

Observe that

$$J_{n3} = \frac{2}{nh_r^{1/2}}\sum_{i=1}^{n}\varsigma_{ni} . \qquad (10)$$

3. Main Assumptions

***Assumptions* (H)**

(**H**1) $h_r = h_r(n), h_d = h_d(n)$, and $h_r \underset{n\to\infty}{\to} 0$, $h_d \underset{n\to\infty}{\to} 0$, but $nh_r \to \infty$, $nh_r \to \infty$ as $n \to \infty$.

(**H**2) $h_d / h_r \underset{n\to\infty}{\to} 0.$

(**H**3) $h_r = c_1 n^{-1/5}$.

(**H**4) $nh_r h_d^{8/3} \underset{n\to\infty}{\to} \infty$.

***Assumptions* (K).**

(**K**1) $K_j(x) \geq 0$, and $K_j(x) = 0, x \notin [-1,1]$, $j = r,d$.

(**K**2) $\int_{-1}^{1} K_r(x)\,dx = 1, \int_{-1}^{1} K_d(x)\,dx = 1,$ $\sup_x K_{r(d)}(x) \leq C_{r(d)}.$

We set

$$\| K \|^2 = \int_{-1}^{1} K^2(x)\,dx . \qquad (11)$$

(**K**3) $K_r(x) = K_r(-x)$, $x \in \mathbf{R}$.

(**K**4) On segment $[-1,1]$ there exist the third continuous bounded derivatives of the functions $K_r(x)$, $K_d(x)$.

(**K**5) $\|K_j\|_\infty = \sup_{x\in\mathbf{R}} |K_j(x)| = \kappa_j < \infty$ for $j = r,d$.

The variation of the function K is defined in the following way.

The variation of a real-valued function $K = K(u)$ a chosen interval (segment) $[a,b] \subset \mathbf{R}$ is the following quantity

$$\mathrm{V}(K) = \mathrm{V}_a^b(K) = \sup_P \sum_{k=0}^{m} |K(u_{k+1}) - K(u_k)|, \qquad (12)$$

where the supremum is taken over the set of all ordered partitions P of the segment $[a,b]$. If K is differentiable and its derivative is Riemann-integrable, then its total variation is the vertical component of the arc-length of its graph, that is to say,

$$\mathrm{V}_a^b(K) = \int_a^b |K'(x)|\, dx. \qquad (13)$$

A real-valued function K on the real line is said to be of *bounded variation* (BV function) on a segment $[a,b] \in \mathbf{R}$ if its variation finite, i.e. $K \in \mathrm{BV}([a,b]) \Leftrightarrow \mathrm{V}_a^b(K) < \infty$. Throughout the work we consider variations of function f on the segment $[0,1]$ and $f \in \mathrm{BV}([0,1])$.

Remark 3.1. *The boundedness of the derivatives of the functions $K_r(u)$, $K_d(u)$ on the segment* $[-1,1]$ (*Assumption* $\mathbf{K}_4$) *imply that their variations are bounded* (see [21]), i.e. $\mathrm{V}(K_{d(r)}) < \infty$.

***Assumptions* (F).**

($\mathbf{F}_1$) There exists the third continuous bounded derivative of the density of the distribution $f(x) = F'(x)$, and $f(x) \ge C_0 > 0$ for $0 \le x \le 1$, i.e. on the segment $[0,1]$, the density $f(x)$ is separated from zero.

***Assumptions* (P).**

($\mathbf{P}_1$) As $n \to \infty$,

$$\max_{k=0,1,...,} \max \left\{ \left| u_k - \frac{k}{n} \right|, \left| u_{k+1} - \frac{k}{n} \right| \right\} = O\left(\frac{1}{n}\right).$$

Assumption (**P**) yields $u_k = \dfrac{k}{n} + O\left(\dfrac{1}{n}\right)$, at that, the sequence $n\left(u_k - \dfrac{k}{n}\right)$ is bounded by C uniformly in $0 \le k \le n$.

Throughout the work (Main) Assumptions **(H), (K), (F), (P)**.

4. Auxiliary results.

In this section we represent the auxiliary results needed to study the asymptotic behavior of the statistics I_1, I_2, I_3.

Let B be the Lebesgue $\sigma-$algebra on $I^s = [0,1]^s$ and ρ is the Lebesgue measure on B. For $P = \{u_0, u_1, ..., u_n, u_{n+1}\}$ and $B \in B$ we define

$$A(B;P) = \sum_{i=1}^{n} \chi_B(u_i),$$

$$D_n(B\,;P) = \sup_{B \in B} \left| \frac{A(B;P)}{n} - \rho(B) \right|, \qquad (14)$$

where $\chi_B(x)$ is the characteristic function of B. The discrepancy $D_n^*(P) = D_n^*(u_1,...,u_n)$ of the point set P is defined by $D_n^*(P) = D_n(J_c^*, P)$ where J_c^* is the family of all subintervals of I^s subset of I of the form $\prod_{i=1}^{n}[0, u_i)$.

For each bounded function $\psi : \mathbf{R} \to \mathbf{R}$ we let $\|\psi\|_I = \sup_{x \in I} |\psi(x)|$.

Theorem 4.1 ([21], Koksma-Hlawka inequality) *If a function $f(u)$ $(0 \le u \le 1)$ has bounded variation $V(f)$ on $[0,1]$, then, for any $0 < u_1 < u_2 < ... < u_n < 1$ we have*

$$\left| \frac{1}{n} \sum_{i=1}^{n} f(u_i) - \int_0^1 f(u)\, du \right| \le \mathrm{V}(f)\, D_n^*(u_1,...,u_n).$$

For $s = 1$, we may arrange the points $u_1,...,u_n$ of a given point set in nondecreasing order. The formula in Theorem 4.1 is due to Niederreiter [21].

Lemma 4.1. *If $0 < u_1 < u_2 < ... < u_n < 1$, then*

$$D_n^*(u_1,...,u_n) = \frac{1}{2n} + \max_{1 \le i \le n} \left| u_i - \frac{2i-1}{2n} \right|. \qquad (15)$$

Remark. *If* , then $\dfrac{i}{n} - \dfrac{2i-1}{2n} = \dfrac{1}{2n}$ *and*

$$D_n^*(u_1,...,u_n) = \frac{1}{n}.$$

In what follows we shall make use of the following auxiliary result.

