

# Pareto-ranking efficient method using dominance-based Hasse diagrams

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**Abstract**—Let a complex multi-objective optimization problem for which we would like to determine near Pareto-optimal sets. Consider a finite set of feasible solutions at random. Performing all pairwise comparisons based on the concept of dominance, we can find which solutions are non-dominated. This subset defines the first rank of non-dominated solutions. Eliminating these solutions, we could repeat the procedure, and thus determine the next class of non-dominated solutions, and so forth. Using the same set of initial data, our method constructs and analyzes the ordinal structure of a directed acyclic graph by using Hasse diagrams. All the ranks of solutions are deduced simultaneously and define an efficient Pareto-ranking. A simple application of engineering optimization illustrates the method.

**Keywords**—Hasse diagram, near Pareto-optimal front, non-dominated set, Pareto ranking method.

## I. INTRODUCTION

THE real-world optimization problems that we consider in this article are continuous multiobjective optimization (MOO) problems which standard form is

$$\underset{\mathbf{x} \in X \subseteq \mathbb{R}^n}{\text{minimize}} \mathbf{f}(\mathbf{x}) \triangleq (f_1(\mathbf{x}), \dots, f_r(\mathbf{x}))^T \quad (1)$$

where the conflicting objectives state that

$f_k : \mathbb{R}^n \mapsto \mathbb{R}$ ,  $k = 1, \dots, r$ . The feasible search space for this problem can be  $X_f = \{\mathbf{x} \in \mathbb{R}^n \mid g_j(\mathbf{x}) \leq 0, j = 1, \dots, m\}$ , where the inequality constraints are  $g_j : \mathbb{R}^n \mapsto \mathbb{R}$ ,  $j = 1, \dots, m$ .

Problem (1) expresses the simultaneous minimization of uncorrelated objectives subject to inequality constraints<sup>1</sup>. We expect a set of ‘best’ solutions for this issue. To compare two solutions and tell which solution is “better” in the line of the objectives, we have to use the concept of Pareto-dominance. Thereafter, the Pareto multi-objective optimization problem is to find the non-dominated set of solutions<sup>2</sup>.

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<sup>1</sup> The decision variables are generally bounded in practice, such as we may add the following bounds  $x_i \in [x_i^l, x_i^u]$ ,  $i = 1, \dots, n$ .

<sup>2</sup> This article expands with new materials and application an issue addressed in the Keller’s eBook (2016) [1] being published.

## II. PARETO-DOMINANCE CONCEPTS

### A. Pareto-Optimality

Let  $\mathbf{x}^i < \mathbf{x}^j$  denote that the solution  $\mathbf{x}^i$  of a minimizing problem is “better” than the solution  $\mathbf{x}^j$  on a particular objective.

**Definition 1** (Pareto-optimal solution). A feasible solution  $\mathbf{x}$  of a minimizing MOO problem is Pareto-optimal if there exists no other better solution  $\mathbf{x}'$ . Then we have  $f_i(\mathbf{x}) < f_i(\mathbf{x}')$ ,  $\exists i \in \{1, \dots, r\}$  for at least one objective. ■

In other words,  $\hat{\mathbf{x}}$  is Pareto-optimal if there exists no feasible  $\mathbf{x}' \in X$  which would improve the performance of some objective without decreasing that of at least one another objective.

**Definition 2** (Pareto optimal set). A Pareto-optimal set is such that  $\mathcal{P} = \{\mathbf{x} \in X \mid \neg \exists \mathbf{x}' \in X : \mathbf{f}(\mathbf{x}') \leq \mathbf{f}(\mathbf{x})\}$ . ■

**Definition 3** (Pareto optimal front). The Pareto optimal front is such that  $\mathcal{F} = \{\mathbf{f}(\mathbf{x}) \mid \mathbf{x} \in \mathcal{P}\}$ . ■

### B. Non-Dominated Solutions

**Definition 4** (Dominance). A solution  $\mathbf{x}^1$  dominates another solution  $\mathbf{x}^2$  (i.e.,  $\mathbf{x}^1 < \mathbf{x}^2$ ) iff two conditions are verified:

- 1) The solution  $\mathbf{x}^1$  is no worse than the solution  $\mathbf{x}^2$  in all the objectives, that is  $f_k(\mathbf{x}^1) \not> f_k(\mathbf{x}^2)$ , for all  $k = 1, \dots, r$ .
- 2) The solution  $\mathbf{x}^1$  is strictly better than the solution  $\mathbf{x}^2$  in at least one objective i.e.,  $f_{\bar{k}}(\mathbf{x}^1) < f_{\bar{k}}(\mathbf{x}^2)$  in at least one  $\bar{k} \in \{1, \dots, r\}$ . ■

Suppose a bi-objective optimization problem. For any  $\mathbf{y}^1, \mathbf{y}^2 \in \mathbb{R}^2$ , we have the dominance  $\mathbf{y}^1 < \mathbf{y}^2$  iff  $y_i^1 \leq y_i^2$ ,  $\forall i = 1, 2$  and  $\exists i \in \{1, 2\} : y_i^1 < y_i^2$ . In condensed form, we can write equivalently

$$\left( y_1^1 < y_1^2 \wedge y_2^1 \leq y_2^2 \right) \vee \left( y_1^1 \leq y_1^2 \wedge y_2^1 < y_2^2 \right).$$

Let the solution space (or objective space) be  $Y = \mathbf{f}(X)$  i.e., the image of search space<sup>3</sup>  $X$  under  $\mathbf{f}$ .

**Definition 5** (Non-dominated set). For any compact subset  $Y \subset \mathbb{R}^r$ , there exist minimal elements w.r.t. the partial order “ $\leq$ ”, also called non-dominated points. Formally, we can write  $Y_N = \{\mathbf{y} \in Y \mid \neg \exists \mathbf{y}' \in Y : \mathbf{y}' < \mathbf{y}\}$ .

The aim of Pareto optimization is to find the non-dominated set of solutions<sup>4</sup>.

C. Domination Degrees

We have different degrees of domination, for which the definitions are as follows

**Definition 6** (Strong domination). The solution  $\mathbf{x}^1$  “strongly dominates” the solution  $\mathbf{x}^2$  (i.e.,  $\mathbf{x}^1 \ll \mathbf{x}^2$  or  $\mathbf{x}^1 < \mathbf{x}^2$ ) if  $f_k(\mathbf{x}^1) < f_k(\mathbf{x}^2), \forall k = 1, \dots, r$ . In other words, the solution  $\mathbf{x}^1$  is better than  $\mathbf{x}^2$  for all objectives.

**Definition 7** (Weak domination). The solution  $\mathbf{x}^1$  “weakly dominates” solution  $\mathbf{x}^2$  (i.e.,  $\mathbf{x}^1 \leq \mathbf{x}^2$ ) if  $f_k(\mathbf{x}^1) \not> f_k(\mathbf{x}^2), \forall k = 1, \dots, r$ . In other words, the solution  $\mathbf{x}^1$  is not worse than  $\mathbf{x}^2$  for all objectives.

**Definition 8** (Incomparable domination). Two solutions  $\mathbf{x}^1$  and  $\mathbf{x}^2$  are “incomparable” w.r.t. the domination (i.e.,  $\mathbf{x}^1 \parallel \mathbf{x}^2$  or  $\mathbf{x}^2 \parallel \mathbf{x}^1$ ) if  $f_k(\mathbf{x}^1) \not\leq f_k(\mathbf{x}^2)$  nor  $f_k(\mathbf{x}^1) \not\geq f_k(\mathbf{x}^2), \forall k = 1, \dots, r$ .

III. PARETO-RANKING

Which elements of a population of potential solutions can we select as members of the next generation? This question is that of the selection mechanism of an evolutionary algorithm (EA), used to solve a complex MOO problem (see Alberto *et al.*, 2003 [6]).

