A two-sided Iterative Method for Solving A Nonlinear Matrix Equation $X = A^* X^r A - I$

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Abstract— An efficient and numerical algorithm is suggested for finding the positive definite solutions of the matrix equation $X = A^* X^r A - I, r \geq 1$, where A is a nonsingular real matrix, if such solutions exist. The suggested technique is called the "Two-sided Iterative Process". Property of solutions is discussed thereof, necessary and sufficient conditions for existence of a positive definite solution are derived, and also the error analysis and the convergence rate are analyzed. Finally, two numerical examples are given to illustrate the effectiveness of the algorithm.

Keywords—convergence rate, non linear matrix equation, positive definite solution, two-sided method.

I. INTRODUCTION

IN recent years, studies of various physical structures of nonlinear equations have attracted much attention in connection with the important problems that arise in scientific applications. These physical structures of nonlinear equations have many forms such as ordinary and partial differential equations [1],[2], matrix equations [3]-[9], and nonlinear programming problems [10],[11].

In this paper, we only focused on matrix equation. The importance of matrix equations and its applications were given in [8] and the references therein. Also, different iterative methods for solving some kinds of matrix equation were given , such as, fixed point iteration [6],[8],[9],[12], [13], Newton method [14], SDA algorithm[7], LR algorithm [15], Butterfly SZ algorithm [3], and Two-sided Iterative Process[12].

In [8], we studied: nonlinear matrix equation of the form

$$X = A^* X^r A - I, r \geq 1, \tag{1}$$

where A is a square real matrix, and X is unknown square matrix. Two properties of a positive definite solution of (1) were discussed; first one, is related with the smallest, the largest eigenvalues of a solution X of (1), and an eigenvalues of A. The second one, gives the relation between the terms of (1).

An iterative method was proposed to compute the unique positive definite solution when $\|A\| > 1$, for $X = A^* X^r A - I$ with real square matrices and $r \geq 1$. The proposed method was based on the fixed point theorem. Moreover, necessary and sufficient conditions for the existence of the positive definite solutions were derived. The error estimation and general convergence results of the iterative method were also provided. For more convenience, we will mention all results in [8].

Theorem 1. If m and M are the smallest and the largest eigenvalues of a solution X of equation (1), respectively, and λ is an eigenvalues of A, then

$$\sqrt{\frac{m-1}{m^r}} \leq |\lambda| \leq \sqrt{\frac{M-1}{M^r}}.$$

Theorem 2. If equation (1) has a positive definite solution X, then

$$A^* X^r A > X \text{ and } X > \sqrt[r]{A^{*-1} A^{-1}}.$$

Algorithm 3.

Take $X_0 = 0$. For $k = 1, 2, \ldots.$ compute

$$X_k = \sqrt[r]{B^*(I + X_{k-1})B},$$

where $B = A^{-1}$, $B^* = A^{*-1}$.

Theorem 4. Let the sequence $\{X_k\}$ be determined by the Algorithm 3, $0 < \alpha I < B^* B < \frac{1}{2} I$ and if (1) has a positive definite solution, then $\{X_k\}$ converges to positive definite solution X. Moreover, if $X_k > 0$ for every k and $0 < \alpha I < B^* B < \frac{1}{2} I$, then (1) has a positive definite solution.

Theorem 5. Let X_k be the iterates in algorithm 3 and $0 < \alpha I < B^* B < \frac{1}{2} I$. If $q = \frac{\sqrt[r]{\alpha}}{2\alpha r} < 1$, then $\|X_k - X\| < q^k \|X\|$, where X is a positive definite solution of (1).

Corollary 6. Assume that (1) has a solution. If $q = \frac{\sqrt[r]{\alpha}}{2\alpha r} < 1$, then $\{X_k\}, k = 0,1,2,\ldots.$ converges to X with at least the linear convergence rate.

Theorem 7. If (1) has a positive definite solution and after k iterative steps of Algorithm 3 and we have $\|I - X_k^{-1} X_{k-1}\| < \varepsilon$, $\varepsilon > 0$, then

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i- $\left\| X_{k+1} - X_k \right\| < \frac{\sqrt[r]{\alpha}}{2\alpha r}\,\varepsilon$, and

ii- $\left\| B^* X_k B - X_k^r + B^* B \right\| < \frac{\sqrt[r]{\alpha}}{4\alpha r}\varepsilon.$

where X_k is the iterates in Algorithm 3.

