Exact analytical formulations for the of symmetric circulant tridiagonal matrices

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Abstract— This study presents a time efficient, exact analytical approach for finding the inverse, decomposition, and solving linear systems of equations where symmetric and symmetric circulant matrices appear is presented. A set of matrices are introduced that any symmetric circulant matrix could be decomposed into them as well as their straightforward inverse. After that, it will be shown how they could be used to find the inverse of the matrix. Moreover, solving related linear equations can be carried out using implemented decomposition for these special, prevalent matrices.

Keywords—: Circulant matrix, Decomposition method, Linear equations, Matrix inversion, Toeplitz matrix, Sparse Matrix.

I. Introduction

A $N \times N$ matrix $A_n = \left[a_{i,j}; i, j = 0,1,....,n-1\right]$ is said to be symmetric if $a_{i,j} = a_{j,i}$ and is said to be tridiagonal if it has nonzero elements only on the diagonal plus or minus one column.

A circulant matrix is a special type of Toeplitz matrix. A $N \times N$ matrix $A_n = \left[a_{i,j}; i, j = 0,1,....,n-1\right]$ is said to be Toeplitz if $a_{i.j} = a_{i+1,j+1}$. A Toeplitz matrix is said to be circulant if the matrix is row-wise wrap-around, or simply the subscripts are taken modulo *n*. Thus a ciculant matrix can be written

$$A = \begin{bmatrix} a_0 & a_1 & a_2 & \dots & a_{n-1} \\ a_{n-1} & a_0 & a_1 & \cdots & a_{n-2} \\ a_{n-2} & a_{n-1} & a_0 & \cdots & a_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_1 & a_2 & a_3 & \cdots & a_0 \end{bmatrix} \quad (1)$$

In cases where only a_0, a_1 and a_{n-1} are nonzero, the matrix denoted by *A* is said circulant tridiagonal and if $a_1 = a_{n-1}$ the matrix is symmetric circulant.

Symmetric linear systems are very popular and prevalently appear in literature, such as electromagnetic scattering problem[1], molecular scattering [2], structural dynamics [3] and quantum mechanics[4]. There are a number of studies in literature that have focused on Toeplitz and circulant systems. Garey and Shaw [5] studied nonsymmetric Toepliz systems and nonsymmetric circulant systems. In the same manner, Nemani and Garey [6] presented a new stable algorithm for the solution of tridiagonal circulant linear systems of equationsBy making the transformation matrices, Zheng and Shon [7] studied the determinant and inverse of a generalized Lucas skew circulant matrix. Also, in very important applications the matrices of coefficients of these resulting systems are symmetric circulant and Toeplitz tridiagonal such as finite difference approximation solution of elliptic equations over a rectangle with periodic boundary conditions[8]. Vidal and Alonso [9] extended Rojo algorithm to the case of symmetric Toeplitz tridiagonal equations. In another study Broughton [10] represented an analytical formula for the inverse of a symmetric circulant tridiagonal matrix as a product of a circulant matrix and its transpose.

The current study aims to present a new and comprehensive approach to decompose, calculate the inverse and solve linear system of equations where symmetric matrices and symmetric circulant matrices appear. It is worth noting that the method proposed in the current study is quite competitive with the Gaussian elimination, both in terms of arithmetic operations and storage requirements. A novel decomposition method is introduced that is used in the procedure. Using present method, the decomposition of symmetric circulant matrices may be found efficiently in $O(n^2)$ and the inverse of that in $O(n^{2.3728639})$ operations.

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II. Decomposition Method

Any matrix $A_{n\times n}$ could be decomposed into n matrices as $A = A_1 A_2 ....A_m ...A_n$ where

$$A_m = \begin{bmatrix} 1 & 0 & \cdots & 0 & g_{1,m} & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & g_{2,m} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 1 & g_{m-1,m} & 0 & \vdots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m-1,m} & a_{m,m} & 0 & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & \cdots & \cdots & 0 & 0 & 1 \end{bmatrix} \tag{2}$$