A. Pareto-Based Ranking Methods

Most commonly ranking methods are Goldberg’s ranking and Fonseca and Fleming’s ranking<sup>5 6</sup>. Goldberg’s ranking in

1989 [9] assign an equal rank 0 to the non-dominated solutions. Removing that solution set from the population of candidates, the new set of non-dominated solution is obtained with rank 1, and so forth. The approach by Fonseca and Fleming (1993) [10] is different. The rank of a solution in the objective space equals the number of other solutions by which it is dominated. The non-dominated solutions are all assigned rank 0. All the dominated solutions will then have a rank between 1 and  $N - 1$ , where  $N$  denotes the population size.

B. Determination of Non-Dominated Solutions

The determination of non-dominated solutions is the direct application of the concept of dominance which computation is illustrated in Fig. 1.

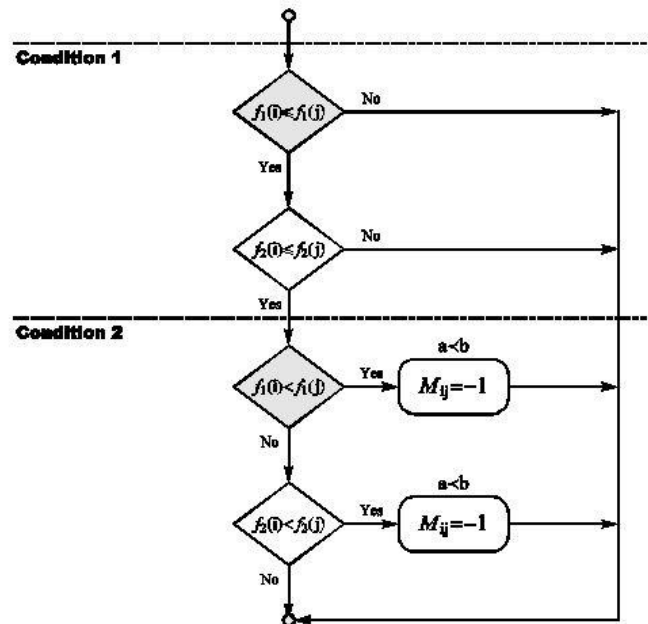


Fig. 1 construction of the dominance matrix based on the conditions of the dominance relation

The data of the following example are drawn from Reyes-Sierra and Coello (2006) [11].

**Example 1** An application is defined by a given set of 18 feasible solutions in the objective space. We suppose a MOO minimization problem with two objectives. The dominance matrix for this example is a  $18 \times 18$  sparse matrix in Fig. 2.

<sup>3</sup> The inverse image of  $Y_N$  defined by  $X_N = \mathbf{f}^{-1}(Y_N)$  is the efficient set of the MOO problem, which members are the efficient solutions.

<sup>4</sup> More theoretical aspects of ordered sets and Pareto optima in MOO can be found in Ehrgott (2000)[2], Emmerich and Deutz (2006) [3], Coello *et al.* (2007) [4], and Rudeanu (2012) [5].

<sup>5</sup> Alberto *et al.* (2003) [6] also presented the Belegundu’s ranking by which all the non-dominated points are assigned rank 0 and the dominated ones rank 1.

<sup>6</sup> Deb (2001) [7] discusses three algorithms for finding a non-dominated set. In the simplest approach, each solution is compared with every other solutions in a population of size  $N$ . Suppose that we have  $M$  objectives.

The total complexity for this procedure is  $\mathcal{O}(MN^2)$ . A more efficient technique was introduced by Kung *et al* (1975) [8].