This paper aims to find the positive definite solution of the matrix equation (1).

In this paper, mathematical induction technique will be used in the most proofs.

The following notations are used throughout [8] and the rest of the paper. The notation $A \ge 0$ $(A > 0)$ means that A is positive semidefinite (positive definite). For matrices A and B, we write $A \ge B (A > B)$ if $A - B \ge 0 (A - B > 0)$, and $\{X_k\}$ denotes the sequence $X_0, X_1, X_2, \ldots$. We denote by $\rho(A) = \|A\|$ the spectral radius of A. The norm used in this paper is the spectral norm of the matrix A unless otherwise noted.

II. PROPERTY OF THE SOLUTIONS AND THE ITERATIVE METHOD

In this section, we shall discuss the property of positive definite solutions of the matrix equation (1).

A two-sided algorithm for solving equation (1) is proposed , then, Lemma is briefly reviewed. Afterwards, it will be used to establish the conditions for the existence of a positive definite solution of (1) when the matrix A is nonsingular matrix and $B = A^{-1}$, $B^* = A^{*-1}$.

A. *Theorem 2.1*

If (1) has a positive definite solution *X*, then

$A^* A - \sqrt[r]{A^{*-1} A^{-1}} > I$.

Proof. Let X be a positive definite solution of equation (1), from **Theorem 2** and Lowner-Heinz inequality [17], we get

$$X > \sqrt[r]{A^{*-1} A^{-1}} \Rightarrow I + X > I + \sqrt[r]{A^{*-1} A^{-1}},$$

$$I > X = \sqrt[r]{A^{*-1}\left(I + X\right)A^{-1}} > \sqrt[r]{A^{*-1}\left(I + \sqrt[r]{A^{*-1}A^{-1}}\right)A^{-1}}.$$

Hence

$$A^* A - \sqrt[r]{A^{*-1} A^{-1}} > I .$$

B. *Algorithm 2.1*

Take $X_0 = \alpha I, Y_0 = \beta I$. For $k = 0,1,2,\ldots.$ compute

$$X_{k+1} = \sqrt[r]{B^*(I + X_k)B}, \tag{2}$$

$$Y_{k+1} = \sqrt[r]{B^*(I + Y_k)B}, \tag{3}$$

where $B = A^{-1}$, $B^* = A^{*-1}$.

C. *Lemma 2.1* [16]

Let f be an operator monotone function on $(0,\infty)$ and let *A*,*B* be two positive operators that are bounded below by a; i.e., $A \ge aI$ and $B \ge aI$ for the positive number *a*. Then for every spectral norm

$$\left\| f(A) - f(B) \right\| \le f'(a) \left\| A - B \right\| .$$

D. *Theorem 2.1.*

If there exist numbers α and β so that $0 < \alpha < \beta$ and the following conditions are satisfied

$$(i)\, \frac{\alpha^r}{(1+\alpha)} I < B^* B < \frac{\beta^r}{(1+\beta)} I,$$

$$(ii)\, q = \frac{\sqrt[r]{\alpha}\,\beta^r}{r\alpha\left(1+\beta\right)} < 1.$$

Then (1) has a positive definite solution X , where X is the same limit of the two sequences $\{X_k\}$ and $\{Y_k\}$ are defined in the Algorithm 2.1.

Proof. To prove the theorem, we will show that

$X_0 = \alpha I < X_k < X_{k+1} \le Y_{k+1} < Y_k < Y_0 = \beta I, k = 1,2,\ldots$ and $\|Y_k - X_k\| \to 0$ as $k \to \infty$.

From algorithm 2.1

$$X_1 = \sqrt[r]{B^*(I + X_0)B} = \sqrt[r]{B^*(I + \alpha I)B}$$

$$= \sqrt[r]{(1+\alpha)B^* B} > \sqrt[r]{\frac{\alpha^r}{(1+\alpha)} \cdot (1+\alpha) I}$$

$$= \alpha I = X_0 \Rightarrow X_1 > X_0.$$

Let $X_i > X_{i-1}$ is true when $k = i$, thus by using Lowner-Heinz inequality we get

$X_{i+1} = \sqrt[r]{B^*(I + X_i)B} > \sqrt[r]{B^*(I + X_{i-1})B} = X_i.$ That is $X_k < X_{k+1}, k = 0,1,2,\ldots$ and $\{X_k\}$ is increasing.

