The common solutions of complementarity problems and a zero point of maximal monotone operators by using the hybrid projection method*

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***Abstract*—**In this paper, we propose a projection method for solving nonlinear complementarity problems and a zero point of maximal monotone operators. Strong convergence theorems are established for solving the common solutions of complementarity problems and a zero point of maximal monotone operators together with a system of generalized equilibrium, variational inequality and fixed point problems in a uniformly smooth and 2-uniformly convex real Banach space. Moreover, we also apply the result to Hilbert spaces.

***Keywords*—**Complementarity problems; a zero point; inverse-strongly monotone mapping; variational inequality problem; equilibrium problem; fixed point problem.

I. Introduction and Preliminaries

THE complementarity problem is one of the important problems and deep connections with nonlinear analysis and by its interesting applications in areas example such as: Optimization Theory, Engineering, Structural Mechanics, Elasticity Theory, Economics, Equilibrium Theory, Nonlinear Dynamics, Stochastic Optimal Control. Some natural connections between the complementarity problem and some special fixed point theorems are used to prove several existance theorems.

Recently, In 2015, Phuangphoo and Kumam [10] introduced a new iterative sequence which is constructed by using the hybrid projection method for solving the common solution for a system of generalized equilibrium problems of inverse strongly monotone mappings and a system of bifunctions satisfying certain the conditions, the common solution for the families of quasi -ϕ- asymptotically nonexpansive and uniformly Lipschitz continuous and the common solution for a variational inequality problem. Strong convergence theorems are proved on approximating a common solution of a system of generalized equilibrium problems, fixed point problems for two countable families and a variational inequality problem in a uniformly smooth and 2-uniformly convex real Banach space.

Let E be a Banach space with norm $\|\cdot\|$, C be a nonempty closed and convex subset of E and let E^* denote by the dual of E. One of the major problems in the theory of monotone operators is as follows: find a point $z \in E$ such that

$$0 \in Az, \tag{I.1}$$

where A is an operator from E into E^*. A point $z \in E$ is called a *zero point* of operator A. The set of zeroes of the operator A is denoted by $A^{-1}0$. This problem contains numerous problems in optimization, economics, physics and several areas of engineering.

The problem of finding a zero point for monotone operators play an important role in modern optimization and nonlinear analysis. This is because it can be related to many kinds of important problems, such as convex optimization problem, image processing, equilibrium problem and variational inequality problems. In order to approximate the solution to this problems, many authors have studies the convergence of such problems in several setting. It is well known that the metric projection operator plays an important role in nonlinear functional analysis, optimization theory, variational inequality, and complementarity problems, etc.

In 2012, Saewan and Kumam [14] introduced a modified hybrid block projection algorithm for finding a common element of the set of the solution of the complementarity problems, the set of solutions of the system of equilibrium problems and the set of common fixed points of an infinite family in a 2-uniformly convex and uniformly smooth Banach space.

In this paper by the previously mentioned above results, we will apply the results of Phuangphoo and Kumam [10], this work bringing in the results to applied in other problems as well complementarity problems and we shall use the result to study the strong convergence theorem of zero point of maximal monotone operators together with a system of generalized equilibrium, variational inequality and fixed point problems in a uniformly smooth and 2-uniformly convex real Banach space. Moreover, we also apply the result to obtain in Hilbert spaces.

Throughout this paper, we assume that $\mathbb{R}$ and $\mathbb{J}$ are denoted by the set of real numbers and the set of $\{1, 2, 3, ..., M\}$, respectively, where M is any given positive integer. Let $\{F_k\}_{k\in\mathbb{J}} : C \times C \to \mathbb{R}$ be a bifunction, and $\{B_k\}_{k\in\mathbb{J}} : C \to E^*$ be a monotone mapping.

The *system of generalized equilibrium problems*, is to find $x \in C$ such that

$$F_k(x, y) + \langle y - x, B_k x\rangle \geq 0, \quad k \in \mathbb{J}, \quad \forall y \in C. \tag{I.2}$$

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The set of solutions of (I.2) is denoted by $SGEP(F_k, B_k)$, that is $SGEP(F_k, B_k) = \{x \in C : F_k(x, y) + \langle y - x, B_k x\rangle \geq 0, \forall y \in C\}, \forall k \in \mathbb{J}$.

If $\mathbb{J}$ is a singleton, then problem (I.2) reduces to the *generalized equilibrium problems*, is to find $x \in C$ such that

$$F(x, y) + \langle y - x, Bx\rangle \geq 0, \quad \forall y \in C. \tag{I.3}$$

The set of solutions of (I.3) is denoted by $GEP(F, B)$, that is

$$GEP(F, B) = \{x \in C : F(x, y) + \langle y - x, Bx\rangle \geq 0,\} \tag{I.4}$$

$\forall y \in C$.

If $B \equiv 0$ the problem (I.3) reduces into the *equilibrium problem* for F, denoted by $EP(F)$, is to find $x \in C$ such that

$$F(x, y) \geq 0, \quad \forall y \in C. \tag{I.5}$$

If $F \equiv 0$ the problem (I.3) reduces into *variational inequality of Browder type*, denoted by $VI(C, B)$, is to find $x \in C$ such that

$$\langle y - x, Bx\rangle \geq 0, \quad \forall y \in C. \tag{I.6}$$

Let $\{x_n\}$ be a sequence in E. We denote by $x_n \to x$ and $x_n \rightharpoonup x$ that the strong convergence and weak convergence of $\{x_n\}$, respectively. The *normalized duality mapping* J from E to 2^{E^*} is defined by $Jx = \{f \in E^* : \langle x, f\rangle = \|x\|^2 = \|f\|^2\}, \quad \forall x \in E$.By the Hahn-Banach theorem, $Jx \neq \emptyset$ for each $x \in E$.

A Banach space E is said to be *strictly convex* if $\left\|\frac{x+y}{2}\right\| < 1$ for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. It is said to be *uniformly convex* if $\lim_{n\to\infty} \|x_n - y_n\| = 0$ for any two sequences $\{x_n\}$ and $\{y_n\}$ in E such that $\|x_n\| \leq 1, \|y_n\| \leq 1$ and

$$\lim_{n\to\infty} \left\|\frac{x_n + y_n}{2}\right\| = 1. \tag{I.7}$$

Let $U_E = \{x \in E : \|x\| = 1\}$ be the *unit sphere* of E. Then, the Banach space E is said to be *smooth* if

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{I.8}$$

exists for each $x, y \in U_E$. It is said to be *uniformly smooth* if the limit (I.8) is attained uniformly for all $x, y \in U_E$.

The function $\rho_E : \mathbb{R}^+ \to \mathbb{R}^+$ is said to be the *modulus of smoothness* of E if $\rho_E(t) = \sup\left\{\frac{\|x+y\|+\|x-y\|}{2} - 1 : \|x\| = 1,\ \|y\| = t\right\}$.

The space E is said to be *smooth* if $\rho_E(t) > 0$, $\forall t > 0$ and is said to be *uniformly smooth* if and only if $\lim_{t\to 0^+} \frac{\rho_E(t)}{t} = 0$.

The *modulus of convexity* of E is the function $\delta_E : [0, 2] \to [0, 1]$ defined by $\delta_E(\epsilon) = \inf\left\{1 - \left\|\frac{x+y}{2}\right\| : \|x\| \leq 1\ ,\ \|y\| \leq 1\ ;\ \|x - y\| \geq \epsilon\right\}$.