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
2	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
3	.	-1	.	.	-1	.	.	.	.	.	.	.	.	.	.	.	.	.
4	-1	-1	.	.	-1	.	.	.	.	.	.	.	.	.	.	.	.	.
5	.	-1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
6	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
7	-1	-1	.	.	-1	.	.	.	.	.	.	.	.	.	.	.	.	.
8	-1	-1	-1	-1	-1	-1	.	.	.	.	.	.	.	-1	.	.	.	.
9	-1	-1	.	.	-1	-1	-1	.	.	.	.	.	.	.	.	.	.	.
10	-1	-1	.	.	-1	-1	-1	-1	.	.	.	.	.	-1	-1	-1	.	.
11	-1	-1	.	.	-1	-1	.	.	.	.	.	.	-1	-1	-1	.	.	.
12	.	.	.	.	-1	.	.	.	.	.	.	.	-1	.	.	.	.	.
13	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
14	-1	.	.	.	-1	.	.	.	.	.	.	.	.	.	.	.	.	.
15	-1	.	.	.	-1	.	.	.	.	.	.	-1	-1	-1	.	.	.	.
16	-1	-1	.	.	-1	-1	.	.	.	.	-1	-1	-1	-1	-1	.	.	-1
17	.	.	.	.	-1	.	.	.	.	.	.	-1	.	.	.	.	.	-1
18	.	.	.	.	.	.	.	.	.	.	-1	.	.	.	.	.	.	.

Fig. 2 dominance matrix for Example 1 where a point “.” stands for zero.

The entries of the dominance matrix  $M$  take the values 0 or -1. An entry  $m_{ij} = 0$  denotes that  $i \not< j$ , and  $i$  dominates  $j$  (i.e.,  $i < j$ ) when  $m_{ij} = -1$ .

The data and the non-dominated solutions are shown in Fig. 3.

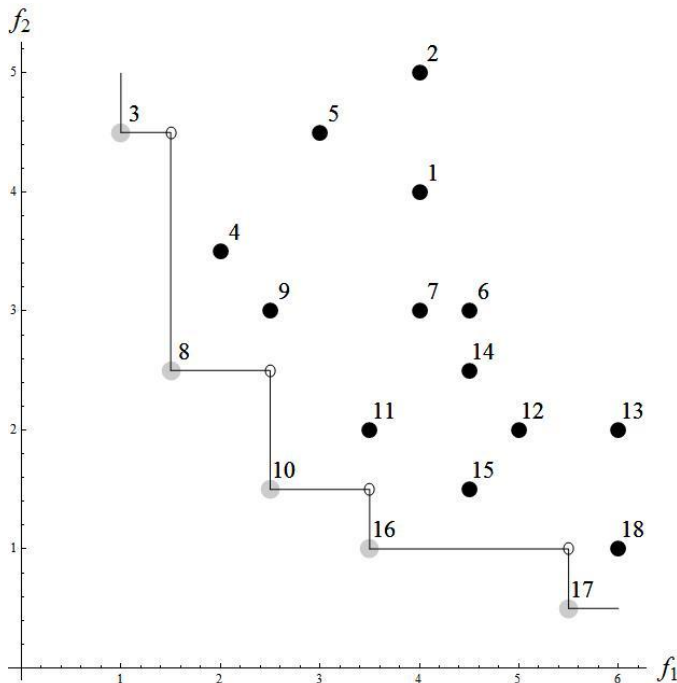


Fig. 3 all potential solutions with non-dominated solutions

IV. EFFICIENT PARETO-RANKING METHOD

Changing the entries  $m_{ij} = -1$  into  $m_{ij} = 1$  for all  $i, j = 1, \dots, 18$ , we obtain the adjacency matrix  $A$  of an oriented digraph (i.e., without loops). Moreover, the digraph is acyclic since its construction is based on a dominance concept. These properties validate the use of Hasse diagrams. A circular embedding of this graph for Example 1 is represented in Fig. 4 (a). In this 18-vertex oriented graph, vertices are placed on a unit circle, and arcs connect pairs of vertices (see Pemmaraju and Skiena, 2003 [12]). Since this graph is acyclic, we can determine the corresponding Hasse diagram in Fig. 4(b).

A. Hasse Diagram

A Hasse diagram is the best embedding for a partially ordered set (i.e., “poset”). In this context, the dominance relation is a strict partial relation. In fact, the dominance relation “ $<$ ” is not reflexive, asymmetric, antisymmetric, and transitive. The Hasse diagram is drawn according to the following rules:

- 1) If  $i < j$  then  $i$  is placed below  $j$ ,
- 2) No edge is implied by transitivity,
- 3) All edges, whose orientation is omitted, go upwards.