We consider the function

$$\tilde{f} = \tilde{f}(u) = \frac{1}{h_d} K\left(\frac{F(u) - \lambda}{h_d}\right), \qquad (16)$$

where $0 < \lambda < 1$.

Lemma 4.2. [9] *Suppose that the main assumptions hold. Then*

$$V(\tilde{f}) = \sup \sum_{j=1}^{l} |\tilde{f}(u_j) - \tilde{f}(u_{j-1})| = O\left(\frac{1}{h_d}\right),$$

where the supremum is taken over all ordered partitions $0 < u_1 < u_2 < ... < u_n < 1$ of the segment $[0,1]$.

We represent the statistics $\hat{x}_{1,n}(\lambda)$ as

$$\hat{x}_{1,n}(\lambda) = x_{\lambda,n} + \Delta, \qquad (17)$$

where $x_{\lambda,n} = \dfrac{1}{nh_d} \displaystyle\sum_{i=1}^{n} H_d\left(\frac{F(i/n) - \lambda}{h_d}\right)$.

We define the statistics

$$\Delta(\lambda)=\frac{1}{n}\sum_{i=1}^{n}\left(H_d\left(\frac{F_{nh_r}(i/n)-\lambda}{h_d}\right)-H_d\left(\frac{F(i/n)-\lambda}{h_d}\right)\right). \quad (18)$$

Then

$$\Delta(\lambda)+\frac{1}{nh_d}\sum_{i=1}^{n}K_d\left(\frac{\lambda-F(i/n)}{h_d}\right)\times \times(F_{nh_r}(i/n)-F(i/n))\xrightarrow[n\to\infty]{p}0. \quad (19)$$

Consider the statistics Δ_1 and represent it as

$$\Delta_1=\Delta_{1,1}+\Delta_{1,2},$$

$$\Delta_{1,1}=-\frac{1}{nh_d}\sum_{i=1}^{n}K_d\left(\frac{F(i/n)-\lambda}{h_d}\right)\times \times(F_{nh_r}(i/n)-\mathbf{E}(F_{nh_r}(i/n))) \quad (20)$$

$$\Delta_{1,2}=-\frac{1}{nh_d}\sum_{i=1}^{n}K_d\left(\frac{F(i/n)-\lambda}{h_d}\right)\times \times(\mathbf{E}(F_{nh_r}(i/n))-F(i/n)) \quad (21)$$

Theorem 4.2. [9] *As* $n\to\infty$,

$$x_{\lambda,n}=\frac{1}{nh_d}\sum_{i=1}^{n}H_d\left(\frac{F(i/n)-\lambda}{h_d}\right)=x_\lambda+a_{2,d}h_d^2+o(h_d^2), \quad (22)$$

where

$$x_\lambda=F^{-1}(\lambda),\ a_{2,d}=-\frac{\nu_d^2 f'(x_\lambda)}{2f^3(x_\lambda)},\ \nu_d^2=\int_{-1}^{1}x^2K_d(x)\,dx. \quad (23)$$

5. Main result.

The main result is the following theorem in which it is proved central limit theorem for integrated square error ρ_n^2, i.e. global integrated measure of deviation between $\hat{x}_n(\lambda)$ and x_λ.

Theorem 5.1. *Under stated assumptions and assuming that* $h_r\to\infty$, $nh_r^5\to\mu$, $0<a<\infty$, *as* $n\to\infty$, *we have*

$$d(n)(\rho_n^2-c(n))\xrightarrow[n\to\infty]{d}N(4\mu^{4/5}\sigma_1^2+\mu^{-1/5}\sigma_2^2), \quad (24)$$

where

$$d(n)=nh_r^{1/2}, \quad (25)$$

$$c(n)=\int_0^1(\mathbf{E}(\hat{x}_{1,n}(\lambda))-x_\lambda)^2\,\omega(\lambda)\,d\lambda+(nh_r)^{-1}\sigma_3^2, \quad (26)$$

$$\sigma_1^2=(1/4)\nu_r^4\|K_r\|^4\int_0^1 f^{-4}(x_\lambda)\lambda(1-\lambda)(f'(x_\lambda))^2\omega(\lambda)\,d\lambda,$$

$$\nu_r^2=\int_{-1}^{1}x^2K_r^2(x)\,dx, \quad (27)$$

$$\sigma_2^2=2\int_0^1 f^{-4}(x_\lambda)\lambda^2(1-\lambda)^2\omega^2(\lambda)\,d\lambda\times \times\int\left(\int K_r(x)K_r(x+y)\,dx\right), \quad (28)$$

$$\sigma_3^2=\|K_r\|^2\int_0^1 f^{-2}(x_\lambda)\lambda(1-\lambda)\,\omega(\lambda)\,d\lambda,$$

$$\|K_r\|^2=\int K_r^2(x)\,dx. \quad (29)$$

We shall consider the terms in expansion (1) individually, via a sequence of lemmas 5.1 – 5.4.