A Banach space E is said to be *uniformly convex* if and only if $\delta_E(\epsilon) > 0$ for all $\epsilon \in (0, 2]$.

Let a real number $p > 1$. Then, E is said to be *p - uniformly convex* if there exists a constant $c > 0$ such that $\delta_E(\epsilon) \geq c\epsilon^p$, for all $\epsilon \in [0, 2]$. Observe that every p-uniformly convex space is uniformly convex. It is well-known for example that

$$L_p(l_p) \text{ or } W_m^p \ is \begin{cases} \text{p-uniformly convex,} & \text{if } p \geq 2; \\ \text{2-uniformly convex,} & \text{if } 1 < p \leq 2. \end{cases}$$

One should note that no a Banach space is p-uniformly convex for $1 < p < 2$. It is known that a Hilbert space is uniformly smooth and 2-uniformly convex.

Now let E be a smooth and strictly convex reflexive Banach space. As Alber (see [1]) and Kamimura and Takahashi (see [7]) did, the *Lyapunov functional* $\phi : E \times E \to \mathbb{R}^+$ is defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2, \quad \forall x, y \in E.$$

It follows from Kohsaka and Takahashi (see [8]) that $\phi(x, y) = 0$ if and only if $x = y$, and that

$$(\|x\| - \|y\|)^2 \leq \phi(x, y) \leq (\|x\| + \|y\|)^2. \tag{I.9}$$

If E is a Hilbert space, then $\phi(x, y) = \|x - y\|^2$, for all $x, y \in E$.

Further suppose that C is nonempty closed convex subset of E. The *generalized projection* (Alber [1]) $\Pi_C : E \to C$ is a map that assigns to an arbitrary point $x \in E$ the minimum point of the functional $\phi(x, y)$, that is, $\Pi_C x = \bar{x}$, where $\bar{x}$ is the solution to the following minimization problem

$$\phi(\bar{x}, x) = \inf_{y\in C} \phi(y, x), \tag{I.10}$$

existence and uniqueness of the operator Π_C follows from the properties of the functional $\phi(x, y)$ and strict monotonicity of the mapping J.

So, the *generalized projection* $\Pi_C : E \to C$ is defined by for each $x \in E$,

$$\Pi_C(x) = \arg\min_{y\in C} \phi(x, y).$$

As well know that if C is a nonempty closed convex subset of a Hilbert space H and $P_C : H \to C$ is the metric projection of H onto C, then P_C is nonexpansive.

Remark I.1. If E is a reflexive, strictly convex and smooth Banach space, then for $x, y \in E$, $\phi(x, y) = 0$ if and only if $x = y$. It is sufficient to show that if $\phi(x, y) = 0$ then $x = y$. From Lyapunov functional, we have $\|x\| = \|y\|$. This implies that $\langle x, Jy\rangle = \|x\|^2 = \|Jy\|^2$. From the definition of J, one has $Jx = Jy$. Therefore, we have $x = y$; see [5] for more details.

The following basic properties can be found in Cioranescu [5].

- If E is a strictly convex, then J is strictly monotone.
- If E is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subset of E.
- If E is reflexive smooth and strictly convex, then the normalized duality mapping J is single valued, one-to-one and onto.
- If E be a reflexive strictly convex and smooth Banach space and J is the duality mapping from E into E^*, then J^{-1} is also single-value, bijective and is also the duality mapping from E^* into E and thus $JJ^{-1} = I_{E^*}$ and $J^{-1}J = I_E$.
- If E is uniformly smooth, then E is smooth and reflexive.

- E is uniformly smooth if and only if E^* is uniformly convex.
- If E is a reflexive and strictly convex Banach space, then J^{-1} is norm-weak*-continuous.

Recall that, a mapping $S : C \to C$ is said to be *nonexpansive* if $\|Sx - Sy\| \leq \|x - y\|, \quad \forall x, y \in C$.

A mapping $A : C \to E^*$ is said to be *δ-inverse-strongly monotone*, if there exists a constant $\delta > 0$ such that

$$\langle x - y, Ax - Ay\rangle \geq \delta\|Ax - Ay\|^2, \quad \forall x, y \in C.$$

A mapping $S : C \to C$ is said to be *closed* if for each $\{x_n\} \subset C, \ x_n \to x$ and $Sx_n \to y$ imply $Sx = y$.

Remark I.2. Let T is a nonexpansive of C into itself and I is the identity mapping of a real Banach space E. Then, a mapping $A = I - T$ is $\frac{1}{2}$-inverse-strongly monotone mapping.

Remark I.3. Let a mapping $A_k : \mathbb{R} \to \mathbb{R}$ by $A_k x = kx$, $\forall x \in \mathbb{R}$ and $k \in \{1, 2, 3, ..., n\}$. Then, a mapping $A_k : \mathbb{R} \to \mathbb{R}$ is a finite family of $\frac{1}{k}$-inverse-strongly monotone.

A mapping $S : C \to C$ is said to be *quasi -ϕ- nonexpansive* (*relatively quasi-nonexpansive*) if $Fix(S) \neq \emptyset$, and

$$\phi(u, Sx) \leq \phi(u, x), \quad \forall x \in C, \ u \in Fix(S).$$

A mapping $S : C \to C$ is said to be *quasi -ϕ- asymptotically nonexpansive* (*asymptotically relatively nonexpansive*) if $Fix(S) \neq \emptyset$, and there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ such that $\phi(u, S^n x) \leq k_n \phi(u, x), \quad \forall x \in C, \ u \in Fix(S), \ \forall n \geq 1$.
It is easy to see that if $A : C \to E^*$ is *δ-inverse-strongly monotone*, then A is *$\frac{1}{\delta}$-Lipschitz continuous*.

Example I.4. Let Π_c be the generalized projection from a smooth, strictly convex and reflexive Banach space E onto a nonempty closed and convex subset C of E. Then, we get Π_c is closed and quasi -ϕ- asymptotically nonexpansive mapping from E onto C with $Fix(\Pi_c) = C$.

Example I.5. Let $C := [\frac{-1}{\pi}, \frac{1}{\pi}]$ and define $T : C \to C$ by

$$Tx = \begin{cases} \frac{x}{2}\sin(\frac{1}{x}), & \text{if } x \neq 0; \\ x, & \text{if } x = 0. \end{cases}$$

Then, T is quasi -ϕ- asymptotically nonexpansive mapping.

The class of quasi -ϕ- asymptotically nonexpansive mappings contains properly the class of relatively nonexpansive mappings (see Matsushita and Takahashi [9]) as a subclass.

Let E be a smooth, strictly convex and reflexive Banach space, C be a nonempty closed convex subset of E, $T : C \to C$ be a mapping and $Fix(T)$ be the set of fixed points of T.

A point $p \in C$ is said to be an *asymptotic fixed point* of T if there exists a sequence $\{x_n\} \subset C$ such that $x_n \rightharpoonup p$ and $\|x_n - Tx_n\| \to 0$. We denoted the set of all asymptotic fixed points of T by $\widehat{Fix}(T)$.

A point $p \in C$ is said to be a *strong asymptotic fixed point* of T, if there exists a sequence $\{x_n\} \subset C$ such that $x_n \to p$ and $\|x_n - Tx_n\| \to 0$. We denoted the set of all strong asymptotic fixed points of T by $\widetilde{Fix}(T)$.