Substituting A_i into Eq. (2) yields a closed formula for g's as follows:

$$g^{(i)}{}_{i,k+1} = \frac{a_{i,k+1} - \left[a_{i,1} ..... a_{i,i-1}\right]\left[g^{i-1}{}_{1,k+1} ..... g^{i-1}{}_{i-1,k-1}\right]}{a_{i,i} - \left[a_{i,1} ..... a_{i,i-1}\right]\left[g_{1,i} ..... g_{i-1,i}\right]} \tag{3}$$

$$g^{(i)}{}_{s,k+1} = g^{(i-1)}{}_{s,k+1} - g^{(i)}{}_{s,i}\, g^{(i)}{}_{i,k+1}$$

For $(i = 1, 2, ..., n-1, s = 1, 2, ..., i-1, k = i)$. If we apply the Gaussian elimination to A^T, we can obtain a lower triangular matrix, L, and an upper triangular matrix, U, as follows:

$$LA^T = U \tag{4}$$

By transposing this equation, we have:

$$AL^T = U^T \tag{5}$$

$g_{1,m}, g_{2,m}, \ldots, g_{m-1,m}$ are found as:

$$L^T = \begin{bmatrix} 1 & -g_{1,2} & -g_{1,3} & \cdots & -g_{1,n} \\ 0 & 1 & -g_{2,3} & \cdots & -g_{2,n} \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & -g_{n-1,n} \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \tag{6}$$

What makes the application of MDM even simpler is that for triangular matrices all g's are zero. Consider a lower triangular matrix of order n to demonstrate the idea:

$$A_{n\times n} = \begin{bmatrix} a_{1,1} & 0 & 0 & \cdots & 0 \\ a_{2,1} & a_{2,2} & 0 & \cdots & 0 \\ a_{3,1} & a_{3,2} & a_{3,3} & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 0 \\ a_{n,1} & a_{n,2} & a_{n,3} & \cdots & a_{n,n} \end{bmatrix} \tag{7}$$

Applying the MDM to this matrix yields:

$$A_{4\times 4} = \begin{bmatrix} a_{1,1} & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & \vdots & \vdots \\ 0 & \ddots & \ddots & 0 & \vdots \\ \vdots & \ddots & \ddots & 1 & 0 \\ 0 & \cdots & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ a_{2,1} & a_{2,2} & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \cdots$$

$$\cdots \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \ddots & \vdots & 0 \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & 1 & 0 \\ a_{n,1} & a_{n,2} & \cdots & \cdots & a_{n,n} \end{bmatrix} \tag{8}$$

The inverse of matrix A is

$$A^{-1}{}_{4\times 4} = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & 0 & 0 \\ 0 & \cdots & 0 & 1 & 0 \\ -\frac{a_{n,1}}{a_{n,n}} & -\frac{a_{n,2}}{a_{n,n}} & \cdots & \cdots & \frac{1}{a_{n,n}} \end{bmatrix} \cdots$$

$$\cdots \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ -\frac{a_{2,1}}{a_{2,2}} & \frac{1}{a_{2,2}} & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{a_{1,1}} & 0 & \cdots & \cdots & 0 \\ 0 & 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \end{bmatrix} \tag{9}$$

According to the above demonstration of the MDM, matrix operations, including finding the inverse of a lower triangular matrix, become much easier

III. Decomposition of a Symmetric Circulant Non-Diagonal Matrix

An optimum strategy to invert a matrix is, first, to reduce the matrix to a simple form, only then beginning a mathematical procedure. For symmetric matrices, the preferred simple form is tridiagonal [11]. The matrices which could be solved with the present method would be all symmetric tridiagonal and symmetric circulant tridiagonal nonsingular matrices which appear in many researches , namely computational fluid dynamics [12], and all symmetric circulant matrices could be transformed to this matrices using Givens rotation or householder method or other computational and analytical algorithms. Using present method tridiagonal symmetric matrix could be solved for inverting problem with one step less than procedure needed for the circulant symmetric matrix. In the case where only a_0, a_1 and a_{n-1} in Eq. (1) are nonzero, the matrix denoted by A is circulant tridiagonal and if $a_1 = a_{n-1}$ the matrix is symmetric. Here we consider $a_0 = c$ and $a_1 = a_{n-1} = a$ for convenience

$$A_{n\times n} = \begin{bmatrix} c & a & 0 & \cdots & 0 & a \\ a & c & a & \ddots & \vdots & 0 \\ 0 & a & \ddots & \ddots & 0 & \vdots \\ \vdots & 0 & \ddots & \ddots & a & 0 \\ 0 & \vdots & \ddots & a & c & a \\ a & 0 & \cdots & 0 & a & c \end{bmatrix}_{n\times n} ,|c| > 2|a| \tag{10}$$