Lemma 5.1. *Under stated assumptions,* $I_{1,n}$ *follows asymptotically a normal distribution (as* $n\to\infty$*) with the parameters* $(0,\sigma_1^2)$, *where*

$$\sigma_1^2=\frac{h_r^4\nu_r^4\|K_r\|^2}{4}\int_0^1\frac{\lambda(1-\lambda)}{f^2(x_\lambda)}\left(\frac{f'(x_\lambda)}{f(x_\lambda)}\right)^2\omega(\lambda)\,d\lambda. \quad (30)$$

Let's notice that if weight function

$$\omega(\lambda)=\begin{cases}1, \text{ for } \lambda\in[F(-A),F(A)],\\ 0, \text{ otherwice,}\end{cases} \quad (31)$$

then

$$\sigma_1^2=\frac{h_r^4\nu_r^4\|K_r\|^2}{4}\int_{-A}^{A}\frac{F(x)(1-F(x))}{f^2(x)}\left(\frac{f'(x)}{f(x)}\right)^2 f(x)\,dx.$$

Define the variables

$$\xi_j=-\frac{1}{n^2h_dh_r}(W_j-F(u_j))\sum_{i=1}^{n}K_d\left(\frac{F(i/n)-\lambda}{h_d}\right)\times \times K_r\left(\frac{i/n-u_j}{h_d}\right). \quad (32)$$

Then $\Delta_{1,1}=\sum_{i=1}^{n}\xi_i$, and

$$\xi_j\sim-\frac{1}{nh_dh_r}(W_j-F(u_j))\times \times\int_{\max(u_j-h_r,F^{-1}(\lambda-h_d))}^{\min(u_j+h_r,F^{-1}(\lambda+h_d))}K_d\left(\frac{F(x)-\lambda}{h_d}\right)\cdot K_r\left(\frac{x-u_j}{h_r}\right)dx. \quad (33)$$

Set $a_{2,r}=-\dfrac{\nu_r^2 f'(x_\lambda)}{f(x_\lambda)}$. From [9] follows that as $n\to\infty$,

$$\mathbf{E}(\hat{x}_{1,n}(\lambda))-x_\lambda\sim-\frac{\nu_r^2h_r^2 f'(x_\lambda)}{2f(x_\lambda)}, \quad (34)$$

$$\mathbf{D}(\hat{x}_{1,n}(\lambda))\sim\frac{\lambda(1-\lambda)}{f^2(x_\lambda)}\|K_r\|^2. \quad (35)$$

Therefore

$$\mathbf{D}\left(\sum_{j=1}^{n}\xi_j\right)=\sum_{j=1}^{n}\mathbf{D}(\xi_j)=\frac{1}{n^2h_d^2h_r^2}\sum_{j=1}^{n}G(u_j)\times$$

$$\times\left(\int_{\max(u_j-h_r,F^{-1}(\lambda-h_d))}^{\min(u_j+h_r,F^{-1}(\lambda+h_d))}K_d\left(\frac{F(x)-\lambda}{h_d}\right)K_r\left(\frac{x-u_j}{h_r}\right)dx\right)^2\sim$$

$$\sim\frac{1}{nh_d^2h_r^2}\int_0^1 G(u)\left(\int_{\max(u-h_r,F^{-1}(\lambda-h_d))}^{\min(u+h_r,F^{-1}(\lambda+h_d))}K_d\left(\frac{F(x)-\lambda}{h_d}\right)\times\right.$$

$$\times K_r\left(\frac{x-u}{h_r}\right)dx\Bigg)^2 du.$$

Making the substitution $\frac{F(x)-\lambda}{h_d}=y$, we will receive

$$\xi_j \sim -\frac{1}{nh_r}\cdot\frac{(W_j-F(u_j))}{f(x_\lambda)}\int K_d(y)K_r\left(\frac{F^{-1}(\lambda+h_d y)-u_j}{h_r}\right)dy \sim$$

$$\sim -\frac{1}{nh_r}\cdot\frac{(W_j-F(u_j))}{f(x_\lambda)}K_r\left(\frac{x_\lambda-u_j}{h_r}\right)\int K_d(y)\,dy =$$
$$-\frac{1}{nh_r}\cdot\frac{(W_j-F(u_j))}{f(x_\lambda)}K_r\left(\frac{x_\lambda-u_j}{h_r}\right)$$

and

$$\sum_{j=1}^{n}\mathbf{D}(\xi_j)\sim\frac{1}{nh_r^2}\int_0^1 G(u)K_r^2\left(\frac{u-x_\lambda}{h_r}\right)du. \quad (36)$$

Once the substitution $\frac{u-x_\lambda}{h_r}=z$ was made the resulting integral became

$$\sum_{j=1}^{n}\mathbf{D}(\xi_j)\sim\frac{1}{nh_r}\int_{-1}^{1}G(x_\lambda+zh_r)K_r^2(z)\,dz\sim$$
$$\sim\frac{1}{f^2(x_\lambda)nh_r}\int_{-1}^{1}G(x_\lambda)K_r^2(z)\,dz=\frac{\lambda(1-\lambda)}{f^2(x_\lambda)nh_r}\|K_r\|^2. \quad (37)$$

Therefore

$$\mathbf{E}(\Delta_1)=\mathbf{E}(\Delta_{1,2})\sim-\frac{\nu_r^2 h_r^2 f'(x_\lambda)}{2f(x_\lambda)}. \quad (38)$$

Thus,

$$\mathbf{D}(\hat{x}_{1,n}(\lambda))\sim\frac{\lambda(1-\lambda)}{f^2(x_\lambda)}, \quad (39)$$

and $\mathbf{E}(\hat{x}_{1,n}(\lambda)-x_\lambda)\sim-\frac{\nu_r^2 h_r^2 f'(x_\lambda)}{2f(x_\lambda)}$.

We have

$$I_{2,n}=\int_0^1(\mathbf{E}(\hat{x}_{1,n}(\lambda))-x_\lambda)^2\omega(\lambda)\,d\lambda\sim$$
$$\sim\frac{\nu_r^4 h_r^4}{4}\int_0^1\left(\frac{f'(x_\lambda)}{f(x_\lambda)}\right)^2\omega(\lambda)\,d\lambda=\frac{\nu_r^4 h_r^4}{4}J_A(f), \quad (40)$$

where $J_A(f)=\int_{-A}^{A}\left(\frac{f'(x)}{f(x)}\right)^2 f(x)\,dx$.