A mapping $T : C \to C$ is said to be *relatively nonexpansive* [9], [11], if $Fix(T) \neq \emptyset$, $Fix(T) = \widehat{Fix}(T)$ and

$$\phi(p, Tx) \leq \phi(p, x), \quad \forall x \in C, \ p \in Fix(T).$$

A mapping $T : C \to C$ is said to be *weak relatively nonexpansive* [15], if $Fix(T) \neq \emptyset$, $Fix(T) = \widetilde{Fix}(T)$ and

$$\phi(p, Tx) \leq \phi(p, x), \quad \forall x \in C, \ p \in Fix(T).$$

Remark I.6. If E is a real Hilbert space H, then $\phi(x, y) = \|x - y\|^2$ and $\Pi_C = P_C$ (the metric projection of H onto C).

Remark I.7. Direct from the definition of a mapping, so it is easy to see that

- Each relatively nonexpansive mapping is closed.
- Every quasi -ϕ- nonexpansive mapping is quasi -ϕ- asymptotically nonexpansive mapping with $\{k_n = 1\}$, but the converse is not true.
- Each weak relatively nonexpansive mapping is a quasi -ϕ- nonexpansive mapping (because it does not require the condition $Fix(T) = \widetilde{Fix}(T)$, but the converse is not true.
- Every relatively nonexpansive mapping is a weak relatively nonexpansive mappings, but the converse is not true.
- Every countable family of weak relatively nonexpansive mappings is a countable family of of uniformly closed and quasi -ϕ- nonexpansive mappings, and so it is a countable family of uniformly closed and quasi -ϕ- asymptotically nonexpansive mappings.

Definition I.8. Let $\{S_i\}_{i=1}^{\infty} : C \to C$ be a sequence of mappings. $\{S_i\}_{i=1}^{\infty}$ is said to be a family of uniformly quasi -ϕ- asymptotically nonexpansive mappings, if $\bigcap_{i=1}^{\infty} Fix(S_i) \neq \emptyset$ and there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ such that for each $i \geq 1$, $\phi(u, S_i^n x) \leq k_n \phi(u, x), \quad \forall u \in \bigcap_{i=1}^{\infty} Fix(S_i), \ x \in C, \ \forall n \geq 1$.

Definition I.9. A mapping $S : C \to C$ is said to be uniformly *L-Lipschitz continuous*, if there exists a constant $L > 0$ such that

$$\|S^n x - S^n y\| \leq L\|x - y\|, \quad \forall x, y \in C, \quad \forall n \geq 1.$$

Lemma I.10. (see Alber [1]). *Let C be a nonempty closed and convex subset of a smooth and strictly convex reflexive Banach space E, and let $x \in E$. Then,*

$$\phi(x, \Pi_C(y)) + \phi(\Pi_C(y), y) \leq \phi(x, y), \forall x \in C, \ y \in E.$$

Lemma I.11. (see Kamimura and Takahashi [7]). *Let C be a nonempty closed and convex subset of a smooth and strictly convex reflexive Banach space E, and let $x \in E$ and $u \in C$. Then,*

$$u = \Pi_C(x) \Leftrightarrow \langle u - y, Jx - Ju\rangle \geq 0, \quad \forall y \in C.$$

We make use of the function $V : E \times E^* \to \mathbb{R}$ defined by

$$V(x, x^*) = \|x\|^2 - 2\langle x, x^*\rangle + \|x^*\|^2, \quad \forall x \in E, \forall x^* \in E^*.$$

Observe that $V(x, x^*) = \phi(x, J^{-1}x^*)$ for all $x \in E$ and $x^* \in E^*$. The following lemma is well-known.

Lemma I.12. (see Alber [1]) *Let E be a smooth and strictly convex reflexive Banach space with E^*as its dual, then*

$$V(x, x^*) + 2\langle J^{-1}x^* - x, y^*\rangle \leq V(x, x^* + y^*),$$

for all $x \in E$ and $x^*, y^* \in E^*$.

Lemma I.13. (see Kamimura and Takahashi [7]). *Let E be a uniformly convex and smooth real Banach space and let $\{x_n\}$ and $\{y_n\}$ be two sequences of E. If $\phi(x_n, y_n) \to 0$ and either $\{x_n\}$ or $\{y_n\}$ is bounded, then $\|x_n - y_n\| \to 0$.*

For solving the generalized equilibrium problem, let us assume that the mapping $B : C \to E^*$ is δ-inverse-strongly monotone mapping and the bifunction $F : C \times C \to \mathbb{R}$ satisfies the following conditions:

(A1) $F(x, x) = 0$, for all $x \in C$;
(A2) F is monotone, i.e., $F(x, y) + F(y, x) \le 0, \quad \forall x, y \in C$;
(A3) $\limsup_{t \downarrow 0} F(x + t(z - x), y) \le F(x, y), \ \forall x, y, z \in C$;
(A4) for any $y \in C$, the function $y \mapsto F(x, y)$ is convex and lower semicontinuous.

Lemma I.14. *Let E be a uniformly smooth and strictly convex Banach space with the Kadec-Klee property, $\{x_n\}$ and $\{y_n\}$ be two sequences of E, and $p \in E$. If $x_n \to p$ and $\phi(x_n, y_n) \to 0$, then $y_n \to p$.*

Lemma I.15. (see Blum and Oettli [2]). *Let C be a nonempty closed and convex subset of a smooth and strictly convex reflexive Banach space E, and let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying the following conditions* (A1) - (A4). *Let $r > 0$ be any given number and $x \in E$ be any point. Then, there exists a $z \in C$ such that*

$$F(z, y) + \frac{1}{r}\langle y - z, Jz - Jx\rangle \ge 0, \quad \forall y \in C. \qquad \text{(I.11)}$$

Lemma I.16. (see Chang et al. [4]) *Let C be a nonempty closed and convex subset of a smooth and strictly convex reflexive Banach space E, and let $B : C \to E^*$ be a δ-inverse-strongly monotone mapping and $F : C \times C \to \mathbb{R}$ be a bifunction satisfying the following conditions* (A1)-(A4). *Let $r > 0$ be any given number and $x \in E$ be any point. Then, there exists a point $z \in C$ such that*

$$F(z, y) + \langle y - z, Bz\rangle + \frac{1}{r}\langle y - z, Jz - Jx\rangle \ge 0, \quad \forall y \in C. \qquad \text{(I.12)}$$

Lemma I.17. (see Chang et al. [4]) *Let C be a nonempty closed and convex subset of a smooth and strictly convex reflexive Banach space E, and let $B : C \to E^*$ be a δ-inverse-strongly monotone mapping and $F : C \times C \to \mathbb{R}$ be a bifunction satisfying the following conditions* (A1)-(A4). *Let $r > 0$ and $x \in E$, and we define a mapping $T_r^F : E \to C$ as follows: for any $x \in C$,*

$$T_r^F x = \{z \in C : F(z, y) + \langle y - z, Bz\rangle$$
$$+\frac{1}{r}\langle y - z, Jz - Jx\rangle \ge 0, \} \quad \forall y \in C.$$

Then, the following conclusions hold:

(1) *T_r^F is single-valued*;
(2) *T_r^F is a firmly nonexpansive type mapping, i.e., $\langle T_r^F x - T_r^F y, JT_r^F x - JT_r^F y\rangle \le \langle T_r^F x - T_r^F y, Jx - Jy\rangle, \forall x, y \in E$*
(3) $Fix(T_r^F) = \widetilde{Fix(T_r^F)} = EP$;
(4) *EP is a closed and convex set of C*;
(5) $\phi(p, T_r^F x) + \phi(T_r^F x, x) \le \phi(p, x), \forall p \in Fix(T_r^F)$;
(6) *for each $n \ge 1$, $r_n > d > 0$ and $u_n \in C$ with $\lim_{n\to\infty} u_n = \lim_{n\to\infty} T_{r_n} u_n = \overline{u}$, we have*

$$F(\overline{u}, y) + \langle y - \overline{u}, B\overline{u}\rangle \ge 0, \quad \forall y \in C.$$

Lemma I.18. (see Cioranescu [6]) *Let C be a nonempty closed and convex subset of a real uniformly smooth and strictly convex Banach space E with the Kadec-Klee property, $S : C \to C$ be a closed and quasi -ϕ- asymptotically nonexpansive mapping with a sequence $\{k_n\} \subset [1, \infty)$ and $k_n \to 1$. Then, $Fix(S)$ is closed and convex in C.*

Lemma I.19. (see Chang et al. [3]) *Let E be a uniformly convex Banach space, $r > 0$ be a positive number and $B_r(0)$ be a closed ball of E. Then, for any given sequence $\{x_n\}_{n=1}^{\infty} \subset B_r(0)$ and for any given $\{r_n\}_{n=1}^{\infty} \subset (0, 1)$ with $\sum_{n=1}^{\infty} r_n = 1$, there exists a continuous, strictly increasing and convex function $g : [0, 2r) \to [0, \infty)$ with $g(0) = 0$ such that for any positive integers i , j with $i < j$,*

$$\left\|\sum_{n=1}^{\infty} r_n x_n\right\|^2 \le \sum_{n=1}^{\infty} r_n \|x_n\|^2 - r_i r_j g(\|x_i - x_j\|).$$

Lemma I.20. (see Xu [17]) *Let E be a 2-uniformly convex real Banach space, then for all $x, y \in E$, we have*

$$\|x - y\| \le \frac{2}{c^2}\|Jx - Jy\|,$$

where J is the normalized duality mapping of E and $0 < c \le 1$, and $\frac{1}{c}$ is called the 2-uniformly convex constant of E.

We note that every uniformly convex Banach space has the Kadec-Klee property. For more details on Kadec-Klee property, the reader is referred to [6], [16].

Example I.21. Let E be a uniformly smooth and strictly convex Banach space and $A \subset E \times E^*$ be a maximal monotone mapping such that its zero set $A^{-1}(0)$ is nonempty. Then, we get $J_r = (J + rA)^{-1}J$ is closed and quasi -ϕ- asymptotically nonexpansive mapping from E onto $D(A)$ and $Fix(J_r) = A^{-1}(0)$.

A monotone A is said to be *maximal* if its graph $G(A) = \{(x, y^*) : y^* \in Ax\}$ is not properly contained in the graph of any other monotone operator. If A is maximal monotone, then the solution set $A^{-1}0$ is closed and convex. Let E be a reflexive, strictly convex and smooth Banach space, it is known that A is a maximal monotone if and only if $R(J + rA) = E^*$ for all $r > 0$. The *resolvent* of monotone operator A is defined by $J_r = (J + rA)^{-1}J, \quad \forall r > 0$.

A well-known method for solving zeroes of maximal monotone operator is *proximal point algorithm*. Let A be a maximal monotone operator in a Hilbert space H. The proximal point algorithm generates, for starting $x_1 = x \in H$, a sequence $\{x_n\}$ in H by

$$x_{n+1} = J_{r_n} x_n, \quad \forall n \ge 1, \qquad \text{(I.13)}$$

where $\{r_n\} \subset (0, \infty)$ and $J_{r_n} = (I + r_n A)^{-1}$.

Rockafellar [13] proved that the sequence $\{x_n\}$ defined by (I.13) converges weakly to an element of $A^{-1}0$.

Let E be a smooth strictly convex and reflexive Banach space, C be a nonempty closed convex subset of E and $A \subset E \times E^*$ be a monotone operator satisfying the following:

$$D(A) \subset C \subset J^{-1}(\cap_{r>0} R(J + rA)).$$

Then the *resolvent* $J_r : C \to D(A)$ of A is defined by

$$J_r x = \{z \in D(A) : Jx \in Jz + rAz,\ \forall x \in C\}.$$

J_r is a single-valued mapping from E to $D(A)$. Also, $A^{-1}(0) = F(J_r)$ for all $r > 0$, where $F(J_r)$ is the set of all fixed points of J_r. For any $r > 0$, the *Yosida approximation* $A_r : C \to E^*$ of A is define by $A_r x = \frac{Jx - JJ_r x}{r}$ for all $x \in C$. We know that $A_r x \in A(J_r x)$ for all $r > 0$ and $x \in E$.

Lemma I.22. (see Kohsaka and Takahashi [8]) *Let E be a smooth strictly convex and reflexive Banach space, C be a nonempty closed convex subset of E and $A \subset E \times E^*$ be a monotone operator satisfying $D(A) \subset C \subset J^{-1}(\cap_{r>0} R(J + rA))$. For any $r > 0$, let J_r and A_r be the resolvent and the Yosida approximation of A, respectively. Then the following hold:*

(1) $\phi(p, J_r x) + \phi(J_r x, x) \le \phi(p, x)$ *for all* $x \in C$ *and* $p \in A^{-1}0$*;*

(2) $(J_r x, A_r x) \in A$ *for all* $x \in C$*;*

(3) $F(J_r) = A^{-1}0$.

Lemma I.23. (see Rockafellar [12]) *Let E be a reflexive strictly convex and smooth Banach space. Then an operator $A \subset E \times E^*$ is maximal monotone if and only if $R(J + rA) = E^*$ for all $r > 0$.*

Let A be an inverse-strongly monotone mapping of C into E^* which is said to be *hemicontinuous* it for all $x, y \in C$, the mapping $F : [0, 1] \to E^*$, defined by $F(t) = A(tx + (1-t)y)$ is continuous with respect to the weak* topology of E^*. We define $N_C(v)$ the *normal cone* for C at a point $v \in C$, that is,

$$N_C(v) = \{x^* \in E^* : \langle v - y, x^* \rangle \ge 0, \quad \forall y \in C\}.$$

Lemma I.24. (see Rockafellar [12]) *Let C be a nonempty, closed and convex subset of a Banach space E and A is monotone, hemicontinuous operator of C into E^*. Let $U \subset E \times E^*$ be an operator defined as follows:*

$$Uv = \begin{cases} Av + N_C(v), & v \in C; \\ \emptyset, & v \notin C. \end{cases}$$

Then, U is maximal monotone and $U^{-1}(0) = VI(C, A)$.

II. The system of generalized equilibrium problems and fixed point problems in Banach spaces

In this section, we refer a strong convergence theorem by [10] which solves the problem of finding a common solution of the system of generalized equilibrium problems and fixed point problems in Banach spaces.

For example the following the condition is satisfied.

Example II.1. Let $A : \mathbb{R} \to \mathbb{R}$ by given by

$$A := \begin{cases} 0, & \text{if } x \le 0; \\ 4x, & \text{if } x > 0. \end{cases}$$

Then, A is $\frac{1}{4}$-inverse-strongly monotone mapping with $VI(\mathbb{R}, A) = A^{-1}(0) = (-\infty, 0]$.