Where $|c| > 2|a|$ denotes that the matrix is strictly diagonal dominant. The normalized form of the matrix is as:

$$A_{n\times n} = a^n \begin{bmatrix} d & 1 & 0 & \ldots & 0 & 1 \\ 1 & d & 1 & \ddots & \vdots & 0 \\ 0 & 1 & \ddots & \ddots & 0 & \vdots \\ \vdots & 0 & 1 & \ddots & 1 & 0 \\ 0 & \vdots & \ddots & 1 & d & 1 \\ 1 & 0 & \ldots & 0 & 1 & d \end{bmatrix}_{n\times n} = a^n \times \overline{A_{n\times n}} \tag{11}$$

Where $d = c/a$. Let matrix $K_{n\times n}$ be of the form

$$K_{n\times n} = \begin{bmatrix} f_1(d) & 0 & \cdots & \cdots & 0 \\ f_1(d) & f_2(d) & \ddots & \ddots & \vdots \\ \vdots & f_2(d) & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & f_{n-1}(d) & 0 \\ f_1(d) & f_2(d) & \cdots & f_{n-1}(d) & f_n(d) \end{bmatrix}_{n\times n} \tag{12}$$

Where matrix K is a lower triangular alternant matrix, an inspired formula derived from Jacobi method, and f_i are obtained from the following recurrence relation:

$$\begin{aligned} & f_{i+1}(d) = -d f_i(d) - f_{i-1}(d), \\ & f_0(d) = 0,\ f_1(d) = 1 \qquad i = 1,2,3,\ldots,n-1 \end{aligned} \tag{13}$$

The bidiagonal inverse of matrix K could be obtained:

$$K_{n\times n} = \begin{bmatrix} \frac{1}{f_1(d)} & 0 & \cdots & \cdots & 0 \\ -\frac{1}{f_2(d)} & \frac{1}{f_2(d)} & \ddots & \ddots & \vdots \\ 0 & -\frac{1}{f_3(d)} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \frac{1}{f_{n-1}(d)} & 0 \\ 0 & \cdots & 0 & -\frac{1}{f_n(d)} & \frac{1}{f_n(d)} \end{bmatrix}_{n\times n} \tag{14}$$

Based on Eq. (13), all entries of matrix $K^{-1}_{n\times n}$ are zero except for the elements of the main diagonal and subdiagonal. It can be seen in Eqs. (12) and (14) that matrix K and its inverse have elements that can be calculated easily using Eq. (13)

In cases where matrix A is tridiagonal the procedure is over here since matrix $A^K{}_{n\times n}$ is an upper triangular matrix without having nonzero component at first column and last row. On the other hand, if matrix A is circulant tridiagonal it is needed to continue the procedure as follows

Let matrix $R_{n\times n}$ be of the form

$$R_{n\times n} = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \ddots & \vdots & 0 \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & 1 & 0 \\ r_1(d) & r_2(d) & \cdots & r_{n-1}(d) & 1 \end{bmatrix}_{n\times n} . \tag{15}$$

Where matrix R is a sparse matrix in a way that the entries of its last row are r_j except for the entry of the n[th] column of that row which is one and this matrix could be derived using Housholder transformation method. In Eq. (16) r_j can be calculated easily:

$$r_j(d) = \frac{f_n(d) f_1(d)}{f_{j+1}(d) f_j(d)}, \quad j = 1,2,3,\ldots,n-1 \tag{16}$$

The inverse of the matrix R can be calculated using decomposition method:

$$R^{-1}{}_{n\times n}=\begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \ddots & \vdots & 0 \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & 1 & 0 \\ -r_1(d) & -r_2(d) & \cdots & -r_{n-1}(d) & 1 \end{bmatrix}. \tag{17}$$