Really, using into consideration that

$$\mathbf{E}(\hat{x}_{1,n}(\lambda))=x_{\lambda,n}+\mathbf{E}(\Delta)=$$
$$=x_\lambda+a_{2,d}h_d^2+o(h_d^2)+\mathbf{E}(\Delta)=$$
$$=x_\lambda+a_{2,d}h_d^2+o(h_d^2)+\Delta_{1,2},$$

this follows from lemma 4.1 and from [5] , that

$$\Delta_{1,2}=-\frac{\nu_r^2}{2}h_r^2\frac{f'(x_\lambda)}{f(x_\lambda)}+o(h_r^2), \quad (41)$$

we deduce

$$(\mathbf{E}(\hat{x}_{1,n}(\lambda))-x_{\lambda,n})^2=$$
$$=\left(a_{2,d}h_d^2-\frac{\nu_r^2}{2}h_r^2(\ln f(x))'|_{x=x_\lambda}+o(h_r^2)+o(h_d^2)\right)^2. \quad (42)$$

From the conditions ($\mathbf{F}_1$), ($\mathbf{H}$) and the boundedness of function $\frac{f'(x_\lambda)}{f(x_\lambda)}$, we have the following lemma.

Lemma 5.2. *Under the stated conditions,*

$$I_{2,n}=\frac{\nu_r^4 h_r^4}{4}J_A(f)(1+o(1)), \quad (43)$$

as $n\to\infty$.

Proof. The result (43) follows from [4].

Lemma 5.3. *Under the stated conditions, we have*

$$J_{4,n}\xrightarrow[n\to\infty]{p}\sigma_2^2=\|K\|^2\int_0^1\frac{\lambda(1-\lambda)}{f^2(x_\lambda)}\omega(\lambda)\,d\lambda, \quad (44)$$

as $n\to\infty$.

Proof. Set $H(u)=F(u)(1-F(u))$. Then

$$\mathbf{E}(J_{4,n})=\frac{1}{n}\sum_{i=1}^{n}\mathbf{D}(W_i)\int_0^1\frac{K_{r,h_r}^2(x_\lambda-u_i)}{f^2(x_\lambda)}\omega(\lambda)\,d\lambda=$$
$$=\frac{1}{n}\sum_{i=1}^{n}H(u_i)\int_0^1\frac{K_{r,h_r}^2(x_\lambda-u_i)}{f^2(x_\lambda)}\omega(\lambda)\,d\lambda\sim$$
$$\sim\int_0^1\frac{1}{f^2(x_\lambda)}\left(\int_0^1 H(u)K_{r,h_r}^2(x_\lambda-u)\,du\right)\omega(\lambda)\,d\lambda=$$
$$=\int_0^1\frac{1}{f^2(x_\lambda)}\left(\int_0^1 H(x_\lambda+th_r)K_r^2(t)\,dt\right)\omega(\lambda)\,d\lambda\sim$$
$$\sim\|K\|^2\int_0^1\frac{\lambda(1-\lambda)}{f^2(x_\lambda)}\omega(\lambda)\,d\lambda. \quad (45)$$

In addition,

$$\mathbf{D}(J_{4,n})=\frac{1}{n^2}\sum_{i=1}^{n}\mathbf{D}((W_i-F(u_i))^2)\left(\int_0^1\frac{K_{r,h_r}^2(x_\lambda-u_i)}{f^2(x_\lambda)}\omega(\lambda)\,d\lambda\right)^2=$$
$$=\frac{1}{n^2}\sum_{i=1}^{n}H(u_i)(1-2F(u_i))\left(\int_0^1\frac{K_{r,h_r}^2(x_\lambda-u_i)}{f^2(x_\lambda)}\omega(\lambda)\,d\lambda\right)^2. \quad (46)$$

Employing the Koksma-Hlawka inequality we obtain

$$\mathbf{D}(J_{4,n})=\frac{1}{n}\int_0^1 H(u)(1-2F(u))\left(\int_0^1\frac{K_{r,h_r}^2(x_\lambda-u)}{f^2(x_\lambda)}\omega(\lambda)\,d\lambda\right)^2 du\times$$
$$\times(1+o(1)),\ \text{ as } n\to\infty. \quad (47)$$

Next,

$$A_n=\frac{1}{n}\int_0^1 H(u)(1-2F(u))\left(\int_0^1\frac{K_{r,h_r}^2(x_\lambda-u)}{f^2(x_\lambda)}\omega(\lambda)\,d\lambda\right)^2 du=$$
$$=\frac{1}{nh_r}\int_0^1 H(x_\lambda+h_r t)(1-2F(x_\lambda+h_r t))\left(\int_{-1}^{1}\frac{K_r^2(t)}{f^2(x_\lambda)}\omega(\lambda)\,d\lambda\right)^2 dt.$$

But $0\le H(u)\le 1/4$, $|1-2F(u)|\le 1$, therefore

$$0 \le A_n \le \frac{1}{4nh_r}\int_{-1}^{1} K_r^4(t)\,dt\left(\int_0^1 \frac{\omega(\lambda)\,d\lambda}{f^2(x_\lambda)}\right)^2 dt \le \frac{\|K_r\|^4}{4nh_r C_0^2} \underset{n\to\infty}{\to} 0\,.$$

That is, $\mathbf{D}(J_{4,n}) \underset{n\to\infty}{\to} 0$. Hence, by Chebyshev inequality, we receive result of the lemma 5.3.

Lemma 5.4. *As* $n \to \infty$ *the sequence* $J_{3,n}$ *is asymptotically normal with parameters* $(0,\sigma_3^2)$, *where*

$$\sigma_3^2 = 2\int \frac{\lambda^2(1-\lambda)^2}{f^4(x_\lambda)}\omega(\lambda)\,d\lambda \int dt\left(\int K_r(u)K_r(u+t)\,du\right)^2$$
.(48)

Proof. Let's consider $J_{n3} = \frac{2}{nh_r^{1/2}}\sum_{i=1}^{n}\varsigma_{ni}$, where

$$\varsigma_{nj} = \sum_{i=j+1}^{n}(W_i - F(u_i))(W_j - F(u_j)) \times \times\int_0^1 \frac{K_{r,h_r}(x_\lambda - u_i)K_{r,h_r}(x_\lambda - u_j)}{f^2(x_\lambda)}\omega(\lambda)\,d\lambda\,. \quad (49)$$