We give an example for nonlinear mappings to illustrate the next results.

Example II.2. Let $S : \mathbb{R} \to \mathbb{R}$ be given by $S := (I + rB)^{-1}$, for $r > 0$, where

$$Bx := \begin{cases} x + 1, & \text{if } x \in (-\infty, -1]; \\ 0, & \text{if } x \in (-1, 0], \\ 2x, & \text{if } x \in (0, \infty). \end{cases}$$

Then, we get that $J_r := (I + rB)^{-1} = S$ is uniformly L- Lipschitz continuous and quasi -ϕ- asymptotically nonexpansive with $\{k_n\} = 1$ for each $n \ge 1$ and $Fix(J_r) = B^{-1}(0) = Fix(S) = [-1, 0]$.

Now, we remark that, let C be a subset of a real Banach space E and $A : C \to E^*$ be an inverse strongly monotone mapping satisfying $\|Ax\| \le \|Ax - Ap\|$, for all $x \in C$ and $p \in VI(C, A)$, then $VI(C, A) = A^{-1}(0) = \{p \in C : Ap = 0\}$.

Theorem II.3. *Let C be a nonempty, closed and convex subset of a uniformly smooth and 2-uniformly convex real Banach space E. Suppose that*

(B1) Let $B_k : C \to E^*$ for each $k = 1, 2, 3, ..., M$ be a finite family of δ_k-inverse-strongly monotone mappings, and let $F_k : C \times C \to \mathbb{R}$ be a bifunction which satisfies conditions (A1)-(A4).

(B2) Let $\{T_i\}_{i=1}^\infty$ and $\{S_j\}_{j=1}^\infty : C \to C$ be countable families of uniformly closed and ω_i,μ_j-Lipschitz continuous and quasi -ϕ- asymptotically nonexpansive mappings with sequences $\{k_n\}, \{l_n\} \subset [1, \infty)$ and $k_n \to 1, l_n \to 1$, respectively.

(B3) Let $A_n : C \to E^*$ for each $n = 1, 2, 3, ..., N$ be a finite family of γ_n-inverse strongly monotone mappings and let $\gamma = min\{\gamma_n : n = 1, 2, 3, ..., N\}$.

(B4) $\Omega := \left(\bigcap_{i=1}^\infty Fix(T_i)\right) \bigcap \left(\bigcap_{j=1}^\infty Fix(S_j)\right) \bigcap$

$$\left(\bigcap_{k=1}^M SGEP(F_k, B_k)\right) \bigcap \left(\bigcap_{n=1}^N VI(C, A_n)\right)$$

is a nonempty and bounded in C.

Let $\{x_n\}_{n=1}^\infty, \{z_n\}_{n=1}^\infty, \{y_n\}_{n=1}^\infty$ and $\{u_n\}_{n=1}^\infty$ be sequences generated by

$$\begin{cases} x_0 \in C \ \ \text{chosen arbitrary},\ C_0 = C \\ z_n = \Pi_C J^{-1}(Jx_n - r_n A_n x_n) \\ y_n = J^{-1}(\beta_{n,0}^{(1)} Jx_n + \sum_{i=1}^\infty \beta_{n,i}^{(2)} JT_i^n x_n \\ \quad + \sum_{j=1}^\infty \beta_{n,j}^{(3)} JS_j^n z_n) \\ u_n = T_{r_{M,n}}^{F_M} T_{r_{M-1,n}}^{F_{M-1}} ... T_{r_{2,n}}^{F_2} T_{r_{1,n}}^{F_1} y_n \\ C_{n+1} = \{v \in C_n : \phi(v, u_n) \le \phi(v, x_n) + \theta_n\} \\ x_{n+1} = \Pi_{C_{n+1}}(x_0), \quad \forall n \ge 0, \end{cases} \tag{II.1}$$

where $T_{r_{k,n}}^{F_k} : E \to C$, $k = 1, 2, 3, ..., M$, is a mapping defined by (I.17) *with $F = F_k$ and $r = r_{k,n}$ and it is the solutions to the following system of generalized equilibrium problem: $F_k(z, y) + \langle y - z, B_k z\rangle + \frac{1}{r}\langle y - z, Jz - Jx\rangle \ge 0$, $\forall y \in C$, $k = 1, 2, 3, ..., M$. $r_{k,n} \in [d, \infty)$, for some $d > 0$, $\theta_n = \sup_{p\in\Omega}(\max\{k_n, l_n\} - 1)\phi(p, x_n)$, $A_n \equiv A_n(mod\ N)$, $\|A_n x\| \le \|A_n x - A_n p\|$, for all $x \in C$ and $p \in \Omega$. Let $\{r_n\}$ be a sequence in* [0,1] *such that $0 < r_n < \frac{c^2\gamma}{2}$, where $\frac{1}{c}$ is the 2-uniformly convex constant of E.*

Let $\{\beta_{n,0}^{(1)}\}, \{\beta_{n,i}^{(2)}\}, \{\beta_{n,j}^{(3)}\}$ be sequences in [0,1] *satisfying the following conditions:*

1) *for each* $n \ge 0$, $\beta_{n,0}^{(1)} + \sum_{i=1}^\infty \beta_{n,i}^{(2)} + \sum_{j=1}^\infty \beta_{n,j}^{(3)} = 1$;

2) $\liminf_{n\to\infty} \beta_{n,0}^{(1)}\beta_{n,i}^{(2)} > 0$ *and* $\liminf_{n\to\infty} \beta_{n,0}^{(1)}\beta_{n,j}^{(3)} > 0$, $\forall i, j \geq 1, i \neq j$.

Then, the sequence $\{x_n\}$ *converges strongly to* $p^* = \Pi_\Omega(x_0)$.

Proof: See the complete this proof in [10]. ∎

III. THE COMPLEMENTARITY PROBLEMS

Complementarity problems are used to model several problems of economics, physics, optimization theory and engineering.

Definition III.1. Let K be a nonempty, closed and convex cone in E. We define the polar K^* of K as follows:

$$K^* = \{y^* \in E^* : \langle x, y^* \rangle \geq 0, \quad \forall x \in K\}.$$

Let a mapping $A : K \to E^*$ is an operator, then an element $u \in K$ is called a solution of complementarity problems if [16]

$$Au \in K^*, \quad \text{and} \quad \langle u, Au \rangle = 0.$$

The set of solutions of the complementarity problem is denoted by $CP(K, A)$.

Several problems arising in different fields such as mathematical programming, mechanics, game theory are to find the solutions of complementarity problem and the variational type problem.

Lemma III.2. *Let X be a nonempty, closed convex cone in a locally convex topological vector space E and let X^* be the polar of X. Let T be a mapping of X into E^*. Then, the set of solutions of the complementarity problem equals the set of solutions of the variational inequality. That is* $\{x \in X : Tx \in X^*$ *and* $\langle x, Tx \rangle = 0\} = \{x \in X : \langle u - x, Tx \rangle \geq 0, \ \forall u \in X\}$.

Proof: Let $x \in X$ be a solution of complementarity problem. Then, for any $u \in X$, we have

$$\begin{aligned} \langle u - x, Tx \rangle &= \langle u, Tx \rangle - \langle x, Tx \rangle \\ &= \langle u, Tx \rangle \\ &\geq 0. \end{aligned}$$

Therefore, x is a solution of variational inequality problem.