Next, we will prove $X_{k+1} < Y_{k+1}$, for $k = 0,1,2,\ldots$ From algorithm 2.1 , $Y_0 = \beta I > \alpha I = X_0$. Since $\beta I > \alpha I$, then $B^*(I + \beta I)B > B^*(I + \alpha I)B$, then by using Lowner-Heinz inequality, we get

$Y_1 = \sqrt[r]{B^*(I + \beta I)B} > \sqrt[r]{B^*(I + \alpha I)B} = X_1.$ Let $Y_i > X_i$ is true when $k = i$, thus by using Lowner-Heinz inequality, we get

$Y_{i+1} = \sqrt[r]{B^*(I + Y_i)B} > \sqrt[r]{B^*(I + X_i)B} = X_{i+1}.$ That is $Y_k > X_k, k = 0,1,2,\ldots$

Finally, we will prove that $Y_{k+1} < Y_k$, $k = 0,1,2,\ldots$. From algorithm 2.1, we get

$$Y_1 = \sqrt[r]{B^*(I + Y_0)B} = \sqrt[r]{B^*(I + \beta I)B} = \sqrt[r]{(1+\beta)B^* B}$$

$$< \sqrt[r]{\frac{\beta^r \cdot (1+\beta)}{1+\beta}} I = \beta I = Y_0 .$$

Let $Y_i < Y_{i-1}$ is true when $k = i$, thus by using Lowner-Heinz inequality, we get

$Y_{i+1} = \sqrt[r]{B^*(I + Y_i)B} < \sqrt[r]{B^*(I + Y_{i-1})B} = Y_i$. That is $Y_{k+1} < Y_k, k = 0,1,2,\ldots$ and $\{Y_k\}$ is decreasing.

Hence, we proved

$X_0 = \alpha I < X_k < X_{k+1} \le Y_{k+1} < Y_k < Y_0 = \beta I, k = 1,2,\ldots$

Now, we shall prove that $\|Y_n - X_n\| \to 0$ as $n \to \infty$ for that we have

$$\|Y_k - X_k\| = \left\| \sqrt[r]{B^*(I+Y_{k-1})B} - \sqrt[r]{B^*(I+X_{k-1})B} \right\|$$
$$= \left\| \sqrt[r]{P} - \sqrt[r]{Q} \right\|,$$

where $P = B^*(I+Y_{k-1})B$ and $Q = B^*(I+X_{k-1})B$. By using lemma 2.1, with a monotone operator $f(x) = \sqrt[r]{x}$, $r \ge 1$, and $Y_{k-1} > X_{k-1} \ge X_0 = \alpha I$ for all $k = 1,2,...$, then

$$P = B^*(I+Y_{k-1})B > B^*(I+\alpha I)B^* > \frac{\alpha^r(1+\alpha)}{(1+\alpha)} I = \alpha^r I = aI,$$

provided $a = \alpha^r \Rightarrow P > aI$, by the same manner, we prove $Q > aI$. Then

$$\|Y_k - X_k\| \le \frac{1}{r}\alpha^{\frac{1}{r}-1} \left\| B^*(I+Y_{k-1})B - B^*(I+X_{k-1})B \right\|$$

$$\le \frac{1}{r}\alpha^{\frac{1}{r}-1} \|B^*B\| . \|Y_{k-1} - X_{k-1}\|,$$

since $B^*B < \frac{\beta^r}{1+\beta} I$, then we have

$$\|Y_k - X_k\| < \frac{1}{r}\alpha^{\frac{1}{r}-1} \frac{\beta^r}{1+\beta} . \|Y_{k-1} - X_{k-1}\|,$$
$$< q . \|Y_{k-1} - X_{k-1}\|, \quad \text{after } k\text{-steps}$$
$$\|Y_k - X_k\| < q^k . \|Y_0 - X_0\| = q^k(\beta - \alpha),$$

where $q = \frac{\beta^r \sqrt[r]{\alpha}}{r\alpha(1+\beta)} < 1$, then $\|Y_n - X_n\| \to 0$ as $n \to \infty$, that is, $\lim\limits_{k\to\infty} Y_k = \lim\limits_{k\to\infty} X_k = X = A^* X^r A - I > 0$.

