Fuzzified weighted OWA (FWOWA) operator

Pavel Holeček, Jana Talašová, Tomáš Talášek

***Abstract*—In practice, it is often necessary to combine information from different sources. For this task, one of aggregation operators can be used. Probably the best-known of them are the weighted average and the OWA (ordered weighted average) operator. The weighted average makes it possible to assign importances to the individual information sources. On the other hand, with the OWA operator, the importances are assigned to the aggregated values according to their order. In cases when we need to combine these two approaches, the weighted OWA (WOWA) operator can be used.**

The situation when the aggregated values are not known precisely is very common in the practice. That is why fuzzified versions of various aggregation operators begun to emerge. In this paper, a fuzzified WOWA operator, which can aggregate values expressed by fuzzy numbers, will be presented. The version studied in this paper is based on Zadeh's extension principle.

The behavior of the presented fuzzified WOWA operator will be demonstrated on an illustrative example and it will be compared to a different approach to the fuzzification of the WOWA and to another aggregation operator generalizing the fuzzy weighted average and the fuzzy OWA operator. Finally, a software tool for the fuzzified WOWA calculation will be mentioned.

***Index Terms*—WOWA, Weighted OWA, Fuzzification, Aggregation operators, Fuzzy numbers, Extension principle**

I. Introduction

Aggregation operators have many uses in the practice. This paper will focus mainly on their application in the multiple-criteria evaluation, but the scope of their use is much wider. Specifically, in the area of multiple-criteria evaluation, they can be used to obtain the overall evaluation from the evaluations according to different criteria.

Probably the best-known and the most often used aggregation operator is a weighted average. For the weighted average, a vector of weights is given. The weights express the importances of the particular information sources (in our case the importances of the particular criteria).

Another widely used aggregation operator is the ordered weighted average (OWA), which has been proposed by Yager in [18]. The OWA operator requires again a vector of weights to be set but, contrary to the weighted average, the weights are not assigned to the particular aggregated criteria, but to the aggregated values according to their order. Different choice of the weights leads to the family of functions between the minimum and the maximum.

Sometimes, a combination of the two mentioned approaches is required. That is why aggregation operators that generalize both the weighted average and the OWA operator have been proposed. The most significant of them is the WOWA operator introduced by Torra [15]. The WOWA operator requires two vectors of weights – the first one is connected to the individual criteria just like in case of the weighted average and the second one is assigned to the order of the aggregated values as in the OWA operator.

The aggregated values are not always known precisely. If the criteria values are estimated by an expert, they always contain some uncertainty. Moreover, if the values have been obtained by some measurement tool (e.g. thermometer), their accuracy is limited by the accuracy of the tool. Therefore, it is meaningful to express the aggregated values by means of fuzzy numbers, which can reflect this uncertainty.

Many fuzzified aggregation operators has been proposed. The fuzzy weighted average has been introduced in [2] and the fuzzification of OWA followed [20]. Later, another version of fuzzy weighted average, based on the extension principle and using a special structure called normalized fuzzy weights, has been introduced together with a computationally effective algorithm [13]. The fuzzification of the OWA operator based on the same ideas has been devised in [14].

In the mentioned paper [14], the authors have divided the fuzzified aggregation operators into two groups – the first-level and the second-level fuzzy aggregation operator. The first group allows the aggregated values to be fuzzy numbers while the rest of the parameters (weights) remain crisp [1]. In the latter group, the parameters of the aggregation operators are also expressed by fuzzy numbers. In the mentioned paper, very simple methods of calculating the fuzzy weighted average and fuzzy OWA for the first group have been devised.

In this paper, a first-level fuzzification of the WOWA operator will be described. The considered fuzzified WOWA operator is able to aggregate the values expressed by fuzzy numbers, but the weights remain crisp. As this is not the first attempt to generalize the WOWA operator so that it could handle uncertain values [8], the comparison with the already existing approaches will be provided. Our version is based on Zadeh's extension principle.

The paper is structured as follows. First, the weighed average, the OWA and the WOWA operator will be described and the behavior of the WOWA operator will be explained briefly. Next, basic notions from the fuzzy set theory will be given. In the following section, the fuzzification of the WOWA operator will be defined and its relationship to the fuzzy weighted average and fuzzy OWA operator will be explained and proven. Subsequently, our approach will be compared to the other approaches and the results will be shown on an illustrative example. Finally, a software tool that uses the proposed fuzzified WOWA will be introduced so that the readers would be able to examine its behavior and apply it in the practice on their own.

[1] In the context of the fuzzy set theory, the word "crisp" means "non-fuzzy". Therefore, by crisp weights, we mean weights expressed by real numbers.

II. Weighted average, OWA and WOWA operators in the crisp case

The weighted average is a simple aggregation operator that has countless applications in the practice. For its use, normalized weights have to be set first.

Definition 1: Real numbers $p_1, \ldots, p_m$ are said to be normalized weights if $p_i \in [0, 1]$ for all $i = 1, \ldots, m$ and $\sum_{i=1}^{m} p_i = 1$.

The weights express the importances of each argument that should be aggregated. Then, the weighed average is defined as follows.

Definition 2: A weighted average of the values $u_1, \ldots, u_m$, $u_i \in \Re$, $i = 1, \ldots, m$, with normalized weights $\vec{p} = (p_1, \ldots, p_m)$ is defined as

$$\mathrm{WA}^{\vec{p}}(u_1, \ldots, u_m) = \sum_{i=1}^{m} p_i \cdot u_i. \tag{1}$$

Another well-known aggregation operator is the OWA proposed by Yager [18]. Again, a vector of weights needs to be given first. In comparison to the weighted average, the weights are, however, connected to the decreasing order of the aggregated evaluations (i.e. the first weight is connected to the largest of the aggregated values, the second weight to the second-largest one, etc.).

Definition 3: The ordered weighted average (OWA) of the values $u_1, \ldots, u_m$, $u_i \in \Re$, $i = 1, \ldots, m$, with normalized weights $\vec{w} = (w_1, \ldots, w_m)$ is defined as

$$\mathrm{OWA}_{\vec{w}}(u_1, \ldots, u_m) = \sum_{i=1}^{m} w_i \cdot u_{\phi(i)}, \tag{2}$$

where ϕ denotes such a permutation of the set of indices $\{1, ..., m\}$ that $u_{\phi(1)} \geq u_{\phi(2)} \geq \ldots \geq u_{\phi(m)}$.

A WOWA proposed in [15] is an aggregation operator that generalizes both the weighted average and the OWA operator. It uses two vectors of weights. The vector $\vec{p} = (p_1, p_2, \ldots, p_m)$ has the same interpretation as in the case of the weighted average – the weights are connected to the individual values that should be aggregated. The second vector of weights, $\vec{w} = (w_1, w_2, \ldots, w_m)$, is connected to the decreasing order of the aggregated values, exactly as in case of the OWA. The WOWA operator is given by the following definition.