Conversely, let $x \in X$ be a solution of variational inequality problem. Then, we have

$$\langle u - x, Tx \rangle \geq 0, \quad \forall u \in X.$$

In the particular, if $u = 0$, we have $\langle x, Tx \rangle \leq 0$.

If $u = rx$, with $r = 1$, we have

$$\langle rx - x, Tx \rangle = (r - 1)\langle x, Tx \rangle \geq 0.$$

Hence, $\langle x, Tx \rangle \geq 0$. Therefore, $\langle x, Tx \rangle = 0$.

Next, we show that $Tx \in X^*$. If not, there exists $u_0 \in X$ such that $\langle x, Tx \rangle = 0$.

On the other hand, $\langle u_0 - x, Tx \rangle \geq 0$, So that, we have

$$0 > \langle u_0, Tx \rangle \geq \langle x, Tx \rangle = 0.$$

This is a contradiction. This implies that x is a solution of complementarity problem. ∎

Theorem III.3. *Let K be a nonempty, closed and convex subset of a uniformly smooth and 2-uniformly convex real Banach space E. Suppose that*

(C1) Let $B_k : K \to E^*$ for each $k = 1, 2, 3, ..., M$ be a finite family of δ_k-inverse-strongly monotone mappings, and let $F_k : K \times K \to \mathbb{R}$ be a bifunction which satisfies conditions (A1)-(A4).

(C2) Let $\{T_i\}_{i=1}^{\infty}$ and $\{S_j\}_{j=1}^{\infty} : K \to K$ be countable families of uniformly closed and ω_i,μ_j-Lipschitz continuous and quasi -ϕ- asymptotically nonexpansive mappings with sequences $\{k_n\}, \{l_n\} \subset [1, \infty)$ and $k_n \to 1, l_n \to 1$, respectively.

(C3) Let $A_n : K \to E^*$ for each $n = 1, 2, 3, ..., N$ be a finite family of γ_n-inverse strongly monotone mappings and let $\gamma = min\{\gamma_n : n = 1, 2, 3, ..., N\}$.

(C4) $\Omega := \left(\bigcap_{i=1}^{\infty} Fix(T_i)\right) \bigcap \left(\bigcap_{j=1}^{\infty} Fix(S_j)\right) \bigcap$

$$\left(\bigcap_{k=1}^{M} SGEP(F_k, B_k)\right) \bigcap \left(\bigcap_{n=1}^{N} CP(K, A_n)\right)$$

is a nonempty and bounded in C.

Let $\{x_n\}_{n=1}^{\infty}, \{z_n\}_{n=1}^{\infty}, \{y_n\}_{n=1}^{\infty}$ and $\{u_n\}_{n=1}^{\infty}$ be sequences generated by

$$\begin{cases} x_0 \in K \ \ \text{chosen arbitrary}, \ K_0 = K \\ z_n = \Pi_K J^{-1}(Jx_n - r_n A_n x_n) \\ y_n = J^{-1}(\beta_{n,0}^{(1)} Jx_n + \sum_{i=1}^{\infty} \beta_{n,i}^{(2)} JT_i^n x_n \\ + \sum_{j=1}^{\infty} \beta_{n,j}^{(3)} JS_j^n z_n) \\ u_n = T_{r_{M,n}}^{F_M} T_{r_{M-1,n}}^{F_{M-1}} ... T_{r_{2,n}}^{F_2} T_{r_{1,n}}^{F_1} y_n \\ K_{n+1} = \{v \in K_n : \phi(v, u_n) \leq \phi(v, x_n) + \theta_n\} \\ x_{n+1} = \Pi_{K_{n+1}}(x_0), \quad \forall n \geq 0, \end{cases} \tag{III.1}$$

where $T_{r_{k,n}}^{F_k} : E \to K$, $k = 1, 2, 3, ..., M$, is a mapping defined by (I.17) *with $F = F_k$ and $r = r_{k,n}$ and it is the solutions to the following system of generalized equilibrium problem:*

$$F_k(z, y) + \langle y - z, B_k z \rangle + \frac{1}{r}\langle y - z, Jz - Jx \rangle \geq 0,$$

$\forall y \in C, \quad k = 1, 2, 3, ..., M.$ *$r_{k,n} \in [d, \infty)$, for some $d > 0$,*

$$\theta_n = \sup_{p \in \Omega} (\max\{k_n, l_n\} - 1)\phi(p, x_n),$$

$A_n \equiv A_n$(mod N), $\|A_n x\| \leq \|A_n x - A_n p\|$, for all $x \in C$ and $p \in \Omega$. Let $\{r_n\}$ be a sequence in [0,1] *such that $0 < r_n < \frac{c^2\gamma}{2}$, where $\frac{1}{c}$ is the 2-uniformly convex constant of E. Let $\{\beta_{n,0}^{(1)}\}, \{\beta_{n,i}^{(2)}\}, \{\beta_{n,j}^{(3)}\}$ be sequences in* [0,1] *satisfying the following conditions:*

1) *for each* $n \geq 0$, $\beta_{n,0}^{(1)} + \sum_{i=1}^{\infty} \beta_{n,i}^{(2)} + \sum_{j=1}^{\infty} \beta_{n,j}^{(3)} = 1$;
2) $\liminf_{n\to\infty} \beta_{n,0}^{(1)}\beta_{n,i}^{(2)} > 0$ *and* $\liminf_{n\to\infty} \beta_{n,0}^{(1)}\beta_{n,j}^{(3)} > 0$, $\forall i, j \geq 1, i \neq j$.

Then, the sequence $\{x_n\}$ *converges strongly to* $p^* = \Pi_\Omega(x_0)$.

Proof: As in the proof of Lemma III.2, we have $VI(C, A_n) = CP(K, A_n)$,

So, we obtain the above the results. ∎

IV. A ZERO POINT OF MAXIMAL MONOTONE OPERATORS

Definition IV.1. Let E be a uniformly smooth and strictly convex Banach space and A be a maximal monotone operator from E to E^*. For each $r > 0$, we can define a single valued mapping $J_r : E \to D(A)$ by $J_r = (J + rA)^{-1}J$ and such a mapping J_r is called the *resolvent* of A. It is easy to show that $Fix(J_r) = A^{-1}(0)$, for all $r > 0$.

It is well known that if E is a smooth strictly convex and reflexive Banach space and A is a continuous monotone operator with $A^{-1}0 \neq \emptyset$, then J_r is weak relatively nonexpansive mappings. We know that $F(J_r)$ is closed and convex.

One of the most interesting and important problems in the theory of maximal monotone operators is to find an efficient iterative algorithm to compute approximately zeroes of maximal monotone operators. The proximal point algorithm of Rockafellar [13] is recognized as a powerful and successful algorithm for finding a solution of maximal monotone operators.

Theorem IV.2. *Let C be a nonempty, closed and convex subset of a uniformly smooth and 2-uniformly convex real Banach space E. Suppose that*

(D1) Let $B_k : C \to E^*$ for each $k = 1,2,3,...,M$ be a finite family of δ_k-inverse-strongly monotone mappings, and let $F_k : C\times C \to \mathbb{R}$ be a bifunction which satisfies conditions (A1)-(A4).

(D2) Let $\{T_i\}_{i=1}^{\infty}$ and $\{S_j\}_{j=1}^{\infty} : C \to C$ be countable families of uniformly closed and weak relatively nonexpansive mappings.