E. Theorem 2.2.

If $X_k > 0$ and $Y_k > 0$ for every k, $0 < \alpha < \beta$ and $\frac{\alpha^r}{(1+\alpha)} I < B^*B < \frac{\beta^r}{(1+\beta)} I$, then (1) has a positive definite solution.

Proof. Suppose $X_k > 0$ and $Y_k > 0$ for $k = 1,2,3,...$, we proved that the limits of $\{X_k\}$ and $\{Y_k\}$ are exist. Let $X_{k+1} = A^* X_k^r A - I > 0$ and $Y_{k+1} = A^* Y_k^r A - I > 0$, by taking the limits as $k \to \infty$, we have $X = A^* X^r A - I > 0$ and $Y = A^* Y^r A - I > 0$, Consequently, equation (1) has positive definite solution.

Hence the theorem is proved.

F. Theorem 2.3

Let X_k and Y_k be the iterates in algorithm 2.1, $\frac{\alpha^r}{(1+\alpha)} I < B^*B < \frac{\beta^r}{(1+\beta)} I$ and $0 < \alpha < \beta$. If $q = \frac{\beta^r \sqrt[r]{\alpha}}{r\alpha(1+\beta)} < 1$. Then $\|X_k - X\| < q^k(\beta - \alpha)$, and $\|Y_k - Y\| < q^k(\beta - \alpha)$, where X is a positive definite solution of (1).

Proof. From Theorem 2.1, it follows that the sequences (2) and (3) are convergent to a positive definite solution X of (1). We compute the spectral norm of the matrix $X_k - X$, we obtain

$$\|X_k - X\| = \left\| \sqrt[r]{B^*(I+X_{k-1})B} - \sqrt[r]{B^*(I+X)B} \right\| = \left\| \sqrt[r]{R} - \sqrt[r]{P} \right\|,$$

where $R = B^*(I+X_{k-1})B$, $P = B^*(I+X)B$. By using lemma 2.1, with a monotone operator $f(x) = \sqrt[r]{x}$; $r > 1$ and $X_{k-1} \ge X_0 = \alpha I$ for $k = 1,2,3,...$, then $R = B^*(I+X_{k-1})B \ge (1+\alpha)B^*B > \alpha^r I$, provided $a = \alpha^r$, $R > aI$ and by the same manner, we get $P > aI$.

Since $f(a) = \sqrt[r]{a}$, $f'(\alpha) = \frac{1}{r} \cdot \alpha^{\frac{1-r}{r}}$,

$$\|X_k - X\| \le \frac{1}{r} \cdot \alpha^{\frac{1-r}{r}} \|B^*B\| \, \|X_{k-1} - X\|,$$
$$< \frac{1}{r} \cdot \alpha^{\frac{1-r}{r}} \frac{\beta^r}{(1+\beta)} \|X_{k-1} - X\|,$$

let $q = \frac{\beta^r \sqrt[r]{\alpha}}{r\alpha(1+\beta)} < 1$, we have

$$\|X_k - X\| < q\|X_{k-1} - X\|, \text{ after } k\text{-steps,}$$
$$< q^k \|X_0 - X\|$$

From theorem 2.1 $X_0 = \alpha I < X_k < X_{k+1} \le Y_{k+1} < Y_k < Y_0 = \beta I$, $k = 0,1,2,...$ and $X_0 = \alpha I$, $Y_0 = \beta I$. Consequently, $\|X_k - X\| \le q^k \|X_0 - X\| < q^k(\beta - \alpha)$.

Similarly, we can prove $\|Y_k - X\| \le q^k \|Y_0 - X\| < q^k(\beta - \alpha)$.

G. Corollary 2.1.

Assume that (1) has a solution. If $q = \frac{\beta^r \sqrt[r]{\alpha}}{r\alpha(1+\beta)} < 1$, then $\{X_k\}$ and $\{Y_k\}$ are converging to X with at least the linear convergence rate.

Proof. As we have

$\|X_k - X\| < \frac{\beta^r \sqrt[r]{\alpha}}{r\alpha(1+\beta)} \|X_{k-1} - X\|$, $q = \frac{\beta^r \sqrt[r]{\alpha}}{r\alpha(1+\beta)} < 1$. Then choose a real number that satisfies $\sqrt{q} < \theta < 1$. Since $X_k \to X$, as $k \to \infty$ there exists a N, such that for any $k \ge N$, $\sqrt{q} \le \theta$.

Hence $\|X_{k+1} - X\| \le \theta^2 \|X_k - X\|$.