Let $\mathsf{F}_k = \sigma(X_1, X_2, ..., X_k)$ be the σ-algebra, generated by the random variables $X_1, X_2, ..., X_k$. Then $\{\varsigma_{nk}, \mathsf{F}_k\}_{1\le k\le n}$, $n \ge 1$, is a martingale-difference (see [22], p.442), since $\mathbf{E}(|\varsigma_{nk}|) < \infty$ and $\mathbf{E}(\varsigma_{nk}|\mathsf{F}_{k-1}) = 0$. To prove the asymptotic normality of J_{n3}, it is necessary to show (see [22], p.442, theorem 8 (II)), that

$$\frac{1}{n^2h^3}\sum_{i=1}^{n-1}\mathbf{E}(\varsigma_{ni}^2 I(|\varsigma_{ni}| > \delta nh^{3/2}) \mid \mathsf{F}_{i-1}) \underset{n\to\infty}{\overset{p}{\to}} 0\,, \quad (50)$$

$\delta \in (0,1)$;

$$\frac{4}{n^2h^3}\sum_{i=1}^{n-1}\mathbf{E}(\varsigma_{ni}^2 \mid \mathsf{F}_{i-1}) \underset{n\to\infty}{\overset{p}{\to}} \sigma_3^2\,. \quad (51)$$

We have

$$\xi_{ni}^2 = (W_i - F(u_i))^2 \times \times\left(\sum_{j=i+1}^{n}(W_j - F(u_j))\int K\left(\frac{x-u_i}{h}\right)K\left(\frac{x-u_j}{h}\right)\omega(x)dx\right)^2.$$

As the random variables W_1, W_2, …, W_{i-1} and W_j for $j \ge i$ are independent and $\mathbf{E}(W_j - F(u_j)) = 0$, then

$$\mathbf{E}(\xi_{ni}^2 \mid A_{i-1}) = F(u_i)(1-F(u_i)), \quad (52)$$

$$\mathbf{D}\left(\sum_{j=i+1}^{n}(W_j - F(u_j))\int K\left(\frac{x-u_i}{h}\right)K\left(\frac{x-u_j}{h}\right)\omega(x)dx\right) =$$

$$= F(u_i)(1-F(u_i))\sum_{j=i+1}^{n}F(u_j)(1-F(u_j)) \times \times\left(\int K\left(\frac{x-u_i}{h}\right)K\left(\frac{x-u_j}{h}\right)\omega(x)dx\right)^2. \quad (53)$$

Hence, as $n \to \infty$,

$$\frac{4}{n^2h^3}\sum_{i=1}^{n}\mathbf{E}(\xi_{ni}^2 \mid \mathsf{F}_{i-1}) =$$

$$= \frac{4}{n^2h^3}\sum_{i=1}^{n-1}F(u_i)(1-F(u_i))\sum_{j=i+1}^{n}F(u_j)(1-F(u_j)) \times$$

$$\times\left(\int K\left(\frac{x-u_i}{h}\right)K\left(\frac{x-u_j}{h}\right)\omega(x)dx\right)^2 \sim$$

$$\sim 4h^{-3}\int F(u)(1-F(u))du\int_u^{+\infty}F(v)(1-F(v))dv \times$$

$$\times\left(\int K\left(\frac{x-u}{h}\right)K\left(\frac{x-v}{h}\right)\omega(x)dx\right)^2 =$$

$$= 4h^{-1}\int F(u)(1-F(u))du\int_u^{+\infty}F(v)(1-F(v))dv \times$$

$$\times\left(\int K(t)K\left(\frac{v-u}{h}+t\right)\omega(u+th)dt\right)^2 =$$

$$= 4\int F(u)(1-F(u))\omega^2(u)du \times$$

$$\times\int_u^{+\infty}F(u+zh)(1-F(u+zh))dz\left(\int K(t)K(z+t)\,dt\right)^2 =$$

$$= 2\int F(u)(1-F(u))\omega^2(u)du \times$$

$$\times\int F(u+zh)(1-F(u+zh))dz\left(\int K(t)K(z+t)\,dt\right)^2.(54)$$

From the condition (**K3**) imply the following: if $\begin{cases}-1 \le t \le 1\\ -1 \le z+t \le 1\end{cases}$, i.e. $-2 \le z \le 2$, then, as $n \to \infty$,

$$\frac{4}{n^2h^3}\sum_{i=1}^{n}E(\xi_{ni}^2 \mid \mathsf{F}_{i-1}) =$$

$$= 2\int F(u)(1-F(u))\omega^2(u)du \times$$

$$\times\int_{-2}^{2}F(u+zh)(1-F(u+zh))dz\left(\int_{-1}^{1}K(t)K(z+t)\,dt\right)^2 =$$

$$= 2\int F^2(u)(1-F(u))^2\omega^2(u)du\int_{-2}^{2}dz$$

$$= 2\int F^2(u)(1-F(u))^2\omega^2(u)du\int dz \times$$

$$\times\left(\int K(t)K(z+t)\,dt\right)^2 = \sigma_3^2\,, \quad (55)$$

therefore condition (51) is satisfied.

Furthermore,

$$\frac{1}{n^2h^3}\sum_{i=1}^{n-1}\mathbf{E}(\xi_{ni}^2 I(|\xi_{ni}| > \delta nh^{3/2}) \mid \mathsf{F}_{i-1}) \le$$

$$\le \frac{1}{\delta^2 n^4 h^6} \sum_{i=1}^{n-1} \mathbf{E}(\xi_{ni}^4 \mid \mathrm{F}_{i-1}). \qquad (56)$$

Consider the sum on the right-hand side of this inequality. By virtue the condition (**A1**) we have $|\omega(x)| \le M$, therefore

$$\mathbf{E}(\xi_{ni}^4 \mid \mathrm{F}_{i-1}) =$$
$$= \mathbf{E}\{W_i - F(u_i)\}^4 \times$$
$$\times \mathbf{E}\left(\sum_{j=i+1}^{n} (W_j - F(u_j)) \int K\left(\frac{x-u_i}{h}\right) K\left(\frac{x-u_j}{h}\right) \omega(x)\,dx \right)^4 =$$

$$= \mathbf{E}\{W_i - F(u_i)\}^4 \mathbf{E}\left(h \sum_{j=i+1}^{n} (W_j - F(u_j)) \times \right.$$
$$\left. \times \int K(z) K\left(\frac{u_i - u_j}{h} + z\right) \omega(u_i + zh)\,dz \right)^4. \qquad (57)$$