Now let introduce matrix A_1^{kR}

$$A_1^{kR}=a\begin{bmatrix} -f_2(d) & 0 & 0 & \cdots & 0 & 1 \\ f_1(d) & -f_3(d) & 0 & \cdots & 0 & 1 \\ 0 & f_2(d) & -f_4(d) & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 & 1 \\ \vdots & \vdots & \ddots & \ddots & -f_n(d) & 1 \\ 0 & \cdots & \cdots & 0 & f_{n-1}(d)+1 & g_{n+1}(d) \end{bmatrix}_{n\times n} \tag{18}$$

In this matrix only one element should be computed which is $g_{n+1}(d)$ to generate matrix $A_1^{kR}{}_{n\times n}$ and it can be obtained as follows:

$$g_{n+1}(d)=1-f_{n+1}(d)+\sum_{j=1}^{n-1}r_j(d)+r_{n-1}(d)f_{n-1}(d). \tag{19}$$

Or

$$g_{n+1}(d)=1+f_1(d)-f_{n+1}(d)+\sum_{j=1}^{n-1}r_j(d), f_1(d)=1. \tag{20}$$

Matrix A_1^{kR} is a lower triangular matrix and its decomposition and inverse can be obtained using Eqs. (6).

$$(A_1^{kR})^{-1}=-a\begin{bmatrix} \frac{1}{f_2(d)} & 0 & 0 & 0 & \cdots & 0 & 0 \\ \frac{f_1(d)}{f_2(d)f_3(d)} & \frac{1}{f_3(d)} & 0 & 0 & \cdots & 0 & 0 \\ \frac{f_1(d)}{f_3(d)f_4(d)} & \frac{f_2(d)}{f_3(d)f_4(d)} & \frac{1}{f_4(d)} & 0 & \ddots & \vdots & \vdots \\ & & & \ddots & \ddots & 0 & 0 \\ \vdots & \vdots & & \ddots & \ddots & \ddots & 0 \\ \frac{f_1(d)}{f_{n-1}(d)f_n(d)} & \frac{f_2(d)}{f_{n-1}(d)f_n(d)} & & \cdots & \frac{f_{n-2}(d)}{f_{n-1}(d)f_n(d)} & \frac{1}{f_n(d)} & 0 \\ \frac{f_1(d)}{f_n(d)g_{n+1}(d)} & \frac{f_2(d)}{f_n(d)g_{n+1}(d)} & & \cdots & \frac{f_{n-2}(d)}{f_n(d)g_{n+1}(d)} & \frac{f_{n-1}(d)}{f_n(d)g_{n+1}(d)} & \frac{1}{g_{n+1}(d)} \end{bmatrix}_{n\times n}$$

Displa

In order to illustrate the decomposition of matrix A to these three matrices, we have:

$$A=\frac{1}{a}K^{-1}R^{-1}\left(A_1^{kR}\right)^T, \tag{22}$$

Eq. (22) represents a decomposition of matrix A into three sparse and easy to calculate matrices. K^{-1} is calculated in Eq. (14), R^{-1} in Eq. (17), and A_1^{KR} using Eq. (18). Using this decomposition, the inverse of matrix A can easily be derived as follow:

$$A^{-1}=\frac{1}{a}\left[\left(A_1^{kR}\right)^{-1}\right]^T RK \tag{23}$$

However, based on Eq. (23), the linear system of $AX=b$ may be solved easily as $X=A^{-1}b$. In addition, as a different approach to solve $AX=b$, we have the following relation based on Eq. (21):

$$AX=b\rightarrow aRK\overline{A}X=\overline{b}\rightarrow\left(A_1^{kR}\right)^T X=\overline{b}\rightarrow A^{kR}X=\overline{b} \tag{24}$$

Where $\overline{b}=aRKb$. With respect to the last equation of Eq. (24) which is an updated form of $AX=b$, the values of X can be easily obtained using gauss method (back substitution).