Similarly, we can prove

$\|Y_{k+1} - X\| \le \theta^2 \|Y_k - X\|$.

H. Theorem 2.4

If matrix equation (1) has a positive definite solution and after k iterative steps of Algorithm (1), the inequalities $\|X_k^{-1} X_{k-1} - I\| < \varepsilon_1$, $\|Y_k^{-1} Y_{k-1} - I\| < \varepsilon_2$ and $\|I - X_k^{-1} Y_k\| < \varepsilon_3$, imply i- $\left\|X_{k+1}-X_k\right\| < \frac{\sqrt[r]{\alpha}\,\beta^{r+1}}{r\alpha(1+\beta)}\ \varepsilon,$

ii- $\left\|Y_{k+1}-Y_k\right\| < \frac{\sqrt[r]{\alpha}\,\beta^{r+1}}{r\alpha(1+\beta)}\ \varepsilon,$

iii- $\left\|X_{k+1}-X_k\right\| \frac{\sqrt[r]{\alpha}\,\beta^{r+1}}{r\alpha(1+\beta)}\ \varepsilon,$

iv- $\left\|X_k - B^* X_k B - B^* B\right\| < \|B\|^2 \beta\varepsilon,$

v- $\left\|Y_k - B^* Y_k B - B^* B\right\| < \beta\varepsilon\|B\|^2,$

where $\{X_k\}$ and $\{Y_k\}$ $,k=0,1,2,....$ are the iterates generated positive definite solution of equation (1).

and $\varepsilon = min\{\varepsilon_1,\varepsilon_2,\varepsilon_3\} > 0$.

Proof. i- From algorithm 2.1 and lemma 3.1, then, take the norms of both sides,

$$\left\|X_{k+1}-X_k\right\| = \left\|\sqrt[r]{B^*(I+X_k)B} - \sqrt[r]{B^*(I+X_{k-1})B}\right\|,$$

$$\le \frac{1}{r}\cdot\alpha^{\frac{1}{r}-1}\left\|B^*B\right\|\left\|X_k - X_{k-1}\right\|,$$

$$< \frac{\alpha^{\frac{1}{r}-1}\beta^r}{r(1+\beta)}\ \left\|X_k\right\|.\left\|I - X_k^{-1}X_{k-1}\right\|.$$

From theorem 2.1 $\|X_k\| < \beta$ and $X_k \to X$, $X_k^{-1} \to X^{-1}$ and $X_{k-1} \to X$ as $k \to \infty$. Consequently, $\left\|I - X_k^{-1}X_{k-1}\right\| \to 0$ as $k \to \infty$, then $\left\|X_{k+1}-X_k\right\| < \frac{\sqrt[r]{\alpha}\,\beta^{r+1}}{r\alpha(1+\beta)}\ \varepsilon.$

ii- From theorem 2.1, $\|Y_k\| < \beta$ and $Y_k \to X$,

$Y_k^{-1} \to X^{-1}$ and $Y_{k-1} \to X$ as $k \to \infty$. By the same manner we can prove that $\left\|Y_{k+1}-Y_k\right\| < \frac{\sqrt[r]{\alpha}\,\beta^{r+1}}{r\alpha(1+\beta)}\ \varepsilon.$

iii- Similarly, we can prove $\left\|X_{k+1}-Y_{k+1}\right\| < \frac{\sqrt[r]{\alpha}\,\beta^{r+1}}{r\alpha(1+\beta)}\ \varepsilon.$

iv- $B^* X_k B - X_k^r + B^* B = B^* X_k B - X_k^r$

$-B^* X_{k-1} B + X_k^r = B^*(X_k - X_{k-1})B,$

then, take the norms of both sides,

$$\left\|B^* X_k B - X_k^r + B^* B\right\| = \left\|B^*(X_k - X_{k-1})B\right\|,$$

$$\le \|B\|^2\|X_k\|.\left\|I - X_k^{-1}X_{k-1}\right\|,$$

$$< \|B\|^2 \beta\varepsilon$$

v- Similarly, we can prove

$$\left\|Y_k - B^* Y_k B - B^* B\right\| < \|B\|^2 \beta\varepsilon.$$

III. Numerical Experiments

In this section, we give two numerical examples to illustrate that the matrix sequences $\{X_k\}$ and $\{Y_k\}$ generated by iterative method 2.1 are converging to the unique positive definite solution X of (1). The unique solution is computed for different nonsingular matrices *A.* All programs are written in MATHAMATICA. For the following examples, the practical stopping criterion is $max\{\|X-X_k\|,\|X-Y_k\|,\|X-Y_k\|\} \le 10^{-10}$, and the solution is $X = X_{500}$.