By virtue the condition (**A1**) and (**L1**) we have $|\omega(x)| \le M$ and $|K(x)| \le K$, from which

$$\mathbf{E}(\xi_{ni}^4 \mid A_{i-1}) \le M^4 K^4 h^4 E\{W_i - F(u_i)\}^4 \times$$
$$\times \mathbf{E}\left(\sum_{j=i+1}^{n} (W_j - F(u_j)) \int K(z)\,dz \right)^4 =$$
$$= M^4 K^4 h^4 \mathbf{E}\{W_i - F(u_i)\}^4 \mathbf{E}\left(\sum_{j=i+1}^{n} (W_j - F(u_j)) \right)^4 \le$$
$$\le 16 M^4 K^4 h^4 \mathbf{E}\left(\sum_{j=i+1}^{n} (W_j - F(u_j)) \right)^4. \qquad (58)$$

Arguing similarly to [22], p.380, and using the independence of rv W_j and W_k at $j \ne k$, we receive

$$\mathbf{E}\left(\sum_{j=i+1}^{n} (W_j - F(u_j)) \right)^4 = \sum_{j=i+1}^{n} \mathbf{E}(W_j - F(u_j))^4 +$$
$$+6 \sum_{i+1 \le j < k \le n} E(W_j - F(u_j))^2 E(W_k - F(u_k))^2 =$$
$$= \sum_{j=i+1}^{n} ((F(u_j))^4 (1 - F(u_j)) + F(u_j)(1 - F(u_j))^4) +$$
$$+6 \sum_{i+1 \le j < k \le n} F(u_j)(1 - F(u_j)) F(u_k)(1 - F(u_k)) \le$$
$$\le \frac{n-i}{12} + \frac{3}{8}(n-i)(n-i+1) \le \frac{3}{8}(n-i+1)^2, \qquad (59)$$

since $\max\limits_{0 \le x \le 1} (x^4(1-x) + x(1-x)^4) = \dfrac{1}{12}$.

Let's notice that

$$\sum_{i=1}^{n} (n-i+1)^2 = \frac{n(n-1)(2n-1)}{6} \le \frac{n^3}{3}. \qquad (60)$$

That is ,

$$\frac{1}{n^2 h^3} \sum_{i=1}^{n-1} \mathbf{E}(\xi_{ni}^2 I(|\xi_{ni}| > \delta n h^{3/2}) \mid A_{i-1}) \le$$

$$\le \frac{6 K^4 M^4 h^4}{\delta^2 n^4 h^6} \sum_{i=1}^{n-1} (n-i+1)^2 \le \frac{2 K^4 M^4}{\delta^2 n h^2} \underset{n \to \infty}{\to} 0. \qquad (61)$$

So, in this case condition (50) is satisfied. Now from [22] that the sequence J_{n3} is asymptotically normal with parameters $(0, \sigma_3^2)$.

Remark 5.1. For Epanechnikov kernel $K(x) = (3/4)(1-x^2) I(|x| \le 1)$, the convolution equals

$$(K * K)(x) = \begin{cases} (3/360)(32 - 40x^2 + 20x^3 - x^5), 0 \le x \le 2, \\ (3/360)(32 - 40x^2 - 20x^3 + x^5), -2 \le x < 0. \end{cases}$$

.

Therefore

$$\int dv \left(\int K(u) K(u+v)\,du \right)^2 = 167/387 \approx 0.434.$$

Let's notice that

$$I_{n1} = \int (F_n(x) - \mathbf{E}(F_n(x)))(\mathbf{E}(F_n(x)) - F(x))\,\omega(x)\,dx,$$
$$I_{n2} + I_{n3} = \int (\mathbf{E}(F_n(x)) - F(x))^2\,\omega(x)\,dx.$$

Hence, $I_n - c(n) = 2I_{n1} + I_{n2} + I_{n3}$, where

$$c(n) = \int \mathbf{E}(F_n(x) - F(x))^2\,dx =$$
$$= \int (\mathbf{E}(F_n(x)) - F(x))^2\,dx + n^{-1} h^{-1} \sigma_2^2.$$

From lemmas 5.1 – 5.4 we derive the theorem 5.1.

In addition, we see that the error ρ_n^2 may be written as

$$\rho_n^2 = 2k\sigma_1 n^{-1/2} h^2 \varsigma_1 + \sigma_2^2 + 2^{1/2} \sigma_3 n^{-1} h^{-1/2} \varsigma_3,$$
(62)

where the random variables ς_1 and ς_3 are each asymptotically normal $N(0,1)$.

The statistics ρ_n^2 is offered to be used for testing the goodness of fit of a statistical model. Asymptotic p-values for statistics can be obtained using the quantile of standard normal distribution.

6 Reduction of a measurement error

Let the dose $\boldsymbol{U}$ is measured with an error, i.e. $\boldsymbol{Y} = \boldsymbol{U} + \boldsymbol{\varepsilon}$, where $\boldsymbol{U}, \boldsymbol{\varepsilon}$ are independent random variables and $\boldsymbol{\varepsilon} \in \boldsymbol{R}^d$ has normal distribution with *d*-dimensional mean vector $\mathbf{0}$ and a known $d \times d$ covariance matrix $\boldsymbol{\Sigma}_0$, and the random vector $\boldsymbol{U}$ has unknown density $g(\boldsymbol{u}) > 0$. The regression curve of $\boldsymbol{U}$ with respect to $\boldsymbol{Y}$ it can be written in form

$$\boldsymbol{u}(\boldsymbol{x}) = \mathbf{E}(\boldsymbol{U} \mid \boldsymbol{Y} = \boldsymbol{x}) = \frac{\boldsymbol{r}(\boldsymbol{x})}{q(\boldsymbol{x})},$$

where

$$\boldsymbol{r}(\boldsymbol{x}) = \int \boldsymbol{u} g(\boldsymbol{u}) \times$$
$$\times \frac{1}{(2\pi)^{d/2} |\boldsymbol{\Sigma}_0|^{1/2}} |\exp\left(-\frac{1}{2} (\boldsymbol{u} - \boldsymbol{x})^T \boldsymbol{\Sigma}_0 (\boldsymbol{u} - \boldsymbol{x}) \right) d\boldsymbol{u},$$

$$q(\boldsymbol{x}) = \int g(\boldsymbol{u}) \times$$

$$\times \frac{1}{(2\pi)^{d/2} |\boldsymbol{\Sigma}_0|^{1/2}} | \exp\left(-\frac{1}{2} (\boldsymbol{u}-\boldsymbol{x})^T \boldsymbol{\Sigma}_0 (\boldsymbol{u}-\boldsymbol{x}) \right) d\boldsymbol{u}.$$

