Comparison of Two Methods of Numerical Solution of Mitchison Biological System of Nonlinear Partial Differential Equations

Temur Jangveladze, Zurab Kiguradze, Mikheil Gagoshidze and Besiki Tabatadze

***Abstract*—**The two-dimensional system of nonlinear partial differential equations is considered. This system arises in process of vein formation of young leaves. Additive splitting and variable directions type finite difference schemes are used. Comparison of numerical calculations of the proposed methods are done.

***Keywords*—**System of nonlinear partial differential equations, additive splitting and variable directions type finite difference schemes.

I. Introduction

THE main purpose of this article is to use the variable directions and additive splitting schemes for one system of nonlinear partial differential equations arising in various fields such as biology (e.g. vein formation of young leaves, see [20]).

The two-dimensional system to be considered here has the following form:

$$\frac{\partial U}{\partial t}=\frac{\partial}{\partial x_1}\left[a_1(V_1)\frac{\partial U}{\partial x_1}\right]+\frac{\partial}{\partial x_2}\left[a_2(V_2)\frac{\partial U}{\partial x_2}\right], \tag{1}$$

$$\frac{\partial V_1}{\partial t}=f_1\left(V_1,\frac{\partial U}{\partial x_1}\right), \tag{2}$$

$$\frac{\partial V_2}{\partial t}=f_2\left(V_2,\frac{\partial U}{\partial x_2}\right), \tag{3}$$

where a_α, f_α, $\alpha=1,2$ are given functions of their arguments.

Motivation for studying such system of equations can be provided as follows. Numerous phenomena, pertaining to physics, biology, medicine and so on are reasonably described in terms of the system of nonlinear partial differential equations of (1) - (3) type. If

$$f_\alpha\left(V_\alpha,\frac{\partial U}{\partial x_\alpha}\right)=-V_\alpha+g_\alpha\left(V_\alpha\,\frac{\partial U}{\partial x_\alpha}\right),$$

$$a_\alpha(V_\alpha)=V_\alpha,\quad \alpha=1,2,$$

where g_α are given sufficiently smooth functions of their arguments, then system (1) - (3) describes the vein-formation in meristematic tissues of young leaves [20]. Here U is the signal concentration and V_1, V_2 are diffusion coefficients for flux parallel to Ox_1 and Ox_2 axes, respectively.

In [20] and [21] some qualitative and structural properties of solutions of the system (1) - (3) are established. In [2] investigations for one-dimensional analog of system (1) - (3) with two unknown functions U and V_1 are carried out. The large theoretical and practical importance of the investigation and construction of approximate solutions of the initial-boundary value problems for systems (1) - (3) are pointed out in [2] and [21]. In biological modeling there are many works where this and many models of similar processes are also presented and discussed (see, for example, [3], [10], [11], [21], [22], [25]-[29] and references therein).

The complexity of the (1) - (3) model, besides of nonlinearity is due to its two-dimensionality. In general, numerical solution of multi-dimensional problems often is carried out by applying decomposition methods.

The study of operator splitting techniques has a long history and has been pursued with various methods. Since alternating-direction methods were introduced by Douglas, Peaceman and Rachford [4] - [7], [24] the methods of constructing of algorithms for the numerical solution of the multi-dimensional

B. Tabatadze thanks Shota Rustaveli National Science Foundation (project PhDF2016_14) for the financial support.

Prof. T. Jangveladze is with the I.Vekua Institute Applied Mathematics of Iv. Javakhishvili Tbilisi State University, University Str. 2, 0186 Tbilisi, Georgia and Georgian Technical University, Kostava Ave. 77, 0175 Tbilisi, Georgia (corresponding author, phone: 995-593-316280; e-mail: tjangv@yahoo.com).

Prof. Z. Kiguradze is with the I.Vekua Institute Applied Mathematics of Iv. Javakhishvili Tbilisi State University, University Str. 2, 0186 Tbilisi, Georgia and Georgian Technical University, Kostava Ave. 77, 0175 Tbilisi, Georgia (e-mail: zkigur@yahoo.com).

Dr. M. Gagoshidze is with the BDO Solutions, Iv. Tarkhnishvili Str. 2, 0179 Tbilisi, Georgia (e-mail: mishagagoshidze@gmail.com).