A. 4.1 Examples

1) Example 1

Consider the nonlinear matrix equation $X = A^* X^r A - I$, where

$$A = 10^{-3}\begin{pmatrix} 103711 & -37462 & -6577 & -422228 \\ -37462 & 30884 & -21447 & -14129 \\ -6577 & -21447 & 6005 & 21733 \\ 42228 & -1429 & 21733 & 22782 \end{pmatrix},$$

$\|A\| = 437.12$,and $\|B\| = 0.419732.$

For $r = 1.04$. Let $\alpha = 0.0003$ and $\beta = 2.01$. After 10 iterations, we get

$$X = 10^{-2}\begin{pmatrix} 0.0469959 & 0.553028 & 0.666889 & 0.476251 \\ 0.553028 & 6.69228 & 7.94655 & 5.75752 \\ 0.666889 & 7.94655 & 9.62647 & 6.8248 \\ 0.476251 & 5.75752 & 6.8248 & 4.99657 \end{pmatrix}.$$

For $r = 50$, Let $\alpha = 0.85$ and $\beta = 0.965$ After 5 iterations, we get

$$X = \begin{pmatrix} 0.854864 & 0.900758 & 0.9003511 & 0.898779 \\ 0.900758 & 0.94985 & 0.952281 & 0.947962 \\ 0.903511 & 0.952281 & 0.95523 & 0.950188 \\ 0.898779 & 0.947962 & 0.950188 & 0.946372 \end{pmatrix}.$$

For $r = 17.4$, Let $\alpha = 0.35$ and $\beta = 0.965$ After 7 iterations, we get

$$X = \begin{pmatrix} 0.637573 & 0.740929 & 0.747546 & 0.736091 \\ 0.740929 & 0.862891 & 0.86943 & 0.857697 \\ 0.747546 & 0.86943 & 0.877289 & 0.86375 \\ 0.736091 & 0.857697 & 0.86375 & 0.853213 \end{pmatrix}.$$

See Table I.

For $r = 100$, Let $\alpha = 0.0000015$ and $\beta = 0.9$ After 5 iterations, we get

$$X = \begin{pmatrix} 0.924561 & 0.949055 & 0.950498 & 0.948022 \\ 0.949055 & .974577 & 0.975815 & 0.973624 \\ 0.950498 & 0.975815 & 0.97732 & 0.974752 \\ 0.948022 & 0.973624 & 0.974752 & 0.972828 \end{pmatrix}.$$

See Table I.

For $r = 1000.5$, Let $\alpha = 0.00001$ and $\beta = 0.8$ After 4 iterations, we get

$$X=\begin{pmatrix} 0.992188 & 0.994784 & 0.994935 & 0.994677 \\ 0.994784 & .997427 & 0.997552 & 0.997331 \\ 0.994935 & 0.997552 & 0.997706 & 0.997445 \\ 0.994677 & 0.997331 & 0.997445 & 0.997251 \end{pmatrix}.$$

See Table I.

For $r = 10000$, Let $\alpha = 0.99$ and $\beta = 1.00098$. After 3 iterations, we get

$$X=\begin{pmatrix} 0.992116 & 0.999477 & 0.999492 & 0.999466 \\ 0.999477 & 0.999742 & 0.999755 & 0.999733 \\ 0.999492 & 0.999755 & 0.99077 & 0.999744 \\ 0.999466 & 0.999733 & 0.999744 & 0.999725 \end{pmatrix}.$$

See Table I.

2) Example 2

Consider the nonlinear matrix equation $X = A^* X^r A - I$, where $A = 50H$, with size m and

$$H=\left(h_{ij}\right):\ h_{ij}=\begin{cases} .01 & i<j, \\ \dfrac{50}{j+1} & i=j, \\ -(i+j) & i>j. \end{cases}$$

I- For $m=10$, $\|A\| = 1459.87$, $\|B\| = 0.044384$, $r = 15.2$. Let $\alpha = 0.6$ and $\beta = 0.95$. After 7 iterations, we get approximate solution. See Table II.