Definition 4: The weighted ordered weighted average (WOWA) of the values $u_1, \ldots, u_m$, $u_i \in \Re$, $i = 1, \ldots, m$, with normalized weights $\vec{p} = (p_1, \ldots, p_m)$ and $\vec{w} = (w_1, \ldots, w_m)$ is defined as:

$$\mathrm{WOWA}^{\vec{p}}_{\vec{w}}(u_1, \ldots, u_m) = \sum_{i=1}^{m} \omega_i \cdot u_{\phi(i)}, \tag{3}$$

where the permutation ϕ is the same as in the OWA, i.e., ϕ denotes such a permutation of the set of indices $\{1, \ldots, m\}$ that $u_{\phi(1)} \geq u_{\phi(2)} \geq \ldots \geq u_{\phi(m)}$. The weight ω_i is defined as

$$\omega_i = z(\sum_{j \leq i} p_{\phi(j)}) - z(\sum_{j < i} p_{\phi(j)}), \tag{4}$$

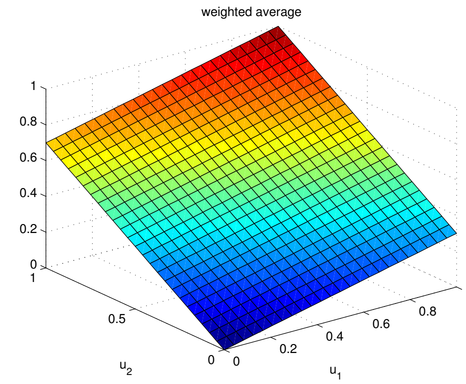

Fig. 1: Weighted average with $\vec{p} = (0.3, 0.7)$

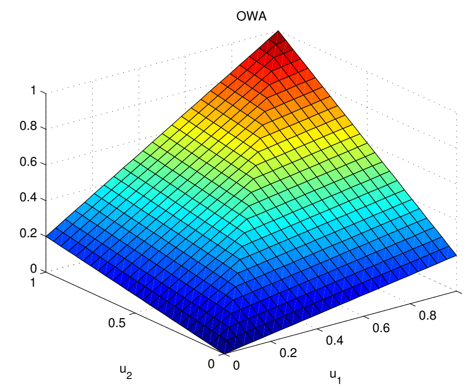

Fig. 2: OWA with the weights $\vec{w} = (0.2, 0.8)$

for $i = 1, \ldots, m$, and z is a nondecreasing function interpolating the following points

$$\{(0, 0)\} \cup \{(i/m, \sum_{j \leq i} w_j)\}_{i=1,\ldots,m}. \tag{5}$$

The function z is required to be linear when the points can be interpolated in that way.

The above-mentioned condition on the function z is rather technical and it is required in order to ensure that the WOWA operator generalizes the weighted average [15].

Although several ways of creating the function z has been discussed in the literature (e.g. [17]), we will use the simplest one in the examples in this paper – z will be a piece-wise linear function connecting the mentioned points.

One feature of the WOWA operator is that it generalizes both the weighted average and OWA. Let $\vec{\eta}$ denotes a vector of uniform weights, i.e. $\vec{\eta} = (\frac{1}{m}, \frac{1}{m}, \ldots, \frac{1}{m})$. Then $\mathrm{WOWA}^{\vec{p}}_{\vec{\eta}} = \mathrm{WA}^{\vec{p}}$ and $\mathrm{WOWA}^{\vec{\eta}}_{\vec{w}} = \mathrm{OWA}_{\vec{w}}$. The proof can be found in [15].

It can appear that WOWA creates the mixture weights in a very complicated way. However, comparing the graphs of the weighted average (Figure 1), OWA (Figure 2) and WOWA (Figure 3) might give an insight into the way how the weights are mixed. Looking at the figures, we can see that the graph of the WOWA is composed of the OWA graph rotated according to the weights for the weighted average.

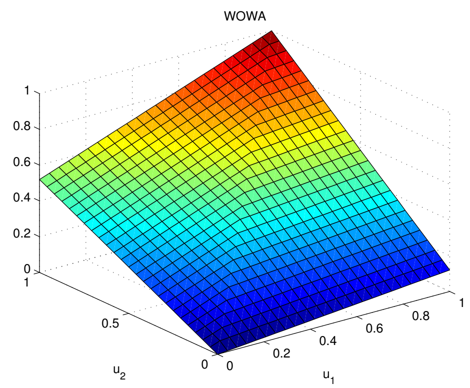

Fig. 3: WOWA with $\vec{p} = (0.3, 0.7)$ and $\vec{w} = (0.2, 0.8)$

Some important properties have been studied in [15] – it has been proven that the WOWA operator is idempotent, monotone in relation to the input variables $u_1, \ldots, u_m$ and that it satisfies a boundary condition (its result remains between the minimum and the maximum). It has been also shown that the WOWA is a special case of the Choquet integral with a particular fuzzy measure [16].

The WOWA has found its application in diverse areas such as medicine [11], reputation management [7], or web search [10]. The FWOWA operator that is studied in this paper broadens the application area even more because it makes it possible to aggregate data that are uncertain or even missing as it will be shown later. Before we proceed to the definition of this operator, we will mention some basic notions from the fuzzy set theory that will be used in this paper

III. Basic notions of the fuzzy set theory

The aggregated values are seldom known precisely. Therefore, it is convenient to express them by fuzzy numbers instead of real numbers. In this paper, a new approach to fuzzification of the WOWA operator will be presented. First, some basic notions from the fuzzy set theory that will be used in this paper will be mentioned. The reader can find more detailed information on the fuzzy set theory e.g. in [3].

A fuzzy set A on a universal set X is characterized by its membership function $A : X \to [0, 1]$. Ker A denotes a kernel of A, Ker $A = \{x \in X \mid A(x) = 1\}$. A support of A is defined as Supp $A = \{x \in X \mid A(x) > 0\}$. For any $\alpha \in [0, 1]$, A_α denotes an α-cut of A, $A_\alpha = \{x \in X \mid A(x) \geq \alpha\}$.

A fuzzy number is a fuzzy set C on the set of all real numbers $\Re$ which satisfies the following conditions: a) the kernel of C is not empty, b) the α-cuts of C are closed intervals for all $\alpha \in (0, 1]$, c) the support of C is bounded. In the following text, the symbol $\mathcal{F}_N$ will denote the set of all fuzzy numbers.

Real numbers $c^1 \leq c^2 \leq c^3 \leq c^4$ are called significant values of the fuzzy number C if the following holds: $[c^1, c^4] = \mathrm{Cl}(\mathrm{Supp}\ C), [c^2, c^3] = \mathrm{Ker}\ C$, where Cl(Supp C) denotes a closure of Supp C.