Differentiating $q(\boldsymbol{x})$ with respect to $\boldsymbol{x}$ yields (see [23])

$$\nabla_q(\boldsymbol{x}) = -\boldsymbol{\Sigma}_0^{-1} \boldsymbol{x} q(\boldsymbol{x}) + \boldsymbol{\Sigma}_0^{-1} \boldsymbol{r}(\boldsymbol{x}),$$

where the symbol $\nabla_q(\boldsymbol{x})$ denote the $1 \times d$ matrix of first-order partial derivatives of the transformation from $\boldsymbol{x}$ to $q(\boldsymbol{x})$.

Let the random vector $\boldsymbol{Y}$ has normal distribution with d-dimensional unknown mean vector $\boldsymbol{a}$ and a known $d \times d$ covariance matrix $\boldsymbol{\Sigma}$. Then

$$\boldsymbol{\Sigma}_0 \frac{\nabla_q(\boldsymbol{x})}{q(\boldsymbol{x}} = -\boldsymbol{x} + \frac{\boldsymbol{r}(\boldsymbol{x})}{q(\boldsymbol{x}} = \nabla_{\ln q}(\boldsymbol{x}) = \boldsymbol{\Sigma}^{-1}(\boldsymbol{x} - \boldsymbol{a}),$$

from where

$$\boldsymbol{\Sigma}_0 \frac{\nabla_q(\boldsymbol{x})}{q(\boldsymbol{x}} + \boldsymbol{x} = (\boldsymbol{\Sigma} - \boldsymbol{\Sigma}_0)\boldsymbol{\Sigma}^{-1}\boldsymbol{x} - \boldsymbol{\Sigma}_0 \boldsymbol{\Sigma}^{-1} \boldsymbol{a}.$$

Since $\boldsymbol{a}$ and $\boldsymbol{\Sigma}$ are unknown, we will estimate them on sample $\boldsymbol{y}_1, \boldsymbol{y}_2, ..., \boldsymbol{y}_n$ with the help of the following the statistics

$$\hat{\boldsymbol{a}} = \bar{\boldsymbol{y}} = \frac{1}{n} \sum_{i=1}^{n} \boldsymbol{y}_i \text{ and}$$

$$\hat{\boldsymbol{\Sigma}} = \mathbf{S} = \frac{1}{n} \sum_{i=1}^{n} (\boldsymbol{y}_i - \bar{\boldsymbol{y}})(\boldsymbol{y}_i - \bar{\boldsymbol{y}})^T.$$

The regression estimation in this case will be equal

$$\hat{\boldsymbol{u}}_n(\boldsymbol{x}) = (\mathbf{S} - \boldsymbol{\Sigma}_0)\mathbf{S}^{-1}\boldsymbol{x} + \boldsymbol{\Sigma}_0 \mathbf{S}^{-1} \bar{\boldsymbol{y}}.$$

If instead of $\boldsymbol{x}$ we will substitute observable value $\boldsymbol{y}_i$, then the corrected value of a vector $\hat{\boldsymbol{u}}_i$ we calculate the corrected value of a vector $\hat{\boldsymbol{u}}_i$ using the formula

$$\hat{\boldsymbol{u}}_i = \hat{\boldsymbol{u}}_n(\boldsymbol{y}_i) = (\mathbf{S} - \boldsymbol{\Sigma}_0)\mathbf{S}^{-1}\boldsymbol{y}_i + \boldsymbol{\Sigma}_0 \mathbf{S}^{-1} \bar{\boldsymbol{y}}.$$

7 Numerical properties

In this section, we report the results of the research of power of our test.

We consider the case when the initial data does not include measurement error, the case when a measurement error is superimposed on the initial data and the case when examines the data with overlay measurable error after conversion

For the error distributions, we consider the normal distributions $N(0{,}0.4^2)$, $N(0{,}0.8^2)$.

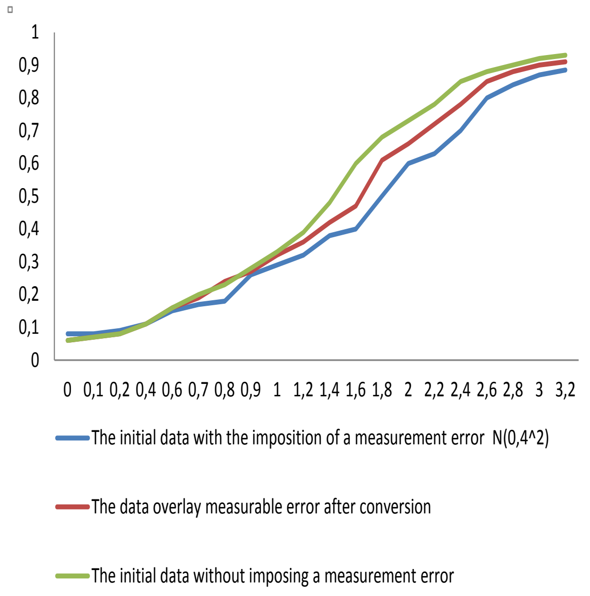

Fig. 6.2. Power functions for the initial data with the imposition a measurement error $N(0{,}0.4^2)$