(D3) Let $A_n : C \to E^*$ for each $n = 1,2,3,...,N$ be a finite family of γ_n-inverse strongly monotone mappings and let $\gamma = min\{\gamma_n : n = 1,2,3,...,N\}$.

(D4) $\Omega := \left(\bigcap_{i=1}^{\infty} Fix(T_i)\right) \bigcap \left(\bigcap_{j=1}^{\infty} Fix(S_j)\right) \bigcap$

$$\left(\bigcap_{k=1}^{M} SGEP(F_k, B_k)\right) \bigcap \left(\bigcap_{n=1}^{N} VI(C, A_n)\right)$$

is a nonempty and bounded in C.

Let $\{x_n\}_{n=1}^{\infty}, \{z_n\}_{n=1}^{\infty}, \{y_n\}_{n=1}^{\infty}$ and $\{u_n\}_{n=1}^{\infty}$ be sequences generated by

$$\begin{cases} x_0 \in C \quad \textit{chosen arbitrary}, \; C_0 = C \\ z_n = \Pi_C J^{-1}(Jx_n - r_n A_n x_n) \\ y_n = J^{-1}(\beta_{n,0}^{(1)} Jx_n + \sum_{i=1}^{\infty} \beta_{n,i}^{(2)} JT_i x_n \\ + \sum_{j=1}^{\infty} \beta_{n,j}^{(3)} JS_j z_n) \\ u_n = T_{r_{M,n}}^{F_M} T_{r_{M-1,n}}^{F_{M-1}} ... T_{r_{2,n}}^{F_2} T_{r_{1,n}}^{F_1} y_n \\ C_{n+1} = \{v \in C_n : \phi(v, u_n) \le \phi(v, x_n)\} \\ x_{n+1} = \Pi_{C_{n+1}}(x_0), \quad \forall n \ge 0, \end{cases} \tag{IV.1}$$

where $T_{r_{k,n}}^{F_k} : E \to C$, $k = 1,2,3,...,M$, is a mapping defined by (I.17) *with $F = F_k$ and $r = r_{k,n}$ and it is the solutions to the following system of generalized equilibrium problem: $F_k(z,y) + \langle y - z, B_k z\rangle + \frac{1}{r}\langle y - z, Jz - Jx\rangle \ge 0, \quad \forall y \in C, \quad k = 1,2,3,...,M$. $r_{k,n} \in [d,\infty)$, for some $d > 0$, $A_n \equiv A_n(mod\ N)$, $\|A_n x\| \le \|A_n x - A_n p\|$, for all $x \in C$ and $p \in \Omega$. Let $\{r_n\}$ be a sequence in* [0,1] *such that $0 < r_n < \frac{c^2\gamma}{2}$, where $\frac{1}{c}$ is the 2-uniformly convex constant of E. Let $\{\beta_{n,0}^{(1)}\}, \{\beta_{n,i}^{(2)}\}, \{\beta_{n,j}^{(3)}\}$ be sequences in* [0,1] *satisfying the following conditions:*

1) *for each $n \ge 0$, $\beta_{n,0}^{(1)} + \sum_{i=1}^{\infty} \beta_{n,i}^{(2)} + \sum_{j=1}^{\infty} \beta_{n,j}^{(3)} = 1$;*
2) *$\liminf_{n\to\infty} \beta_{n,0}^{(1)}\beta_{n,i}^{(2)} > 0$ and $\liminf_{n\to\infty} \beta_{n,0}^{(1)}\beta_{n,j}^{(3)} > 0$, $\forall i,j \ge 1, i \ne j$.*

Then, the sequence $\{x_n\}$ converges strongly to $p^* = \Pi_\Omega(x_0)$.

Proof: Since $\{T_i\}_{i=1}^{\infty}$ and $\{S_j\}_{j=1}^{\infty}$ are countable families of uniformly closed and weak relatively nonexpansive mappings, By Remark I.7, it is countable families of uniformly closed and quasi -ϕ- nonexpansive mappings, and it is countable families of uniformly closed and quasi -ϕ- asymptotically nonexpansive mappings. Therefore, it can be obtained from Theorem II.3 immediately. ■

If $T_i = T$, $S_j = S$, $F_k = F$, $B_k = B$ and $A_n = A$ where $\forall i,j \in \mathbb{N}$, $k = 1,2,3,...,M$ and $\forall n = 1,2,3,...,N$ in Theorem II.3, then the Theorem II.3 is reduced to the following corollary.

Theorem IV.3. *Let C be a nonempty, closed and convex subset of a uniformly smooth and 2-uniformly convex real Banach space E. Suppose that*

(E1) Let $B_k : C \to E^*$ for each $k = 1,2,3,...,M$ be a finite family of δ_k-inverse-strongly monotone mappings, and let $F_k : C\times C \to \mathbb{R}$ be a bifunction which satisfies conditions (A1)-(A4).

(E2) Let A and B be two maximal monotone operators from E to E^* and let J_r^A and J_r^B be the resolvent of A and B, respectively, where $r > 0$.

(E3) Let $A_n : C \to E^*$ for each $n = 1,2,3,...,N$ be a finite family of γ_n-inverse strongly monotone mappings and let $\gamma = min\{\gamma_n : n = 1,2,3,...,N\}$.

(E4)

$$\Omega := A^{-1}(0) \bigcap B^{-1}(0) \bigcap \left(\bigcap_{k=1}^{M} SGEP(F_k, B_k)\right)$$

$\bigcap \left(\bigcap_{n=1}^{N} VI(C, A_n)\right)$ *is a nonempty and bounded in C.*

Let $\{x_n\}_{n=1}^{\infty}, \{z_n\}_{n=1}^{\infty}, \{y_n\}_{n=1}^{\infty}$ and $\{u_n\}_{n=1}^{\infty}$ be sequences generated by

$$\begin{cases} x_0 \in C \quad \textit{chosen arbitrary}, \; C_0 = C \\ z_n = \Pi_C J^{-1}(Jx_n - r_n A_n x_n) \\ y_n = J^{-1}(\beta_{n,0}^{(1)} Jx_n + \sum_{i=1}^{\infty} \beta_{n,i}^{(2)} JJ_{r_i}^A x_n \\ + \sum_{j=1}^{\infty} \beta_{n,j}^{(3)} JJ_{r_j}^B z_n) \\ u_n = T_{r_{M,n}}^{F_M} T_{r_{M-1,n}}^{F_{M-1}} ... T_{r_{2,n}}^{F_2} T_{r_{1,n}}^{F_1} y_n \\ C_{n+1} = \{v \in C_n : \phi(v, u_n) \le \phi(v, x_n) + \theta_n\} \\ x_{n+1} = \Pi_{C_{n+1}}(x_0), \quad \forall n \ge 0, \end{cases} \tag{IV.2}$$

where $T_{r_{k,n}}^{F_k} : E \to C$, $k = 1,2,3,...,M$, is a mapping defined by (I.17) *with $F = F_k$ and $r = r_{k,n}$ and it is the solutions to the following system of generalized equilibrium problem: $F_k(z,y) + \langle y - z, B_k z\rangle + \frac{1}{r}\langle y - z, Jz - Jx\rangle \ge 0, \quad \forall y \in C, \quad k = 1,2,3,...,M$. $r_{k,n} \in [d,\infty)$, for some $d > 0$, $\theta_n = \sup_{p\in\Omega}(\max\{k_n, l_n\} - 1)\phi(p, x_n)$, $A_n \equiv A_n(mod\ N)$, $\|A_n x\| \le \|A_n x - A_n p\|$, for all $x \in C$ and $p \in \Omega$. Let $\{r_n\}$ be a sequence in* [0,1] *such that $0 < r_n < \frac{c^2\gamma}{2}$, where $\frac{1}{c}$ is the 2-uniformly convex constant of E. Let $\{\beta_{n,0}^{(1)}\}, \{\beta_{n,i}^{(2)}\}, \{\beta_{n,j}^{(3)}\}$ be sequences in* [0,1] *satisfying the following conditions:*