A fuzzy number C is said to be linear, if its membership function between each pair of the neighboring significant values is linear. The linear fuzzy number C is called triangular if $c^2 = c^3$, otherwise it is called trapezoidal.

In the examples, linear fuzzy numbers will be described by their significant values. Therefore, we will write such a fuzzy number as $C = (c^1, c^2, c^3, c^4)$ if C is trapezoidal, or simply $C = (c^1, c^2, c^4)$ if C is a triangular fuzzy number.

Obviously, fuzzy numbers can be also used to express real numbers. If for the fuzzy number C it holds that $c^1 = c^2 = c^3 = c^4 = c$, for some $c \in \Re$, this fuzzy number can be seen as a representation of a real number c. Such a fuzzy number will be called a fuzzy singleton containing the element c

Any fuzzy numbers C can be described in an alternative way, which is often very convenient for calculations. It can be characterized by a pair of functions $\underline{c} : [0, 1] \to \Re$, $\overline{c} : [0, 1] \to \Re$ defined as $C_\alpha = [\underline{c}(\alpha), \overline{c}(\alpha)]$ for all $\alpha \in (0, 1]$ and $\mathrm{Cl}(\mathrm{Supp}\ C) = [\underline{c}(0), \overline{c}(0)]$. The functions $\underline{c}(\alpha)$ and $\overline{c}(\alpha)$ thus represent the lower and the upper bounds of the respective α-cuts with a technical exception for $\alpha = 0$.

A fuzzy number C can be then denoted as $C = \big\{[\underline{c}(\alpha), \overline{c}(\alpha)], \alpha \in [0, 1]\big\}$ using these two mentioned functions. For example a triangular fuzzy number $C = (1, 3, 4)$ can be written as $C = \big\{[1 + 2\alpha,\ 4 - \alpha],\ \alpha \in [0, 1]\big\}$.

In the rest of the text, $U_1, \ldots, U_m$ will denote the fuzzy numbers that should be aggregated and U will denote the result of such an aggregation. For all of these fuzzy numbers, the mentioned notation will be used, i.e. $U_i = \{[\underline{u}_i(\alpha), \overline{u}_i(\alpha)], \alpha \in [0, 1]\}$, for $i = 1, \ldots, m$, and $U = \{[\underline{u}(\alpha), \overline{u}(\alpha)], \alpha \in [0, 1]\}$.

An important notion in the fuzzy set theory is the extension principle introduced by Zadeh [19]. In this paper we will use its special case [14], according to which, any real continuous function f of n real arguments can be extended to a FNV-function (a fuzzy-number-valued function) of n FNV-arguments.

Definition 5: Let $f : \Re^n \to \Re$ be a real continuous function of n variables. Then its fuzzy extension is a mapping $F : \mathcal{F}_N^n \to \mathcal{F}_N$ assigning to any n-tuple of fuzzy numbers $C_1, \ldots, C_n$ a fuzzy number $D = F(C_1, \ldots, C_n)$ whose membership function is for any $y \in \Re$ defined as follows:

$$D(y) = \begin{cases} \max\big\{\min\{C_1(x_1), \ldots, C_n(x_n)\} \mid \\ \qquad y = f(x_1, \ldots, x_n),\ x_i \in \Re, \\ \qquad i = 1, 2, \ldots, n\big\} & \text{if } f^{-1}(y) \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases} \tag{6}$$

The requirement that f is continuous guarantees that the result of the function F will be always a fuzzy number (without this requirement the result would be, generally, a fuzzy set). The following theorem (see [12] for more information) makes it possible to obtain the function F in an easier way, providing that the original function f is moreover non-decreasing.

Theorem 1: Let $f : \Re^n \to \Re$ be a real continuous function of n variables non-decreasing in those variables, and F be the fuzzy extension of f. Let $C_i = \{[\underline{c}_i(\alpha), \overline{c}_i(\alpha)], \alpha \in [0, 1]\}$,

$i = 1, \dots, n$, be fuzzy numbers. Then for the value $D = F(C_1, \dots, C_n)$, $D = \{[\underline{d}(\alpha), \overline{d}(\alpha)], \alpha \in [0,1]\}$, it holds that

$$\underline{d}(\alpha) = f(\underline{c}_1(\alpha), \underline{c}_2(\alpha), \dots, \underline{c}_n(\alpha)), \quad (7)$$
$$\overline{d}(\alpha) = f(\overline{c}_1(\alpha), \overline{c}_2(\alpha), \dots, \overline{c}_n(\alpha)). \quad (8)$$

IV. Weighted average and OWA in the fuzzy case

Using the extension principle, the fuzzified weighted average has been introduced [2], [13]. Even though it is possible to express both the aggregated values and the weights in form of fuzzy numbers, for our purposes, we will focus on a simpler so-called first-level fuzzification [14] where the aggregated values are expressed by fuzzy numbers, but the weights remain crisp. The extension of this approach, the second-level fuzzy weighted average, which allows both the aggregated values and the weights to be expressed by fuzzy number is studied e.g. in [13]. In the same paper, an effective algorithm for its calculation is proposed.

The first-level fuzzy weighted average is defined as follows.

Definition 6: Let $U_1, \dots, U_m$ be fuzzy numbers and let $p_1, \dots, p_m$ be normalized (real) weights. Then the first-level fuzzy weighted average of the values $U_1, \dots, U_m$ with the weights $p_1, \dots, p_m$ is a fuzzy number U with the membership function given for any $y \in \Re$ as follows:

$$U(y) = \max\big\{ \min\{U_1(u_1), \dots, U_m(u_m)\} \mid u_i \in \Re,\ i = 1, \dots, m,\ y = \sum_{i=1}^{m} p_i u_i \big\}. \quad (9)$$

Similarly, the fuzzy OWA operator has been defined [20]. Again, for our purposes we will consider only its first-level fuzzification [14] where the aggregated values are in form of fuzzy numbers, but the weights are crisp. A definition of the second-level fuzzy OWA, which uses the weights expressed by fuzzy numbers, together with a computationally effective algorithm can be found in [14].

Definition 7: Let $U_1, \dots, U_m$ be fuzzy numbers and let $w_1, \dots, w_m$ be normalized (real) weights. Then the first-level fuzzy OWA of the values $U_1, \dots, U_m$ with the weights $w_1, \dots, w_m$ is a fuzzy number U with the membership function given for any $y \in \Re$ as follows:

$$U(y) = \max\big\{ \min\{U_1(u_1), \dots, U_m(u_m)\} \mid u_i \in \Re,\ i = 1, \dots, m,\ y = \sum_{i=1}^{m} w_i u_{\phi(i)} \big\}, \quad (10)$$

where ϕ denotes such a permutation of the set of indices $\{1, \dots, m\}$ that $u_{\phi(1)} \geq u_{\phi(2)} \geq \dots \geq u_{\phi(m)}$.