Mr. B. Tabatadze is with the Sokhumi State University (PhD Student), Politkovskaia Str. 2, 0186 Tbilisi, Georgia and Georgian Technical University (Senior Teacher), Kostava Ave. 77, 0175 Tbilisi, Georgia (e-mail: besotabatadze84@gmail.com).

problems of the mathematical physics and the sphere of problems solvable with the help of these algorithms were essentially extended. These procedures, which reduce the time-stepping of multi-dimensional problems to locally one-dimensional computations, have been applied in the numerical simulation of many physically important problems. At present, there are some effective algorithms for solving the multi-dimensional problems (see, for example, [12], [19], [30], [31] and references therein). These algorithms mainly belong to the methods of splitting-up or sum approximation according to their approximative properties. Some schemes of the variable directions are constructed and studied in [1].

We should note that some questions of construction and investigation of the variable directions scheme and the average model of sum approximation as well as difference schemes for one-dimensional case for the (1) - (3) type systems are discussed in the papers [8], [9], [13] - [18], [23].

Our note is oriented on study of such questions.

This article is organized as follows. In the Section 2 the differential problem is formulated and some its properties are given. In the Section 3 the variable directions difference scheme is constructed and its stability and convergence are given. The averaged model of sum approximation is also considered. Numerical examples are given in the Section 4 to compare the exact and numerical solutions and to show the efficiency of the constructed schemes. Comparison of numerical results for those schemes are done. We close with some concluding remarks in the last Section 5.

II. Differential Problem and Some of its Properties

Nonlinear systems of partial differential equations describing various processes of diffusion are the subject of investigating for many scientists.

The main features of such systems often are expressed in fact that they contain equations of different kinds, which are strongly connected to each other. Mentioned condition for each concrete system determines the usage of respective methods of research, because general theory is incompletely developed for such systems even in linear case. Naturally arises the questions of approximate solution of these problems which also are connected with serious complexities as well.

The considered model, as we remarked in an introduction is connected with process of vein formation in meristematic tissues of young leaves. Mentioned model has the following form [20]:

$$\begin{gathered}\frac{\partial U}{\partial t}=\frac{\partial}{\partial x}\left(V\frac{\partial U}{\partial x}\right)+\frac{\partial}{\partial y}\left(W\frac{\partial U}{\partial y}\right),\\ \frac{\partial V}{\partial t}=-V+f\left(V\frac{\partial U}{\partial x}\right),\\ \frac{\partial W}{\partial t}=-W+g\left(W\frac{\partial U}{\partial y}\right).\end{gathered}\tag{4}$$

Here U is the signal concentration and V, W are diffusion coefficients for flux parallel to Ox and Oy axes, respectively, f and g are given sufficiently smooth functions of their arguments, which satisfy the following conditions:

$$0<d\le f(r)\le D,\ 0<d\le g(s)\le D,$$
$$|f'(r)|<D,\qquad |g'(s)|<D,$$

where d and D are constants.

In the parallelepiped $Q=\Omega\times[0,T]$, where $\Omega=[0,1]\times[0,1]$ and T is a given positive constant, consider the system (4) with following boundary and initial conditions:

$$\begin{gathered}U(x,0,t)=U(x,1,t)=0,\\ U(0,y,t)=U(1,y,t)=0,\\ (x,y)\in\partial\Omega\times[0,T],\end{gathered}\tag{5}$$

$$\begin{gathered}U(x,y,0)=U_0(x,y),\\ V(x,y,0)=V_0(x,y),\\ W(x,y,0)=W_0(x,y),\\ (x,y)\in\Omega,\end{gathered}\tag{6}$$

where $\partial\Omega$ is the boundary of Ω.

The essential difficulties arise in the processes of constructing, investigating and realization of the numerical algorithms for problem (4) - (6). Besides nonlinearity the complexity of studying such problems are conditioned also by its two-dimensionality. Therefore, naturally arises the question of reduction this problem to easier ones. In particular, it is very important to reduce the two-dimensional problem to the set of one-dimensional problems.