1) *for each $n \ge 0$, $\beta_{n,0}^{(1)} + \sum_{i=1}^{\infty} \beta_{n,i}^{(2)} + \sum_{j=1}^{\infty} \beta_{n,j}^{(3)} = 1$;*
2) *$\liminf_{n\to\infty} \beta_{n,0}^{(1)}\beta_{n,i}^{(2)} > 0$ and $\liminf_{n\to\infty} \beta_{n,0}^{(1)}\beta_{n,j}^{(3)} > 0$, $\forall i,j \ge 1, i \ne j$.*

Then, the sequence $\{x_n\}$ converges strongly to $p^* = \Pi_\Omega(x_0)$.

Proof: It is well-known that for each $i \geq 1$, $J^A_{r_i}$ is a relatively nonexpansive mapping (see, for example [9], [11]).

Therefore, for each $p \in Fix(J^A_{r_i})$ and $w \in E$, we have

$$\phi(p, J^A_{r_i} w) \leq \phi(p, w).$$

Again, by the same method we can prove that the set of strong asymptotic fixed points, so that

$$\widetilde{Fix}(\{J^A_{r_i}\}_{i=1}^{\infty}) = \bigcap_{i=1}^{\infty} Fix(J^A_{r_i}) = A^{-1}(0).$$

This implies that $\{J^A_{r_i}\}_{i=1}^{\infty}$ is a countable family of weak relatively nonexpansive mapping with the common fixed point set $\bigcap_{i=1}^{\infty} Fix(J^A_{r_i}) = A^{-1}(0)$.

By the similar way, we can prove that $\{J^B_{r_j}\}_{j=1}^{\infty}$ is a countable family of weak relatively nonexpansive mapping with the common fixed point set $\bigcap_{j=1}^{\infty} Fix(J^B_{r_j}) = B^{-1}(0)$.

Hence, the conclusion of the Theorem IV.3 can be obtained from Theorem IV.2 immediately. ■

A. Application to Hilbert spaces

Theorem IV.4. *Let C be a nonempty, closed and convex subset of a Hilbert space H. Suppose that*

(F1) Let $B_k : C \to H$ for each $k = 1, 2, 3, ..., M$ be a finite family of δ_k-inverse-strongly monotone mappings, and let $F_k : C \times C \to \mathbb{R}$ be a bifunction which satisfies conditions (A1)-(A4).

(F2) Let $\{T_i\}_{i=1}^{\infty}$ and $\{S_j\}_{j=1}^{\infty} : C \to C$ be countable families of uniformly closed and ω_i,μ_j-Lipschitz continuous and quasi -ϕ- asymptotically nonexpansive mappings with sequences $\{k_n\}, \{l_n\} \subset [1, \infty)$ and $k_n \to 1, l_n \to 1$, respectively.

(F3) Let $A_n : C \to H$ for each $n = 1, 2, 3, ..., N$ be a finite family of γ_n-inverse strongly monotone mappings and let $\gamma = min\{\gamma_n : n = 1, 2, 3, ..., N\}$.

(F4) $\Omega := \Big(\bigcap_{i=1}^{\infty} Fix(T_i)\Big) \bigcap \Big(\bigcap_{j=1}^{\infty} Fix(S_j)\Big) \bigcap$

$$\Big(\bigcap_{k=1}^{M} SGEP(F_k, B_k)\Big) \bigcap \Big(\bigcap_{n=1}^{N} VI(C, A_n)\Big)$$

is a nonempty and bounded in C.

Let $\{x_n\}_{n=1}^{\infty}, \{z_n\}_{n=1}^{\infty}, \{y_n\}_{n=1}^{\infty}$ and $\{u_n\}_{n=1}^{\infty}$ be sequences generated by

$$\begin{cases} x_0 \in C \ \ \text{chosen arbitrary}, \ C_0 = C \\ z_n = P_C(x_n - r_n A_n x_n) \\ y_n = \beta^{(1)}_{n,0} J x_n + \sum_{i=1}^{\infty} \beta^{(2)}_{n,i} T_i^n x_n + \sum_{j=1}^{\infty} \beta^{(3)}_{n,j} S_j^n z_n \\ u_n = T^{F_M}_{r_{M,n}} T^{F_{M-1}}_{r_{M-1,n}} ... T^{F_2}_{r_{2,n}} T^{F_1}_{r_{1,n}} y_n \\ C_{n+1} = \{v \in C_n : \|v - u_n\| \leq \|v - x_n\| + \theta_n\} \\ x_{n+1} = P_{C_{n+1}}(x_0), \quad \forall n \geq 0, \end{cases} \tag{IV.3}$$

where $T^{F_k}_{r_{k,n}} : H \to C$, $k = 1, 2, 3, ..., M$, is a mapping defined by (I.17) *with $F = F_k$ and $r = r_{k,n}$ and it is the solutions to the following system of generalized equilibrium problem: $F_k(z, y) + \langle y - z, B_k z\rangle + \frac{1}{r}\langle y - z, Jz - Jx\rangle \geq 0$, $\forall y \in C$, $k = 1, 2, 3, ..., M$. $r_{k,n} \in [d, \infty)$, for some $d > 0$, $\theta_n = \sup_{p\in\Omega}(\max\{k_n, l_n\} - 1)\|p - x_n\|$, $A_n \equiv A_n$(mod N), $\|A_n x\| \leq \|A_n x - A_n p\|$, for all $x \in C$ and $p \in \Omega$. Let $\{r_n\}$ be a sequence in* [a , b] *for some a, b such that $0 < a < b < \frac{r_n}{2}$. Let $\{\beta^{(1)}_{n,0}\}, \{\beta^{(2)}_{n,i}\}, \{\beta^{(3)}_{n,j}\}$ be sequences in* [0,1] *satisfying the following conditions:*

1) *for each $n \geq 0$, $\beta^{(1)}_{n,0} + \sum_{i=1}^{\infty} \beta^{(2)}_{n,i} + \sum_{j=1}^{\infty} \beta^{(3)}_{n,j} = 1$;*
2) $\liminf_{n\to\infty} \beta^{(1)}_{n,0}\beta^{(2)}_{n,i} > 0$ *and* $\liminf_{n\to\infty} \beta^{(1)}_{n,0}\beta^{(3)}_{n,j} > 0$, $\forall i, j \geq 1, i \neq j$.

Then, the sequence $\{x_n\}$ converges strongly to $p^* = P_\Omega(x_0)$.

Proof: If $E = H$, a Hilbert space, then E is 2-uniformly convex (we can choose $c = 1$) and uniformly smooth Banach space.

Moreover, $J = I$, identity mapping on H and $\Pi_C = P_C$, projection mapping from H to C.

Thus, the conclusion of the Theorem IV.4 can be obtained from Theorem II.3 immediately. ■

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