IV. Numerical Example

Now let us consider the following 5x5 ciculant tridiagonal symmetric matrix:

$$A=\begin{bmatrix} 5 & 2 & 0 & 0 & 2 \\ 2 & 5 & 2 & 0 & 0 \\ 0 & 2 & 5 & 2 & 0 \\ 0 & 0 & 2 & 5 & 2 \\ 2 & 0 & 0 & 2 & 5 \end{bmatrix}, \overline{A}=\begin{bmatrix} 2.5 & 1 & 0 & 0 & 1 \\ 1 & 2.5 & 1 & 0 & 0 \\ 0 & 1 & 2.5 & 1 & 0 \\ 0 & 0 & 1 & 2.5 & 1 \\ 1 & 0 & 0 & 1 & 2.5 \end{bmatrix} d=2.5, a=2$$

From Eq. (10) f_i are calculated as follow:

$$f_2=-2.5, f_3=5.25, f_4=-10.625, f_5=21.3125,$$
$$f_6=-42.6566$$

Subsequently matrix K is calculated by using Eqs. (10) and (12):

$$K=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & -2.5 & 0 & 0 & 0 \\ 1 & -2.5 & 5.25 & 0 & 0 \\ 1 & -2.5 & 5.25 & -10.625 & 0 \\ 1 & -2.5 & 5.25 & -10.625 & 21.3125 \end{bmatrix}$$

Inverse of this matrix can easily be obtained by using Eq. (13):

$$K^{-1}=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0.4 & -0.4 & 0 & 0 & 0 \\ 0 & -0.1905 & 0.1905 & 0 & 0 \\ 0 & 0 & 0.0941 & -0.0941 & 0 \\ 0 & 0 & 0 & -0.0469 & 0.0469 \end{bmatrix}$$

Matrix R is obtained using Eqs . (17) and (18):

$$R=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -8.525 & -1.6238 & -0.3821 & -0.0941 & 1 \end{bmatrix}$$

Considering Eq. (19) its inverse is

$$R^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 8.525 & 1.6238 & 0.3821 & 0.0941 & 1 \end{bmatrix}$$

By multiplying matrixes R and K from left to matrix A or using Eq. (20), matrix A^{kR} A^{kR} is obtained ($g_6(d) = 34.0312$):

$$A^{KR} = \begin{bmatrix} 5 & 2 & 0 & 0 & 2 \\ 0 & -10.5 & -5 & 0 & 2 \\ 0 & 0 & 21.25 & 10.5 & 2 \\ 0 & 0 & 0 & -42.625 & -19.25 \\ 0 & 0 & 0 & 0 & 68.0625 \end{bmatrix}$$

And its transpose is defined as:

$$A_1^{KR} = \begin{bmatrix} 5 & 0 & 0 & 0 & 0 \\ 2 & -10.5 & 0 & 0 & 0 \\ 0 & -5 & 21.25 & 0 & 0 \\ 0 & 0 & 10.5 & -42.625 & 0 \\ 2 & 2 & 2 & -19.25 & 68.0625 \end{bmatrix}$$

Finally inverse of matrix A is calculated using Eq. (25):

$$A^{-1} = \begin{bmatrix} 0.3131 & -0.1414 & 0.0404 & 0.0404 & -0.1414 \\ -0.1414 & 0.3131 & -0.1414 & 0.0404 & 0.0404 \\ 0.1414 & -0.1414 & 0.3131 & -0.1414 & 0.0404 \\ 0.1414 & 0.0404 & -0.1414 & 0.3131 & -0.1414 \\ -0.1414 & 0.0404 & 0.0404 & -0.1414 & 0.3131 \end{bmatrix}$$

V. Conclusion

The amount of labor and time has always been an issue for computations of circulant matrices appearing in numerous researches. For instance, consider a circulant matrix of order greater than 100. It is almost impossible to compute inverse of this matrix with its corresponding determinant or solve linear systems of equations with circulant matrix as coefficient matrix in a straightforward way. However, the matrix decomposition method presented in section 2 is a timely solution to this problem. This study presented a new comprehensive approach using matrix decomposition method for calculation of the symmetric circulant tridiagonal matrices. As such, the two R and K matrices were used in order to make the original matrix upper triangular. The suggested decomposition method for special matrices could be utilized for solving related linear equations. It is also worth noting that the other decompositions and methods applicable in making the matrices lower (upper) triangular such as the Householder transformation or QR factorization are much more difficult and complicated to work with than the presented method. Additionally, other sparse matrices with a small change can be factorized to the presented decomposition. The proposed method in this study is even simpler in some cases including tridiagonal matrices since there is no need for computing matrix R. By only obtaining K, we can make the original matrix upper triangular. Further studies on the other special case of toeplitz and sparse matrices would be helpful to find a lower (upper) triangular form of them and use the matrix decomposition method presented in section 2 of this study to decompose them.

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