Let us assume that U_0, V_0 and W_0 are given sufficiently smooth functions, such that

$$U_0(x,y)\ge c, V_0(x,y)\ge c, W_0(x,y)\ge c,$$

where c is positive constant. Suppose that all necessary consistence conditions are satisfied and there exists the sufficiently smooth solution of the problem (1) - (3). It should be noted that the uniqueness of the solution of the problem (1) - (3) is studied in [8].

Under the conditions on functions f, g and V_0, W_0 it is not difficult to obtain the following estimates:

$$c<V(x,y,t)<C,\quad c<W(x,y,t)<C,$$
$$\left|\frac{\partial V}{\partial t}\right|<C,\left|\frac{\partial W}{\partial t}\right|<C,$$

where c and C are positive constants.

III. Two Type Economical Schemes

Later we shall follow notations from [30]. Introduce on the domain Q the grids:

$$\overline{\omega}_{h\tau} = \overline{\omega}_h \times \overline{\omega}_h \times \omega_\tau,$$
$$\overline{\omega}_{1h\tau} = \overline{\omega}_{1h} \times \omega_\tau,$$
$$\overline{\omega}_{2h\tau} = \overline{\omega}_{2h} \times \omega_\tau,$$

where

$$\overline{\omega}_h = \{(x_i, y_j) = (ih, jh), i, j = 0,\dots, M, Mh = 1\},$$
$$\overline{\omega}_{1h} = \left\{(x_i, y_j) = ((i-1/2)h, jh), i, j = 1,\dots, M\right\},$$
$$\overline{\omega}_{2h} = \left\{(x_i, y_j) = (ih, (j-1/2)h), i, j = 1,\dots, M\right\},$$
$$\omega_\tau = \left\{t_k = k\tau, k = 0,1,\dots, K, K\tau = T\right\},$$
$$u_x^k = \frac{u_{i+1}^k - u_i^k}{h}, \quad u_{\bar{x}}^k = \frac{u_i^k - u_{i-1}^k}{h}, \quad u_t = \frac{u_i^{k+1} - u_i^k}{\tau}.$$

Let us correspond to the problem (4) - (6) following scheme of variable directions [13]:

$$\begin{gathered}
\frac{u_1^{k+1} - u_1^k}{\tau} = (v^{k+1} u_{1\bar{x}}^{k+1})_x + (w^k u_{1\bar{y}}^k)_y, \\
\frac{v^{k+1} - v^k}{\tau} = -v^{k+1} + f(v^k u_{1\bar{x}}^k), \\
u_1(x_i, y_j, t_k) = u_2\left(x_i, y_j, t_{k+1}\right), \\
u_1(x_i, y_j, 0) = U_0\left(x_i, y_j\right), \quad (x_i, y_j) \in \overline{\omega}_h, \\
v(x_i, y_j, 0) = V_0(x, y), \quad (x_i, y_j) \in \overline{\omega}_{1h}, \\
u_1(0, y_j, t_{k+1}) = 0, \quad u_1(1, y_j, t_{k+1}) = 0, \\
j = 0,1,\dots, M,\ k = 0,1,\dots, K-1;
\end{gathered} \tag{10}$$

$$\begin{gathered}
\frac{u_2^{k+1} - u_2^k}{\tau} = (v^{k+1} u_{1\bar{x}}^{k+1})_x + (w^{k+1} u_{2\bar{y}}^{k+1})_y, \\
\frac{w^{k+1} - w^k}{\tau} = -w^{k+1} + g(w^k u_{2\bar{y}}^k), \\
u_2(x_i, y_j, t_k) = u_1\left(x_i, y_j, t_{k+1}\right), (x_i, y_j) \in \overline{\omega}_h, \\
w(x_i, y_j, 0) = W_0(x, y), \quad (x_i, y_j) \in \overline{\omega}_{2h}, \\
u_2(x_i, 0, t_{k+1}) = 0, \quad u_1(x_i, 1, t_{k+1}) = 0, \\
i = 0,1,\dots, M, k = 0,1,\dots, K-1.
\end{gathered} \tag{11}$$

Under the sufficiently smoothness of exact solution of the problem (4) - (6) the difference schemes (10), (11) approximate the problem (4) - (6) with the rate $O(\tau + h^2)$.

Let us introduce following notations for the errors: $Z_1 = u_1 - U$, $Z_2 = u_2 - U$, $S_1 = v - V$, $S_2 = w - W$.