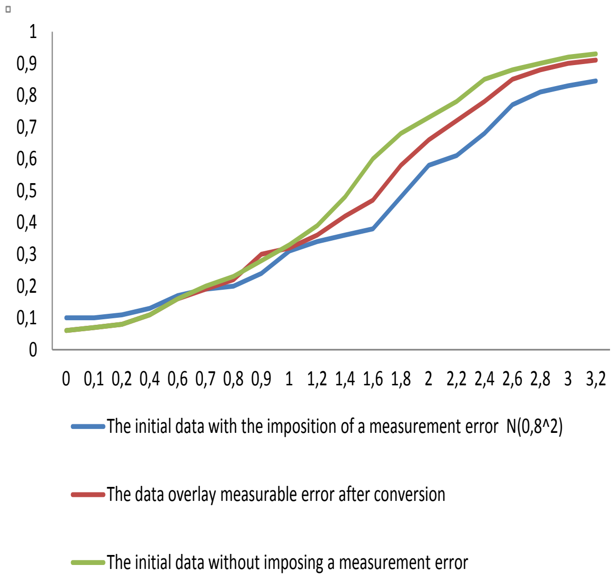

Fig. 6.4. Power functions for the initial data with the imposition a measurement error $N(0{,}0.8^2)$

By construction, in the application package MatLab graphs easy to see that in the case when a measurement error is superimposed on the initial data, the power of the test is less than in the case of direct obesrvations. But after converting the data with the superimposed measurement error power function shows good results (better than in the case with the imposition error) for different values of dispersion of the distribution error as seen from the Fig.6.1 - Fig.6.4.

8 Discussion

Numerical simulations shows that if at fixing the observations there is a measurement error, the power of the test is reduced and also it becomes displaced. To reduce the influence of errors, we apply the procedure to reduce the error by the algorithm described in paragraph 6. The graphs show that the statistical characteristics of these tests after this procedure improved. Namely, the capacity of the test becomes larger, the offset of the test decreases.

References:

[1] N.V. Smirnov, About distribution of Mises test $n\omega^2$. *Math. transactions*, Vol. 2(44), No.3, 1937, pp.973-993.

[2] T.W. Anderson, D.A. Darling, Asymptotic theory of certain goodness-of-fit criteria based on stochastic process. *Ann. Math. Statist.*, Vol. 23., No.2, 1952, pp.193-212.

[3] P. Hall, Central Limit Theorem for Integrated Square Error of Multivariate Nonparametric Density Estimators. *J. Multivar. Analysis*, Vol.14., 1984, pp.1-16.

[4] P. Hall, Integrated square error properties of kernel estimators of regression functions. *Ann. Statist.*, Vol.12, 1984, pp.241-260.

[5] M.S. Tikhov, D.S. Krishtopenko, Asymptotic Normality of the integrated square error of distribution function estimators in dependence dose-response indirect observations. *ASMDA: book of abstracts*. – Chania, Crete, Greece: ed. C.H.Skiadis, 2007, p.180.

[6] H. Cramér, On the composition of elementary errors. *Scandinavian Actuarial Journal*, Vol.11, 1928, pp.141-180.

[7] R.E. von Mises, *Wahrsheinlicheinkeit, Statistik und Wahrheit,* Julius Springer, 1928.

[8] M. Tikhov, D. Krishtopenko, Asymptotic normality of the integrated square error at the fixed plan of experiment for indirect observations. In «*Computer Modeling and New Technologies*», Riga, Latvia, 2007, Vol. 11, No.1, pp.46-56:

[9] M.S. Tikhov, Nonparametric estimation of effective doses at quantal response. *Ufa math. journal*, Vol.5, No.2, 2013, pp. 94-108.

[10] D.J. Finney, *Probit Analysis*, Cambridge Univer. Press , 2009, – 256 p.

[11] S. Krishtopenko, M. Tikhov, E. Popova, *Dose-effect*, Moskva, Medicina, 2008. (In Russian)

[12] M.S. Tikhov, Statistical Estimation on the Basis of Interval-Censored Data. *Journal Math. Sciences*, Vol. 119, No.3, 2004, pp. 321-335.

[13] H. Dette, N. Neumeyer, K.F. Pilz, A note on nonparametric estimation of the effective dose in quantal bioassay. *Journal Amer. Statist. Assoc*., Vol.100, No.470, pp.503-510.

[14] M. Tikhov, M. Ivkin, Multivariate k-Nearest Neighbors Distribution Function Estimates in Dose-effect Relationship. *Recent Advances in Mathematical Methods in Applied Sciences*: Proceedings of the 2014 International Conference on Mathematical Models and Methods in Applied Sciences (MMAS'2014), 2014, pp.325-329.

[15] M. Tikhov, M. Ivkin, A new Yang-type estimator of Distribution Function at Quantal Response over Indirect Data. *Wseas Transactions on Mathematics*, Vol. 13, 2014, pp. 684-693.

[16] M. Tikhov, T. Borodina, M. Ivkin, On Reduction of Measurement Errors at Estimation of Distributions in Dose-Effect Relationships. In *Advances in Mathematics and Statistical Sciences* Proceedings of the 2015 International Conference on Mathematical Models and Methods in Applied Sciences, Dubai 2015.

[17] H. Midi, S. Rana, S.K. Sarkar, Binary Response Modeling and Validation of its Predictive Ability. *Wseas Transactions on Mathematics*, Vol.9, No.6, 2010, pp. 436-447.

[18] J. Yang, L. Fan, Weighted Generalized Kernel Discriminant Analysis Using Fuzzy Memberships, *Wseas Transactions on Mathematics*, Vol.10, No.10, 2011, pp. 346-357.

[19] E.A. Nadaraya, On estimating regression – *Theory of Probability and its Applications*, Vol.15, No.1, 1964, pp.134-137.

[20] G.S. Watson, Smooth regression analysis – *Sankya*, Ser. A, Vol.26, 1964, pp.359-372.

[21] H. Niederreiter, *Random Number Generation and Quasi-Monte Carlo-Method*, Pensylvania, SIAM Philadelphia, 1992.

[22] R. Liptser, A. Shiryayev, *Theory of Martingales*, Kluwer Academic Publishers, Dortrecht, 1989 – 792 p.