The following two theorems present an easy way for computation of the (first-level) fuzzy weighted average and the (first-level) fuzzy OWA operator.

Theorem 2: The (first-level) fuzzy weighted average of the fuzzy numbers $U_1, \dots, U_m$ with normalized weights $p_1, \dots, p_m$ is a fuzzy number U that can be calculated, for any $\alpha \in [0,1]$, as follows

$$\underline{u}(\alpha) = \sum_{i=1}^{m} p_i \cdot \underline{u}_i(\alpha), \quad (11)$$
$$\overline{u}(\alpha) = \sum_{i=1}^{m} p_i \cdot \overline{u}_i(\alpha). \quad (12)$$

Proof: The proof can be found in [1]. □

Theorem 3: The (first-level) fuzzy OWA of the fuzzy numbers $U_1, \dots, U_m$ with normalized weights $w_1, \dots, w_m$ is a fuzzy number U, which can be obtained by the following formulae for any $\alpha \in [0,1]$:

$$\underline{u}(\alpha) = \sum_{i=1}^{m} w_i \cdot \underline{u}_{\sigma(i)}(\alpha), \quad (13)$$
$$\overline{u}(\alpha) = \sum_{i=1}^{m} w_i \cdot \overline{u}_{\chi(i)}(\alpha), \quad (14)$$

where σ and χ are such permutations of the set of indices $\{1, \dots, m\}$ that $\underline{u}_{\sigma(1)} \geq \underline{u}_{\sigma(2)} \geq \dots \geq \underline{u}_{\sigma(m)}$ and $\overline{u}_{\chi(1)} \geq \overline{u}_{\chi(2)} \geq \dots \geq \overline{u}_{\chi(m)}$.

Proof: The proof can be found in [1]. □

V. The fuzzified WOWA operator

The aim of the paper is to introduce the fuzzified WOWA operator that generalizes the both of the previous operators. Its first-level fuzzification, considered in this text, is able to aggregate the values given by fuzzy numbers $U_1, \dots, U_m$. However, the weights $\vec{p} = (p_1, \dots, p_m)$ and $\vec{w} = (w_1, \dots, w_m)$ are crisp.

This aggregation operator has been already mentioned in [5] in an intuitive way. However, this paper strives to define it properly and study some of its properties. The fuzzified WOWA is defined according to the extension principle as follows.

Definition 8: Let $U_1, \dots, U_m$ be fuzzy numbers and let $\vec{p} = (p_1, \dots, p_m)$ and $\vec{w} = (w_1, \dots, w_m)$ be two vectors of normalized (real) weights. Then the result of the aggregation by a fuzzified WOWA operator is a fuzzy number U with the membership function defined for any $y \in \Re$ as

$$U(y) = \max\big\{ \min\{U_1(u_1), \dots, U_m(u_m)\} \mid u_i \in [0,1], i = 1, \dots, m,\ y = \mathrm{WOWA}_{\vec{w}}^{\vec{p}}(u_1, u_2, \dots, u_m) \big\}. \quad (15)$$

The definition is not very convenient for calculations but the following theorem makes it possible to calculate the fuzzified WOWA directly.

Theorem 4: The result of the fuzzified WOWA of the fuzzy numbers $U_1, \dots, U_m$ with the weights $\vec{p} = (p_1, \dots, p_m)$ and $\vec{w} = (w_1, \dots, w_m)$ is a fuzzy number U defined for any $\alpha \in [0,1]$ as follows

$$\underline{u}(\alpha) = \mathrm{WOWA}_{\vec{w}}^{\vec{p}}(\underline{u}_1(\alpha), \underline{u}_2(\alpha), \dots, \underline{u}_m(\alpha)), \quad (16)$$
$$\overline{u}(\alpha) = \mathrm{WOWA}_{\vec{w}}^{\vec{p}}(\overline{u}_1(\alpha), \overline{u}_2(\alpha), \dots, \overline{u}_m(\alpha)). \quad (17)$$

Proof: Applying Theorem 1 to Definition 8, we obtain Formulae 16 and 17. The required monotonicity has been proven in [15] and continuity is derived from the fact that the WOWA is a special case of the Choquet integral with a particular fuzzy measure [16], which is monotone [4]. □

When the fuzzified WOWA is calculated, the used mixtures weights for WOWA in Formulae 16 and 17 can differ and, moreover, they need not to be the same throughout different α-cuts. This is a significant difference to the other approaches. For instance, the approaches based on the fuzzy numbers arithmetic struggle with the fact that the fuzzy numbers can be incomparable and that, in these cases, it is not possible to find a single permutation that would order them. They usually deal with the problem by replacing the fuzzy numbers with some of their characteristics (such as centers of gravity), which are real numbers and therefore can be easily ordered. This way, one "average" ordering is used neglecting the possible incomparability completely.

In the next step, we will show that the presented fuzzified WOWA operator reduces to the first-level fuzzy weighted average if the weights $w_1, \ldots, w_m$ are uniform.

Theorem 5: Let $U_1, \ldots, U_m$ be fuzzy numbers. and $\vec{p}$ be a vector of normalized weights. Further, let $\vec{w}$ be a vector of uniform real weights, $\vec{w} = \vec{\eta}$. Then the result of the fuzzified WOWA of $U_1, \ldots, U_m$ with the weights $\vec{p}$ and $\vec{w}$ is identical to the result of fuzzy weighted average of $U_1, \ldots, U_m$ with the weights $\vec{p}$.

Proof: Because $\mathrm{WOWA}_{\vec{\eta}}^{\vec{p}} = \mathrm{WA}^{\vec{p}}$ (for proof see [15]), we obtain

$$\begin{aligned}\underline{u}(\alpha) &= \mathrm{WOWA}_{\vec{\eta}}^{\vec{p}}(\underline{u}_1(\alpha), \underline{u}_2(\alpha), \ldots, \underline{u}_m(\alpha)) \\ &= \mathrm{WA}^{\vec{p}}(\underline{u}_1(\alpha), \underline{u}_2(\alpha), \ldots, \underline{u}_m(\alpha)) \\ &= \sum_{i=1}^{m} p_i \cdot \underline{u}_i(\alpha), \qquad (18)\\ \overline{u}(\alpha) &= \mathrm{WOWA}_{\vec{\eta}}^{\vec{p}}(\overline{u}_1(\alpha), \overline{u}_2(\alpha), \ldots, \overline{u}_m(\alpha)) \\ &= \mathrm{WA}^{\vec{p}}(\overline{u}_1(\alpha), \overline{u}_2(\alpha), \ldots, \overline{u}_m(\alpha)) \\ &= \sum_{i=1}^{m} p_i \cdot \overline{u}_i(\alpha). \qquad (19)\end{aligned}$$

These two formulae correspond to the first-level fuzzification of the weighed average from Theorem 2. □

Similarly, it can be shown that the fuzzified WOWA is also a generalization of the fuzzy OWA operator.