The following statement takes place.

Theorem. *If the problem (4) - (6) has the sufficiently smooth solution then finite difference scheme (10), (11) is stable and converges to the exact solution of the problem (4) - (6) when* $\tau \to 0$, $h \to 0$ *and the following estimate holds*

$$\|Z_1\|_{\overline{\omega}_h} + \|Z_2\|_{\overline{\omega}_h} + \|S_1\|_{\overline{\omega}_{1h}} + \|S_2\|_{\overline{\omega}_{2h}} \le C(\tau + h^2).$$

Here C is a positive constant independent of τ and h, norms are discrete analogous of the norm of space L_2.

Using continuous variant of the averaged model of sum approximation [9] let us correspond to problem (4) - (6) following decomposition finite difference scheme:

$$\begin{gathered}
\frac{u_1^{k+1} - u_1^k}{\tau} = (v^{k+1} u_{1\bar{x}}^{k+1})_x, \\
\frac{v^{k+1} - v^k}{\tau} = -v^{k+1} + f(v^k u_{1\bar{x}}^k), \\
u_1(x_i, y_j, t_k) = u_2\left(x_i, y_j, t_{k+1}\right), \\
u_1(x_i, y_j, 0) = U_0\left(x_i, y_j\right), \quad (x_i, y_j) \in \overline{\omega}_h, \\
v(x_i, y_j, 0) = V_0(x, y), \quad (x_i, y_j) \in \overline{\omega}_{1h}, \\
u_1(0, y_j, t_{k+1}) = 0, \quad u_1(1, y_j, t_{k+1}) = 0, \\
j = 0,1,\dots, M,\ k = 0,1,\dots, K-1;
\end{gathered} \tag{12}$$

$$\begin{gathered}
\frac{u_2^{k+1} - u_2^k}{\tau} = (w^{k+1} u_{2\bar{y}}^{k+1})_y, \\
\frac{w^{k+1} - w^k}{\tau} = -w^{k+1} + g(w^k u_{2\bar{y}}^k), \\
u_2(x_i, y_j, t_k) = u_1\left(x_i, y_j, t_{k+1}\right), \quad (x_i, y_j) \in \overline{\omega}_h, \\
w(x_i, y_j, 0) = W_0(x, y), \quad (x_i, y_j) \in \overline{\omega}_{2h}, \\
u_2(x_i, 0, t_{k+1}) = 0, \quad u_1(x_i, 1, t_{k+1}) = 0, \\
i = 0,1,\dots, M, k = 0,1,\dots, K-1, \\
u = \eta_1 u_1 + \eta_2 u_2, \\
\eta_1 > 0, \quad \eta_2 > 0, \quad \eta_1 + \eta_2 = 1.
\end{gathered} \tag{13}$$

Here functions u_1, u_2 are defined on $\overline{\omega}_{h\tau}$; v, w - on $\overline{\omega}_{1h\tau}$ and $\overline{\omega}_{2h\tau}$ respectively.

The statement analogical to Theorem above is true for the scheme (12), (13) too. The problem similar to (4) - (6) with Dirichlet boundary conditions on part of boundary and Neumman boundary conditions on other side is also studied. In this case instead of (2) the following boundary conditions are considered:

$$\begin{gathered}
U(0, y, t) = 0, \quad U(x, 0, t) = 0, \\
V(x, y, t) \left.\frac{\partial U(x, y, t)}{\partial x}\right|_{x=1} = \varphi_1(y, t), \\
W(x, y, t) \left.\frac{\partial U(x, y, t)}{\partial y}\right|_{y=1} = \varphi_2(x, t),
\end{gathered} \tag{14}$$

where φ_1 and φ_2 are given functions.

Let us note that boundary conditions here are dictated by biological viewpoint [20].

IV. RESULTS OF NUMERICAL EXPERIMENTS

Numerous numerical computations are carried out by variable directions difference schemes (10), (11) and (12), (13). The numerical experiments agree with theoretical researches. We give here some of them.