II- For $m=100$, $\|A\| = 127087$, $\|B\| = 0.0197004$, $r = 120.5$. Let $\alpha = 0.005$ and $\beta = 1.001$. After 7 iterations, we get approximate solution. See Table II.

III- For $m=500$, $\|A\| = 1.645E{+}6$, $\|B\| = 0.0199563$, $r = 25$. Let $\alpha = 0.9$ and $\beta = 1.09$. After 7 iterations, we get approximate solution. See Table II.

B. 4.2 Tables

In the following tables we denote $\delta_x = \| X - X_n \|$, $\delta_y = \| X - Y_n \|$ and $\delta_{xy} = \| X_n - Y_n \|$.

Table I: Error analysis for Example 1 for different values of *r*

r	K	δ_x	δ_y	δ_{xy}
1.04	1	1.5932E-2	0.354075	0.369909
	3	6.66571E-5	1.76403E-3	1. 883069E-3
	5	2.74056E-7	7.27049E-6	7.54454E-6
	7	1.12642E-9	2.98844E-8	3.10108E-8
	9	4.6298E-12	1.2283E-10	1.3746E-10
	10	2. 96831E-13	7.8747E-12	8.17153E-12*
17.4	1	9.73142E-3	7.08819E-2	7.03942E-2
	3	2.1816E-6	1.45552E-5	1.23752E-5
	5	2.7137E-10	2.11828E-9	1.84691E-9
	6	3.43809E-12	2.64854E-11*	2.30473E-12
50	1	2.45169E-2	2.89306E-2	4.50058E-3
	3	5.64664E-7	6.51877E-7	8.7217E-8
	5	9.25516E-12	1.07506E-11*	1.49548E-12
100	1	0.0114875	0.013866	0.0247108
	3	3.96776E-8	7.6275E-8	1.15944E-7
	5	1.82295E-13	3.06488E-13	4.8781E-13*
1000.5	1	1.1774E-3	1.23687E-3	2.34085E-3
	2	1.97269E-7	1.97269E-7	5.91476E-7
	3	8.4096E-11	8.4096E-11	1.06143E-10
	4	8.4096E-16	1.24263E-14	2.08354E-14*
10000	1	1.63614E-4	1.6578E-4	2.2002E-6
	2	4.99117E-9	5.04667E-9	5.55592E-11
	3	8.53209E-14	8.62992E-14*	9.83845E-16

Table II: Error analysis for Example 2 for different values of *r*

r	m	K	δ_x	δ_y	δ_{xy}
15.2	**10**	1	0.182905	0.265667	0.0828854
		3	1.529297E-4	1.94859E-4	4.19349E-5
		5	1.15423E-7	1.39777E-7	2.44109E-8
		7	9.45465E-11	1.13843E-10*	1.93216E-11
25	**100**	1	6.99179	8.07667	1.08488
		3	1.25592E-5	1.43782E-5	1.87561E-6
		5	2.46831E-9	1.95518E-9	5.18404E-10
		7	4.97063E-12	4.01735E-12	9.54027E-12*
100.5	**500**	1	2.61788E-3	0.525154	.0522543
		3	9.98343E-8	3.97474E-8	2.01066E-8
		5	1.57493E-11	1.01759E-11	5.57798E-11*

IV. Conclusion

In this paper, a two-sided iterative process for the matrix equation was investigated. The novel idea here is that the two sequences were obtained by starting with two different provided values (a) an interval in which the solution is located, that is, $X_k < X < Y_k$ for all k; and (b) a better stopping criterion. Property of solution was discussed, as well, and sufficient solvability conditions on a matrix A were derived. Moreover, general convergence results for the suggested iteration for equation1 were given. Some numerical examples were presented to show the usefulness of the iterations.

The two-sided iteration method described above possesses some advantages. We can compute X_{k+1} and Y_{k+1} in parallel [12] and if the conditions of Theorems 2.1 are satisfied, we can calculate the solution X of (1) for any power of X in (1) as we see in examples 1 and 2, while this cannot be calculated for one-sided iteration method. It is also easy to propose a stopping criteria, using

$max\left\{\|X_k - X\|, \|Y_k - X\|\right\} < \|Y_k - X_k\|$, or

$max\left\{\|X_k - X\|, \|Y_k - X\|, \|Y_k - X_k\|\right\} < tolerance$,

which are not applicable for one-sided iteration methods. Here we consider the case when A is a non singular.

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