Theorem 6: Let $U_1, \ldots, U_m$ be fuzzy numbers, $\vec{w}$ be a vector of normalized weights and $\vec{p}$ be a vector of uniform real weights, $\vec{p} = \vec{\eta}$. Then the result of the fuzzified WOWA of $U_1, \ldots, U_m$ with the weights $\vec{p}$ and $\vec{w}$ is identical to the fuzzy OWA of $U_1, \ldots, U_m$ with the weights $\vec{w}$.

Proof: Because $\vec{p}$ is uniform and it holds that $\mathrm{WOWA}_{\vec{w}}^{\vec{\eta}} = \mathrm{OWA}_{\vec{w}}$ [15], we can write

$$\begin{aligned}\underline{u}(\alpha) &= \mathrm{WOWA}_{\vec{w}}^{\vec{\eta}}(\underline{u}_1(\alpha), \underline{u}_2(\alpha), \ldots, \underline{u}_m(\alpha)) \\ &= \mathrm{OWA}_{\vec{w}}(\underline{u}_1(\alpha), \underline{u}_2(\alpha), \ldots, \underline{u}_m(\alpha)) \\ &= \sum_{i=1}^{m} w_i \cdot \underline{u}_{\sigma(i)}(\alpha), \qquad (20)\\ \overline{u}(\alpha) &= \mathrm{WOWA}_{\vec{w}}^{\vec{\eta}}(\overline{u}_1(\alpha), \overline{u}_2(\alpha), \ldots, \overline{u}_m(\alpha)) \\ &= \mathrm{OWA}_{\vec{w}}(\overline{u}_1(\alpha), \overline{u}_2(\alpha), \ldots, \overline{u}_m(\alpha)) \\ &= \sum_{i=1}^{m} w_i \cdot \overline{u}_{\chi(i)}(\alpha), \qquad (21)\end{aligned}$$

where σ and χ are such permutations of the set of indices $\{1, \ldots, m\}$ that $\underline{u}_{\sigma(1)} \geq \underline{u}_{\sigma(2)} \geq \ldots \geq \underline{u}_{\sigma(m)}$ and $\overline{u}_{\chi(1)} \geq \overline{u}_{\chi(2)} \geq \ldots \geq \overline{u}_{\chi(m)}$. Again, it can be seen that these two formulae are identical to those in Theorem 3. □

It can be easily shown that the fuzzified WOWA (FWOWA) generalizes the WOWA operator.

Theorem 7: Let U_i, $i = 1, \ldots, m$, be fuzzy singletons containing only single elements $u_i \in \Re$. Then for any vectors of normalized weights $\vec{p}$ and $\vec{w}$ it holds that $\mathrm{FWOWA}_{\vec{w}}^{\vec{p}}(U_1, \ldots, U_m) = \mathrm{WOWA}_{\vec{w}}^{\vec{p}}(u_1, \ldots, u_m)$.

Proof: If the fuzzy numbers U_i are fuzzy singletons containing only a single element $u_i \in \Re$, $i = 1, \ldots, m$, then $\underline{u}_i(\alpha) = \overline{u}_i(\alpha)$, for all $\alpha \in [0, 1]$ and any $i = 1, \ldots, m$. Then, in Formulae 16 and 17, we can see that $\underline{u}(\alpha) = \overline{u}(\alpha) = \mathrm{WOWA}_{\vec{w}}^{\vec{p}}(u_1, \ldots, u_m)$. □

The presented fuzzified WOWA operator has some important properties. First, it will be shown that it is idempotent. Next, a boundary condition for the FWOWA will be presented. It guarantees that the result lies between the minimum and the maximum input value in the described way.

Theorem 8: The fuzzified WOWA is idempotent, i.e. if U is a fuzzy number and $\vec{p}$ and $\vec{w}$ are two vectors of normalized weights, then it holds that $U = \mathrm{FWOWA}_{\vec{w}}^{\vec{p}}(U, \ldots, U)$.

Proof: The theorem is a result of Theorem 4 and the fact that the crisp WOWA is idempotent. □

Theorem 9: Let $U_1, \ldots, U_m$ be fuzzy numbers and $\vec{p}$ and $\vec{w}$ be two vectors of normalized weights. For the fuzzy number $U = FWOWA_{\vec{w}}^{\vec{p}}(U_1, \ldots, U_m)$, it holds that for any $\alpha \in [0, 1]$:

$$\underline{u}(\alpha) \geq \min\{\underline{u}_1(\alpha), \underline{u}_2(\alpha), \ldots, \underline{u}_m(\alpha)\}, \qquad (22)$$

$$\overline{u}(\alpha) \leq \max\{\overline{u}_1(\alpha), \overline{u}_2(\alpha), \ldots, \overline{u}_m(\alpha)\}. \qquad (23)$$

Proof: The theorem follows from the fact that $\underline{u}(\alpha)$ and $\overline{u}(\alpha)$ can be calculated by Formulae 16 and 17 as a pair of crisp WOWA and, moreover, that for the WOWA operator of values $u_1, \ldots, u_m \in \Re$, it holds that

$$\begin{aligned}\min\{u_1, \ldots, u_m\} &\leq \mathrm{WOWA}_{\vec{w}}^{\vec{p}}(u_1, \ldots, u_m) \\ &\leq \max\{u_1, \ldots, u_m\}, \qquad (24)\end{aligned}$$

which has been proven in [15]. □

The fuzzy OWA does not preserve the linearity. The fuzzified WOWA has the same property. This means that even

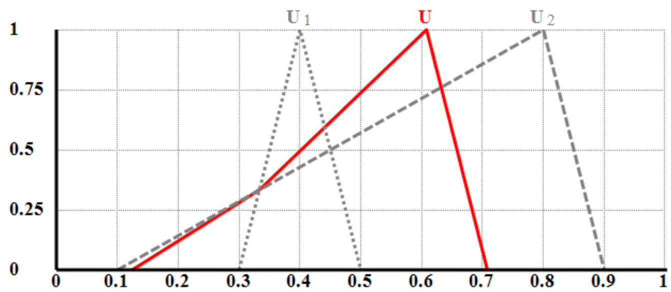

Fig. 4: An example of the fuzzified WOWA where the result is not a linear fuzzy number.

though the fuzzy numbers $U_1, \ldots, U_m$ are linear, generally, the resulting fuzzy number U is not.