Let us consider the following problem:

$$\frac{\partial U}{\partial t}-\frac{\partial}{\partial x}\left(V\frac{\partial U}{\partial x}\right)-\frac{\partial}{\partial y}\left(W\frac{\partial U}{\partial y}\right)=F(x,y,t),$$
$$\frac{\partial V}{\partial t}+V-f\left(V\frac{\partial U}{\partial x}\right)=G_1(x,y,t),\qquad(15)$$
$$\frac{\partial W}{\partial t}+W-g\left(W\frac{\partial U}{\partial y}\right)=G_2(x,y,t)$$

with corresponding initial and boundary conditions:

$$U(x,0,t)=U(x,1,t)=0,$$
$$U(0,y,t)=U(1,y,t)=0,$$
$$U(x,y,0)=U_0(x,y),\qquad(16)$$
$$V(x,y,0)=V_0(x,y),$$
$$W(x,y,0)=W_0(x,y).$$

For numerical experiments, with different kind functions f, g and exact solutions, we use schemes (10), (11) and (12), (13) corresponding to problem (16), (17) with suitable right sides.

In our numerical experiment we have chosen the right side so that the exact solution is given by

$$U(x,y,t)=x(1-x)y(1-y)(1+t),$$
$$V(x,y,t)=x(1-x)y(1-y)(1+t+t^2),$$
$$W(x,y,t)=x(1-x)y(1-y)(1+t+t^3)$$

and

$$f(\xi)=g(\xi)=\frac{1}{1+(1+\xi)^2}.$$

The parameters used are $M=10$ and $K=250$. Differences between exact and numerical solutions as well as CPU time at different time values are given in Tables 1 – 6. In Tables 1 – 3 results are obtained using scheme (10), (11) while in Tables 4 – 6 results are obtained using scheme (12), (13).

Table 1: Absolute value of maximum errors and CPU time for u applying scheme (10), (11).

t	CPU time	Error for u
0.2	0.074	0.00013912790131447
0.4	0.148	0.00022425859907783
0.6	0.224	0.00031286373416026
0.8	0.301	0.00040788793632886
1	0.378	0.00051151056363487

Table 2: Absolute value of maximum errors and CPU time for v applying scheme (10), (11).

t	CPU time	Error for v
0.2	0.074	0.00000712766408961
0.4	0.148	0.00001730244454379
0.6	0.224	0.00004804529821700
0.8	0.301	0.00009668298990784
1	0.378	0.00016425091499817

Table 3: Absolute value of maximum errors and CPU time for w applying scheme (10), (11).

t	CPU time	Error for w
0.2	0.074	0.00002916084998672
0.4	0.148	0.00009005618525060
0.6	0.224	0.00017715471240609
0.8	0.301	0.00028758192640277
1	0.378	0.00041715052893787

Table 4: Absolute value of maximum errors and CPU time for u applying scheme (12), (13).

t	CPU time	Error for u
0.2	0.072	0.00006973950435170
0.4	0.146	0.00007422011594080
0.6	0.221	0.00007890208614024
0.8	0.295	0.00008480943243865
1	0.369	0.00009205402490850

Table 5: Absolute value of maximum errors and CPU time for v applying scheme (12), (13).

t	CPU time	Error for v
0.2	0.072	0.00001634140038553
0.4	0.146	0.00003786305693865
0.6	0.221	0.00006202878270467
0.8	0.295	0.00008875495749039
1	0.369	0.00011818090303972

Table 6: Absolute value of maximum errors and CPU time for w applying scheme (12), (13).

t	CPU time	Error for w
0.2	0.072	0.00001662571352523
0.4	0.146	0.00003781271060488
0.6	0.221	0.00005790906416947
0.8	0.295	0.00007978157763566
1	0.369	0.00010625023389577

As we see from Tables 1 – 6 the approximation error for variable direction difference scheme (10), (11) is smaller compared with scheme (12), (13). However, CPU time is better for the scheme (12), (13) than scheme (10), (11). We have carried out number of other experiments and observed the same situations. CPU time difference is more visible for more complex test functions.

We also computed errors for different values of time and space steps applying schemes (10), (11) and obtained rate of convergence confirming the theoretical result in Theorem from Section 3. Corresponding data are given in Tables 7 – 24.