The Hasse diagram shows all the ranks. At the bottom, we find the subset of non-dominated with rank [0], i.e.,  $\{3, 8, 10, 16, 17\}$ . At other levels, we find subsets of vertices with different ranks<sup>7</sup>.

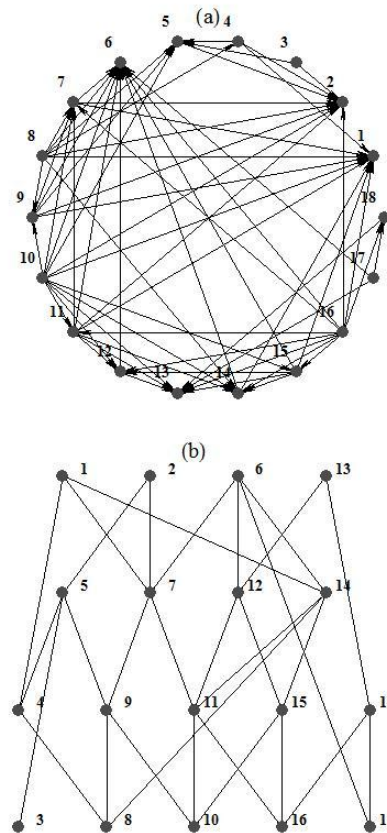


Fig. 4 oriented digraph in (a) and Hasse diagram in (b) of Example 1.

B. Pareto Ranking

The Pareto-ranking of Example 1 is deduced from the Hasse diagram. The result is shown in Fig. 5.

<sup>7</sup> Partial order applications were presented by Brüggermann and Patil (2011) [13].

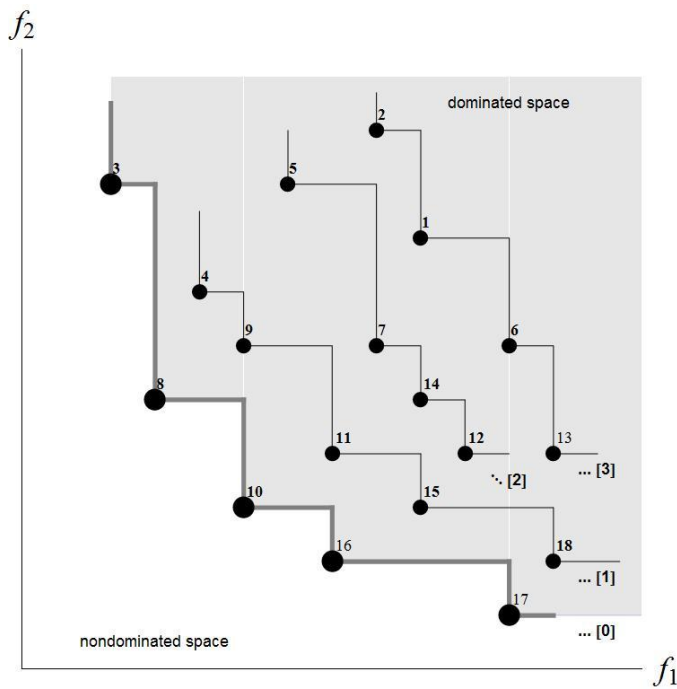


Fig. 5 Pareto-ranking the Example 1

V. APPLICATION TO AN ENGINEERING PROBLEM

The following application is the simple four-bar plane truss. It was retained by Coello (2001) [14] to illustrate a MOO problem.

A. Engineering Problem

An illustration of the four-bar plane problem is shown in Fig. 6.