This can be demonstrated on an example. Let us consider $U_1 = (0.3, 0.4, 0.5)$, $U_2 = (0.1, 0.8, 0.9)$, $p_1 = 0.3$, $p_2 = 0.7$, $w_1 = 0.2$ and $w_2 = 0.8$. Then the result of the fuzzified WOWA is shown in Figure 4 and it can be seen that it is not linear.

VI. Other approaches combining fuzzy weighted average and fuzzy OWA operator

In this section, we will describe an existing approach to the WOWA fuzzification and another approach combining the weighted average and the OWA operator for fuzzy numbers – UIWOWA (uncertain induced weighted OWA) and UIOWAWA (uncertain induced ordered weighted averaging – weighted averaging) operators [8]. In the next section, they will be compared with the aggregation operator proposed in this paper on illustrative examples. It will be shown that the results are quite different.

While the fuzzified WOWA described in this paper is based on Zadeh's extension principle, the other approaches exploit fuzzy numbers arithmetic. For example, if two triangular fuzzy numbers are expressed by triplets, $A = (a_1, a_2, a_3)$ and $B = (b_1, b_2, b_3)$, the common arithmetic operations can be performed by the following formulae (see e.g. [8]):

$$A + B = (a_1 + b_1,\ a_2 + b_2,\ a_3 + b_3), \tag{25}$$

$$k \cdot A = (k \cdot a_1, k \cdot a_2, k \cdot a_3) \text{ for } k \geq 0, \tag{26}$$

$$\begin{aligned} A \cdot B = (&\min\{a_1 b_1, a_1 b_2, a_2 b_1, a_2 b_2\},\ a_2 b_2, \\ &\max\{a_1 b_1, a_1 b_2, a_2 b_1, a_2 b_2\}). \end{aligned} \tag{27}$$

Similar formulae can be easily derived for A and B represented by intervals or by trapezoidal fuzzy numbers. It should be noted that Formula 27 do not provide the exact result but just its approximation by a linear fuzzy number because the multiplication of the linear fuzzy numbers is not, generally, a linear fuzzy number.

The OWA and WOWA aggregation operators require a permutation that orders the aggregated values from the largest to the lowest one. In case of fuzzy numbers, this presents a great problem. The fuzzy numbers can be, and in the practice usually are, incomparable. This obstacle is usually overcome by replacing the fuzzy numbers with some of their characteristics (e.g. their centers of gravity), which are real numbers and can be ordered easily. In practical cases, the validity of the ordering created this way can be questionable as it will be seen in the example. One way to avoid the problem is by introducing an order-inducing variables. The aggregated values are then composed of a pair containing the fuzzy number and a real number which is used for the ordering. This way does not solve the problem that some fuzzy numbers can be hard to compare. Instead, the expert is required to provide, besides the aggregated values and the weights, more information – the order-inducing variables.

In the paper [8], the authors recommend to compare fuzzy numbers that are intervals by their centers and triangular fuzzy numbers by a weighted average of their significant values – for a triangular fuzzy number $A = (a_1, a_2, a_3)$, the value $(a_1 + 4a_2 + a_3)/6$ is used for the comparison. We will use this method to derive the order-inducing variables in the examples.

The two approaches UIWOWA and UIOWAWA are according to [8] defined as follows.

Definition 9: Let fuzzy numbers $U_1, \ldots, U_m$ and the corresponding order-inducing variables $o_1, \ldots, o_m$, $o_i \in \Re$, $i = 1, \ldots, m$, be given. Let $\vec{p} = (p_1, \ldots, p_m)$ and $\vec{w} = (w_1, \ldots, w_m)$ be two vectors of normalized (real) weights. Then the result of the aggregation by an UIWOWA operator is a fuzzy number U given by the following formula

$$\mathrm{UIWOWA}^{\vec{p}}_{\vec{w}} \left(\langle o_1, U_1 \rangle, \ldots, \langle o_m, U_m \rangle \right) = \sum_{i=1}^{m} \omega_i \cdot U_{\phi(i)}, \tag{28}$$

where ϕ denotes such a permutation of the set of indices $\{1, \ldots, m\}$ that $o_{\phi(1)} \geq o_{\phi(2)} \geq \ldots \geq o_{\phi(m)}$. The weight ω_i is defined as

$$\omega_i = z(\sum_{j \leq i} p_{\phi(j)}) - z(\sum_{j<i} p_{\phi(j)}), \tag{29}$$

for $i = 1, \ldots, m$, and z is a nondecreasing function interpolating the following points

$$\{(0,0)\} \ \cup \ \{(i/m, \ \sum_{j \leq i} w_j)\}_{i=1,\ldots,m}. \tag{30}$$

The function z is required to be linear when the points can be interpolated in that way.

Definition 10: Let $U_1, \ldots, U_m$ be fuzzy numbers and $o_1, \ldots, o_m$ be the corresponding order-inducing variables, $o_i \in \Re$, $i = 1, \ldots, m$. Let $\vec{p} = (p_1, \ldots, p_m)$ and $\vec{w} = (w_1, \ldots, w_m)$ be two vectors of normalized (real) weights. Further, let a parameter $\beta \in [0, 1]$ be given. Then the result of the aggregation by an UIOWAWA operator is a fuzzy number U given by the following formula

$$\mathrm{UIOWAWA}^{\vec{p}}_{\vec{w},\beta} \left(\langle o_1, U_1 \rangle, \ldots, \langle o_m, U_m \rangle \right) = \sum_{i=1}^{m} \omega_i \cdot U_{\phi(i)}, \tag{31}$$

where

$$\omega_i = \beta p_i + (1 - \beta) w_{\phi(i)}, \tag{32}$$

and ϕ denotes such a permutation of the set of indices $\{1, \ldots, m\}$ that $o_{\phi(1)} \geq o_{\phi(2)} \geq \ldots \geq o_{\phi(m)}$.

The both approaches are very similar. Contrary to the presented FWOWA operator, both UIWOWA and UIOWAWA considers only a single permutation ϕ that is given by the order-inducing variables (i.e. it should be set by the expert). The two aggregation operators differ in the way how the mixture weights are obtained – the UIWOWA uses the same approach as the WOWA, while UIOWAWA uses a weighted average of the weighs assigned to the value by the weighted average and by the OWA operator. If $\beta = 0$, the weights $\vec{p}$ are ignored and the operator behaves as an uncertain OWA [8] and, vice versa, for the $\beta = 1$, the weights $\vec{w}$ and the order-inducing variables $o_1, \ldots, o_m$ are not used by the UIOWAWA and the operator reduces to an uncertain weighted average [8].