Table 7: Absolute value of maximum errors when $t = 0.5$ for u.

h	τ	Error
0.05	0.00125	0.00008575709159125
0.04	0.0008	0.00005620840479086
0.025	0.0003125	0.00002307500617572
0.02	0.0002	0.00001502515952632
0.01	0.00005	0.00000392916752282

Table 8: The rate of convergence when $t = 0.5$ for u.

h	τ	Rate for h
0.05	0.00125	2.00121819226109000
0.04	0.0008	1.99110315962800000
0.025	0.0003125	1.99717930994037000
0.02	0.0002	1.98529955671341000
0.01	0.00005	

Table 9: The rate of convergence when $t = 0.5$ for u.

h	τ	Rate for τ
0.05	0.00125	1.00060909613054000
0.04	0.0008	0.99555157981399900
0.025	0.0003125	0.99858965497018500
0.02	0.0002	0.99264977835670700
0.01	0.00005	

Table 10: Absolute value of maximum errors when $t = 0.5$ for v.

h	τ	Error
0.05	0.00125	0.00008575709159125
0.04	0.0008	0.00005620840479086
0.025	0.0003125	0.00002307500617572
0.02	0.0002	0.00001502515952632
0.01	0.00005	0.00000392916752282

Table 11: The rate of convergence when $t = 0.5$ for v.

h	τ	Rate for h
0.05	0.00125	1.99511955776888000
0.04	0.0008	1.99752072425378000
0.025	0.0003125	1.99878865036986000
0.02	0.0002	1.99948010796618000
0.01	0.00005	

Table 12: The rate of convergence when $t = 0.5$ for v.

h	τ	Rate for τ
0.05	0.00125	0.99755977888444200
0.04	0.0008	0.99879576821464700
0.025	0.0003125	0.99939432518493100
0.02	0.0002	0.99974005398308900
0.01	0.00005	

Table 13: Absolute value of maximum errors when $t = 0.5$ for w.

h	τ	Error
0.05	0.00125	0.00008575709159125
0.04	0.0008	0.00005620840479086
0.025	0.0003125	0.00002307500617572
0.02	0.0002	0.00001502515952632
0.01	0.00005	0.00000392916752282

Table 14: The rate of convergence when $t = 0.5$ for w.

h	τ	Rate for h
0.05	0.00125	1.99455428960597000
0.04	0.0008	1.99726462386849000
0.025	0.0003125	1.99865609149270000
0.02	0.0002	1.99942320548199000
0.01	0.00005	

Table 15: The rate of convergence when $t = 0.5$ for v.

h	τ	Rate for τ
0.05	0.00125	0.99727714480298600
0.04	0.0008	0.99863231193424400
0.025	0.0003125	0.99932804574634800
0.02	0.0002	0.99971160274099400
0.01	0.00005	

Table 16: Absolute value of maximum errors when $t = 1$ for u.

h	τ	Error
0.05	0.00125	0.00024074087939129
0.04	0.0008	0.00015728407178949
0.025	0.0003125	0.00006418736860213
0.02	0.0002	0.00004172715815061
0.01	0.00005	0.00001084525005050

Table 17: The rate of convergence when $t = 1$ for u.

h	τ	Rate for h
0.05	0.00125	1.98351010779040000
0.04	0.0008	1.97259353943770000
0.025	0.0003125	1.98408840973232000
0.02	0.0002	1.98843583267871000
0.01	0.00005	

Table 18: The rate of convergence when $t = 1$ for u.

h	τ	Rate for τ
0.05	0.00125	0.99175505389520200
0.04	0.0008	0.98629676971885200
0.025	0.0003125	0.99204420486615900
0.02	0.0002	0.99421791633935300
0.01	0.00005	

Table 19: Absolute value of maximum errors when $t = 1$ for v.

h	τ	Error
0.05	0.00125	0.00015579938599405
0.04	0.0008	0.00009981476150336
0.025	0.0003125	0.00003903430252067
0.02	0.0002	0.00002498844720772
0.01	0.00005	0.00002498844720772

Table 20: The rate of convergence when $t = 1$ for v.

h	τ	Rate for h
0.05	0.00125	1.99536624107410000
0.04	0.0008	1.99759153642929000
0.025	0.0003125	1.99883528950439000
0.02	0.0002	1.99949990589306000
0.01	0.00005	

Table 21: The rate of convergence when $t = 1$ for v.