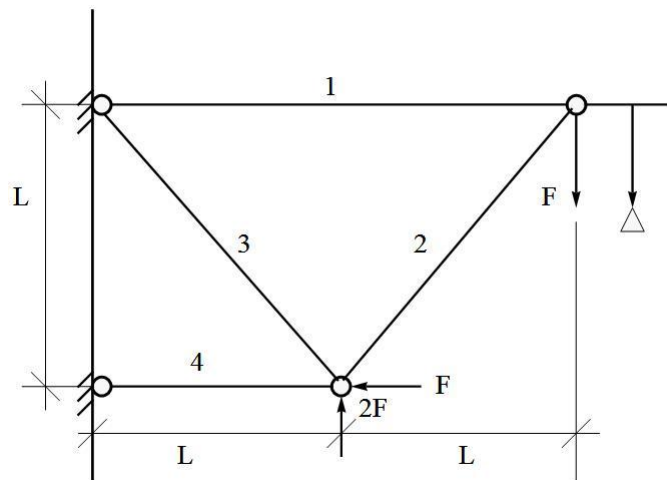


Fig. 6 a four-bar plane truss [adapted from Coello (2001) [14], Fig.1]

Example 2 (four-bar plane truss). The MOO problem consists of two objectives and four bounded decision variables. We have the following program

$$\begin{cases} \text{minimize } f_1(\mathbf{x}) = L(2x_1 + \sqrt{2}x_2 + \sqrt{x_3 + x_4}) \\ \mathbf{x} \in \mathbb{R}^4 \\ \text{minimize } f_2(\mathbf{x}) = \frac{FL}{E} \left( \frac{2}{x_1} + \frac{2\sqrt{2}}{x_2} - \frac{2\sqrt{2}}{x_3} + \frac{2}{x_4} \right) \quad (2) \\ \mathbf{x} \in \mathbb{R}^4 \\ \text{such that : } x_1, x_4 \in [F / \sigma, 3F / \sigma] \\ x_2, x_3 \in [\sqrt{2}F / \sigma, 3F / \sigma] \end{cases}$$

where the parameters are  $F = 10\text{kN}$ ,  $E = 2 \times 10^5 \text{ kN / cm}^2$  (Young's modulus),  $L = 200\text{cm}$ , and  $\sigma = 10\text{kN / cm}^2$ .

The engineering problem is to minimize simultaneously the structural volume of the truss ( $f_1$ ) and its joint displacement ( $f_2$ ). The design variables  $x_1, \dots, x_4$  denote the cross-sectional areas of the four bars. The four variables are bounded as in (2).

B. Pareto Ranking Based on a Small-size Data Set

Example 2 has no analytical Pareto-optimal front in the objective space. Therefore, the shape was specified by using the images of 1,000 points drawn randomly in the search space of the programming problem.

Using the Hasse diagram approach, we obtain the following non-dominated Pareto-optimal front in Fig. 7. This figure shows the image of 1,000 random feasible solutions. The Hasse diagram analysis is based on a subset of 20 points in the objective space (see big colored points). These points are drawn randomly in the initial list of the 1,000 points.

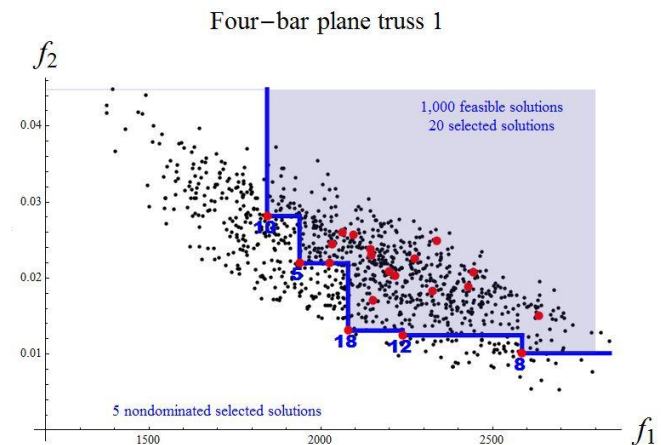
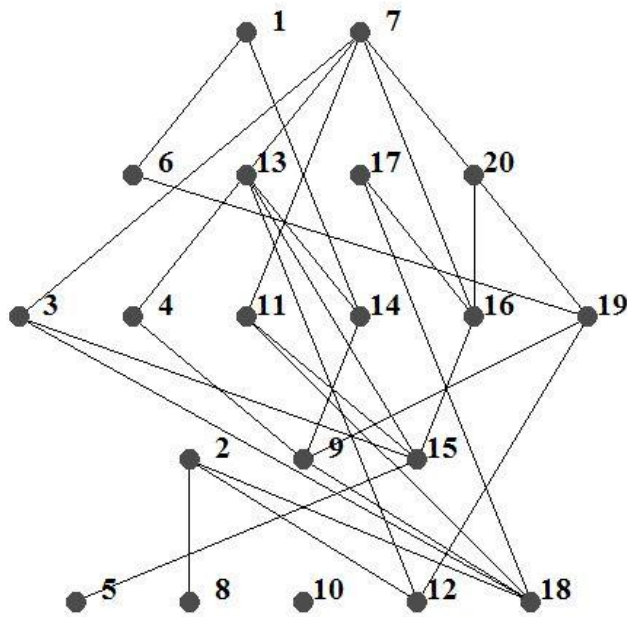