The mentioned approaches are not the only attempts to incorporate fuzziness into the WOWA. For example, versions of WOWA using hesitant fuzzy sets and intuitionistic fuzzy sets has been proposed [9], [6]. However a comparison with FWOWA would be very difficult because of a completely different nature of the input data.

VII. Example

Let us consider the following scenario. We evaluate various resorts for holidays according to three criteria – *hotel quality*, *sport possibilities*, and *culture possibilities*. We will study two examples. The first one presents the simplest possible setting. The aggregation function should behave as a minimum (which is a special case of WOWA) and the evaluations will be expressed in form of intervals (which can be viewed as a special case of fuzzy numbers). Because of the simplicity of the example, it will be easier to observe the difference in the behavior of the compared aggregation operators. It will be seen that the results of UIWOWA and UIOWAWA can be against intuition.

In the next example, we will use the full power of the operators – both vectors of weights will be used and the evaluations will be represented by fuzzy numbers. This will allow us to study the performance of the selected aggregation operator in the settings that are close to real situations.

A. Example 1

In this simplest settings, let us assume that we require all three criteria to be satisfied at the same time with the same importance for all three areas. This corresponds to the minimum, which is a special case of the WOWA with the weights $\vec{p} = (1/3,\ 1/3,\ 1/3)$ and $\vec{w} = (0, 0, 1)$. For UIOWAWA, we set $\beta = 0$.

We will ask three people who visited one of the considered resorts about their opinion on the resort according each of the three areas. These evaluation will be modeled by fuzzy numbers on the interval $[0, 1]$, where 0 means that the resort is completely unsatisfactory according to the criterion, and 1 that it is fully satisfactory.

In this example, we consider only evaluations in form of intervals. Intervals can be understood as a special case of fuzzy numbers (an interval $[a, b]$ can be modeled by a trapezoidal fuzzy number (a, a, b, b)). The evaluations according to the criteria and the results are summarized in Table I.

	Criteria values		
Person	Hotel	Sport	Culture
P_1	$[0.7, 0.8]$	$[0.5, 0.6]$	$[0.9, 1]$
P_2	$[0.3, 0.7]$	$[0.7, 0.8]$	$[0.4, 0.5]$
P_3	$[0.5, 0.6]$	$[0, 1]$	$[0.4, 0.8]$

	Results		
Person	FWOWA	UIWOWA	UIOWAWA
P_1	$[0.5, 0.6]$	$[0.5, 0.6]$	$[0.5, 0.6]$
P_2	$[0.3, 0.5]$	$[0, 4, 0.5]$	$[0, 4, 0.5]$
P_3	$[0, 0.6]$	$[0, 1]$	$[0, 1]$

TABLE I: The results of FWOWA, IOWAWA and UIOWAWA with parameters set so that they would behave as a minimum.

The first person P_1 assess the particular resort by the following fuzzy evaluations in form of intervals: $U_{hotel} = [0.7, 0.8]$, $U_{sport} = [0.5, 0.6]$, and $U_{culture} = [0.9, 1]$, where the order of their upper and lower bounds is the same. In this simple case all three aggregation operators return, in unison, the lowest of the three evaluations $U = [0.5, 0.6]$ as the result.

For the second person P_2, the values of the order-inducing variable will be $\vec{o} = (0.5, 0.75, 0.45)$ and we obtain the following results: $[0.3, 0.5]$ for FWOWA, and $[0, 4, 0.5]$ for both IOWAWA and UIOWAWA. We will show that the results returned by IOWAWA and UIOWAWA are against intuition. Let us mention that the weights were set so that all three aggregation operators would behave as a fuzzy minimum. We have estimations of the values given by the intervals U_{hotel}, U_{sport}, and $U_{culture}$. Let us assume that those estimations were correct and the real evaluations are within the given intervals, e.g. $u_{hotel} = 0.3$, $u_{sport} = 0.75$, and $u_{culture} = 0.5$. Then it is easy to see, that the minimum of them is 0.3. This number is included only in the solution given by FWOWA.

Finally, the statement given by the third person P_3 presents one more obstacle. Let us say that the person is not a sport type and therefore he/she did not survey the sport possibilities in this area at all. The value for U_{sport} is therefore unknown. The advantage of fuzzified aggregation operators is that they can cope even with this situation. Because the real value of U_{sport} can by any number from 0 to 1, we can set $U_{sport} = [0, 1]$. Other evaluations given by the person are $U_{hotel} = [0.5, 0.6]$ and $U_{culture} = [0.4, 0.8]$.

As the value one of the variables is unknown, it would be really difficult for the expert to set one order inducing variable. We will use again centers of the intervals and obtain $\vec{o} = (0.5, 0.55, 0.6)$. With this variable, both IOWAWA and UIOWAWA considers U_{sport} to be the lowest and return $[0, 1]$ as the result expressing that the final evaluation can be anything, i.e. even 1 representing the perfect evaluation. However, this is not true. If we consider the real evaluations from the intervals, the evaluation of the hotel cannot be greater than 0.6 and so the minimum of the three evaluations will never exceeds 0.6. The FWOWA operators takes this fact into account and returns the value $[0, 0.6]$.

We can see that even in the simplest case, when the aggregated values are represented only by intervals, the results calculated IOWAWA and UIOWAWA can be against intuition. This is caused by the fact that both operators can take into

	Criteria values		
Person	Hotel	Sport	Culture
P_1	$(0.7, 0.75, 0.8)$	$(0.5, 0.55, 0.6)$	$(0.9, 0.95, 1)$
P_2	$(0.3, 0.3, 0.7)$	$(0.7, 0.75, 0.8)$	$(0.4, 0.45, 0.5)$
P_3	$(0.5, 0.55, 0.6)$	$[0, 1]$	$(0.4, 0.7, 0.8)$

	Results		
Person	FWOWA	UIWOWA	UIOWAWA
P_1	$(0.59, 0.64, 0.69)$	$(0.59, 0.64, 0.69)$	$(0.64, 0.69, 0.74)$
P_2	$(0.36, 0.38, 0.64)$	$(0.36, 0.38, 0.68)$	$(0.41, 0.44, 0.67)$
P_3	$(0.17, 0.22, 0.63, 0.68)$	$(0.19, 0.22, 0.84, 0.86)$	$(0.25, 0.31, 0.79, 0.82)$

TABLE II: The results of FWOWA, IOWAWA and UIOWAWA with parameters $\vec{p} = (0.45,\ 0.35\ 0.2)$, $\vec{w} = (0.1, 0.3, 0.6)$ and $\beta = 0.5$ for the second example.