h	τ	Rate for τ
0.05	0.00125	0.99768312053704900
0.04	0.0008	0.99879576821464700
0.025	0.0003125	0.99941764475219500
0.02	0.0002	0.99974995294653000
0.01	0.00005	

Table 22: Absolute value of maximum errors when $t = 1$ for w.

h	τ	Error
0.05	0.00125	0.00015579938599405
0.04	0.0008	0.00009981476150336
0.025	0.0003125	0.00003903430252067
0.02	0.0002	0.00002498844720772
0.01	0.00005	0.00002498844720772

Table 23: The rate of convergence when $t = 1$ for w.

h	τ	Rate for h
0.05	0.00125	1.99465428371514000
0.04	0.0008	1.99746978244626000
0.025	0.0003125	1.99871907598386000
0.02	0.0002	1.99945008701027000
0.01	0.00005	

Table 24: The rate of convergence when $t = 1$ for w.

h	τ	Rate for τ
0.05	0.00125	0.99732714185756800
0.04	0.0008	0.99873489122313100
0.025	0.0003125	0.99935953799193100
0.02	0.0002	0.99972504350513300
0.01	0.00005	

V. CONCLUSION

Numerous numerical experiments are done for the problem (4) - (6) using studied (10), (11) and (12), (13) schemes. Carried out numerical experiments show that in all cases numerical solutions fully agree with the theoretical results. The approximation error for variable direction difference scheme (10), (11) is smaller compared with scheme (12), (13). However, CPU time is better for the scheme (12), (13) than scheme (10), (11). We have experimented number of experiments and computed absolute value of maximum errors for different time and space steps and calculated rate of convergence of the (10), (11) scheme. In all cases results of numerical experiments are in accordance to the theoretical findings.

REFERENCES

[1] V.N. Abrashin, "A variant of the method of variable directions for the solution of multi-dimensional problems in mathematical physics. I, (Russian)," *Diff. Uravn.*, vol. 26, 1990, pp. 314–323. English translation: *Diff. Equ.*, vol. 26, pp. 243–250, 1990.

[2] J. Bell, C. Cosner and W. Bertiger, "Solution for a flux-dependent diffusion model," *SIAM J. Math. Anal.* vol. 13, 1982, pp. 758–769.

[3] H. Candela, A. Martinez-Laborda and J. Luis Micol, "Venation pattern formation in arabidopsis thaliana vegetative leaves," *Develop. Biol.*, vol. 205, pp. 205–216, 1999.

[4] J. Douglas, "On the numerical integration of $u_{xx} + u_{yy} = u_t$ by implicit methods," *J. Soc. Industr. Appl. Math.*, vol. 3, pp. 42–65, 1955.

[5] J. Douglas and D.W. Peaceman, "Numerical solution of two-dimensional heat flow problems," *Amer. Inst. Chem. Engin. J.*, vol. 1, pp. 505–512, 1955.

[6] J. Douglas and H.H. Rachford, On the Numerical solution of heat conduction problems in two and three space variables, *Trans. Amer. Math. Soc.*, vol. 82, pp. 421–439, 1960.

[7] J. Douglas, D.W. Peaceman and H.H. Rachford, "A method for calculating multidimensional immiscible displacement," *Trans. of AIME*, 216, pp. 297–308, 1959.

[8] T. A. Dzhangveladze, "Avaraged model of sum approximation for a system of nonlinear partial differential equations (Russian)," *Proc. I.Vekua Inst. Appl. Math.* vol. 19, 1987, pp. 60–73.

[9] T.A. Dzhangveladze and T.G. Tagvarelia, "Convergence of a difference scheme for a system of nonlinear partial differential equations, that arise in biology (Russian)," *Proc. I. Vekua Inst. Appl. Math.*, vol. 40, 1990, pp. 77–83.

[10] D.C. Freeman, J.H. Graham and J.M. Emlen, Developmental stability in plants: symmetries," *Stress and Epigenesis, Genetica,* vol. 89, pp. 97–119, 1993.

[11] R.C. Hardwick, "Physiological consequences of modular growth in plants," *Philosophical Transactions of the Royal Society of London. B, Biological Sciences,* vol. 313, , pp. 161–173 1986.

[12] N. N. Janenko, *The Method of Fractional Steps for Multi-Dimensional Problems of Mathematical Physics (Russian),* M.: Nauka, 1967.