Fig. 7 a non-dominated set based on small-size data set for Example 2

The Pareto-ranking for this subset is shown in Fig. 8. We can draw the set of non-dominated solutions that is  $\{5, 8, 10, 12, 18\}$ . This set is reordered by increasing values of  $f_1$  as  $\{10, 5, 18, 12, 8\}$  to get the cone-representation of the non-dominated solutions at rank [0]. The shaded region of the stepwise near Pareto-optimal front includes all the dominated solutions.



**Fig. 8** Hasse diagram based on small-size data set of [Example 2](#)

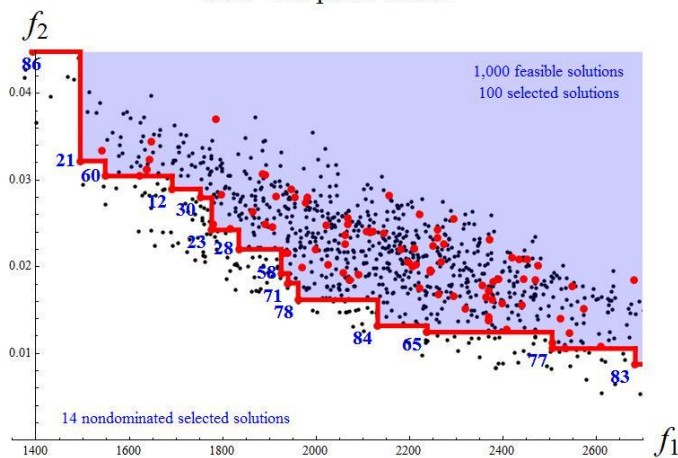
### C. Pareto Ranking Based on a Large-Size Data Set

Using the same data set (i.e., 1,000 images of random points), we draw a larger random subset of 100 points in the objective space (see the big colored points). [Fig. 9](#) shows better performances. The reordered set of non-dominated solutions consists of 14 elements (instead of 5 elements in the small-size case).

This reordered set is

$$\{86, 21, 60, 12, 30, 23, 28, 58, 71, 78, 84, 65, 77, 83\}.$$

Four-bar plane truss 2



**Fig. 9** nondominated set based on large-size data set of [Example 2](#)

## VI. CONCLUSION

The ranking method uses a dominance-based Hasse diagram to construct and analyze the ordinal structure of an acyclic digraph. A simple four-bar plane truss consists of two nonlinear objective functions and four bounded decision variables. We aim at approximating the Pareto-optimal front in

the objective space.

The image of 1,000 random feasible solutions approximates the shape of Pareto-optimal front to the border of the feasible set in the solution space. We define a partial order by means of the Pareto domination concept.

A random subset of 100 points is selected to construct an acyclic graph  $G(100,995)$  with 100 vertices and 995 oriented edges. The Hasse diagram  $H(100,267)$  shows a ranked ordinal structure with 9 ranks. At rank zero, we find the non-dominated solutions. Reordering the non-dominated set, we can represent a stepwise linear function which approximates the Pareto-optimal front for this problem.

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