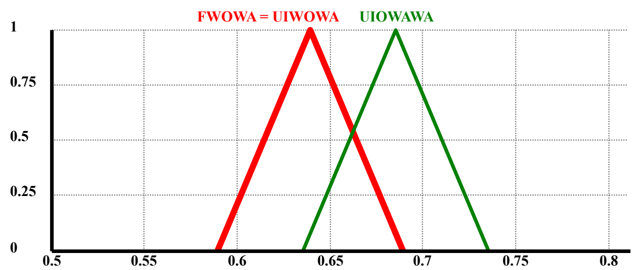

Fig. 5: Comparison of the FWOWA, UIWOWA (left) and UIOWAWA (right) evaluations of the first resort.

account only a single ordering of the values.

B. Example 2

In this section, we will study the three resorts' evaluation under more realistic requirements on the criteria importances. Let us say that we require a good accommodation, we are quite keen on culture and the sport possibilities have a lower importance for us. That could be expressed by the weights $\vec{p} = (0.45,\ 0.35,\ 0.2)$. We assign the greatest weight to the worst of the evaluations but the other two are taken into account, too, i.e. $\vec{w} = (0.1, 0.3, 0.6)$. For UIOWAWA, we expect that both weights vectors have the same importance and we set $\beta = 0.5$.

The evaluations will no longer be limited to intervals – they can be fuzzy numbers. The input values and the results are summarized in Table II.

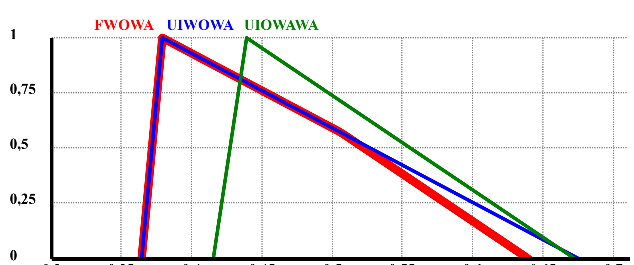

Fig. 6: Comparison of the FWOWA, UIWOWA and UIOWAWA evaluations of the second resort.

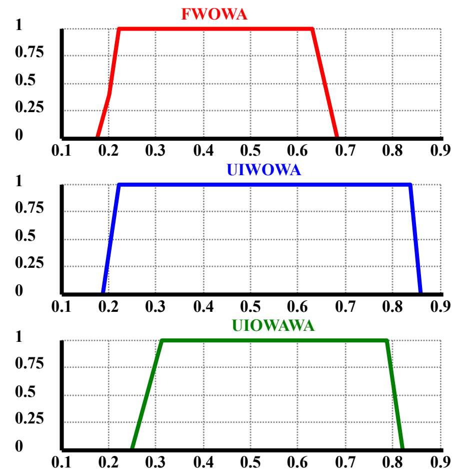

Fig. 7: Comparison of the FWOWA, UIWOWA and UIOWAWA evaluations of the third resort.

The first person P_1 could reformulate his/her description of the resorts and use triangular fuzzy numbers (see Table II). The results are then depicted in Figure 5. The UIOWAWA is the most optimistic. However the difference is not big. It can be seen that FWOWA and UIWOWA are equivalent in this case.

The results of evaluation by the second person P_2 are depicted in Figure 6. The UIOWAWA is again more optimistic. The results of the FWOWA and UIWOWA are very similar except for the fact that the FWOWA result is no more a linear fuzzy number.

In case of the third person P_3, the value of U_{sport} is unknown. Figure 7 reveals that the results of the compared aggregation operators are substantially different.

From this example the following conclusions can be drawn. If the aggregated values are comparable, the FWOWA based on the extension principle and UIWOWA based on the fuzzy numbers arithmetic are equivalent. The UIOWAWA differs due to another way of combining both of the weights vectors but the difference is not substantial. However, if the aggregated values are incomparable, the results of the three aggregation operators can differ significantly. This is caused by the fact that UIWOWA and UIOWAWA use only a single permutation while FWOWA considers all feasible permutations of the input values. Therefore, the result of the FWOWA is more realistic.

VIII. Software

The FWOWA requires two WOWA calculations for each of the α-cuts. This involves a large number of calculations and therefore a software implementation of the FWOWA is necessary in order to be able to apply it in the practice. The calculation of the FWOWA is supported by the FuzzME software.

The FuzzME (Figure 8) is a software tool that makes it possible to design complex multiple-criteria evaluation models. The solved problem is organized into a tree structure called a goals tree. Both quantitative and qualitative criteria can be used. For aggregation, multiple methods are supported –

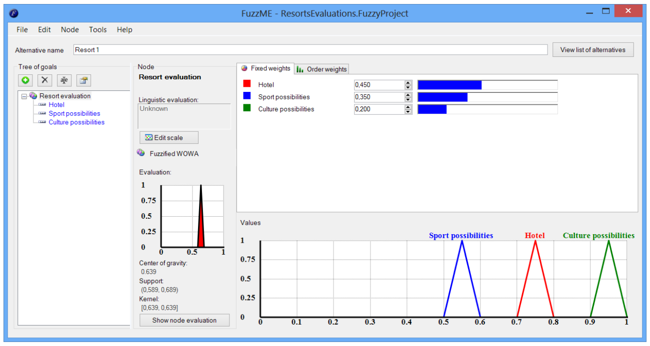

Fig. 8: The FuzzME software

fuzzy weighted average, fuzzy OWA, fuzzified WOWA, fuzzy Choquet integral, or a fuzzy expert system.

The FuzzME has been designed to be able to evaluate a large set of alternatives. More information on the software, the used evaluation type and the system of the supported methods can be found in [5]. The demo version of FuzzME can be downloaded at *http://www.FuzzME.net*.

IX. Conclusion

In this paper, we have introduced a fuzzification of the WOWA operator based on the extension principle. The new operator has been compared to two existing approaches – the UIWOWA and UIOWAWA. On an illustrative example, it is show that, when the input variables are incomparable (which is quite common case in the practice), the approaches can give significantly different results. It has been pointed out that the results provided by the UIWOWA and UIOWAWA can be against intuition because these operators simplify the problem by taking into account only a single ordering, which need not to be valid for all feasible values of the input variables. On the other side, the FWOWA based on Zadeh's extension principle considers all feasible orderings of the values and, thus, it gives more realistic result.

The paper pursued to lay the theoretical foundations so that the presented FWOWA operator could be applied in practice. Generally, any practical problem where the WOWA operator has been used and the input values might be uncertain can be extended to employ FWOWA instead. The benefit of such an extension is that the information on the uncertainty is preserved. The illustrative example showed another advantage of FWOWA over WOWA in the area of multiple-criteria evaluation and decision–making – the FWOWA is applicable even when some of the aggregated values are unknown.

The calculation of the FWOWA can be performed by the FuzzME software, which has been presented briefly. The software makes it possible to create complex multiple-criteria evaluation models using fuzzy methods including, among the others, also the presented FWOWA operator. Therefore, there is a ready-made solution for applying the proposed operator in the practice.

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