[13] T. A. Jangveladze, "The difference scheme of the type of variable directions for one system of nonlinear partial differential equations," *Proc. I.Vekua Inst. Appl. Math.* vol. 42, 1992, pp. 45–66.

[14] T. Jangveladze, "Variable directions difference scheme for one system of nonlinear partial differential equations," *Recent Advances in Mathematical Methods and Computational Techniques in Modern Science, Proc. 2nd Int. Conf. Appl. Numer. Computat.*, 2013, pp. 119–123.

[15] T. Jangveladze, Z. Kiguradze and M. Nikolishvili, "On aproximate solution of one nonlinear two-dimensional diffusion system," *Rep. Enl. Sess. Sem. I.Vekua Inst. Appl. Math.* vol. 23, pp. 42–45, 2009.

[16] T. Jangveladze, M. Nikolishvili and B. Tabatadze, "On one nonlinear two-dimensional diffusion system," *Proc. 15th WSEAS Int. Conf. Applied Math. (MATH 10)*, 2010, pp. 105–108.

[17] T. Jangveladze and T. G. Tagvarelia, "On the convergence of the difference scheme for one nonlinear system of partial differential equations, arising in biology (Russian), *Proc. I.Vekua Inst. Appl. Math.* vol. 40, 1990, pp. 77–83.

[18] Z. Kiguradze, M. Nikolishvili and B. Tabatadze, "Numerical resolution of one system of nonlinear partial differential equations," *Rep. Enlarged Sess. Semin. I.Vekua Appl. Math.,* vol. 25, pp. 76–79, 2011.

[19] G. I. Marchuk, *The Splitting-up Methods (Russian)*, M.: Nauka 1988.

[20] G. I. Mitchison, "A Model for vein formation in higher plants.," *Proc. R. Soc. Lond. B.* vol. 207, pp. 79–109, 1980.

[21] G.J. Mitchison, "The polar transport of auxin and vein patterns in plants," *Philos. Trans. R. Soc. Lond. B Biol. Sci.,* vol. 295, pp. 461-471, 1981.

[22] T. Nelson and N. Dengler, "Leaf vascular pattern formation," *The Plant Cell,* vol. 9, pp. 1121–1135, 1997.

[23] M. Nikolishvili, "The Discrete Analogues for One Two-dimensional Nonlinear Diffusion System," *Rep. Enl. Sess. Sem. I.Vekua Inst. Appl. Math.* vol. 24, pp. 46–50, 2010.

[24] D. Peaceman and H. Rachford, "The numerical solution of parabolic and elliptic differential equations," *J. Soc. Industr. Appl. Math.* vol. 3, pp. 28–42, 1955.

[25] P. Prusinkiewicz, S. Crawford, R.S. Smith, K. Ljung, T. Bennett, V. Ongaro and O. Leyser, "Control of Bud Activation by an Auxin Transport Switch," *Proc. Nat. Acad. Sci.*, vol. 106, pp. 17431–17436, 2009.

[26] P. Prusinkiewicz and A.G. Rolland-Lagan, "Modeling plant morphogenesis," *Current Opinion in Plant Biology,* vol. 9, pp. 83–88, 2006.

[27] A. Runions, M. Fuhrer, B. Lane, P. Federl, A.G. Rolland-Lagan and P. Prusinkiewicz, "Modeling and visualization of leaf venation patterns," *ACM Transactions on Graphics*, vol. 24, pp. 702–711, 2005.

[28] C.J. Roussel and M.R. Roussel, Reaction–Diffusion, "Models of development with state-dependent chemical diffusion coefficients," *Progress in Biophysics and Molecular Biology,* vol. 86, pp. 113–160, 2004.

[29] M.R. Roussel and M.J. Slingerland, "A biochemically semi-detailed model of auxin-mediated vein formation in plant leaves," *Biosystems,* vol. 109, pp. 475–487, 2012.

[30] A. A. Samarskii, *Theory of Difference Schemes (Russian)*, M.: Nauka, 1977.

[31] A.A. Samarskii, P.P. Matus and P.N. Vabishchevich, *Difference Schemes with Operator Factors*, Kluwer Academic, Dordrecht, 2002.