# Interval Estimation Of Polynomial Splines of the Fifth Order and its Derivatives and Errors of Approximation

Burova I.G., Vartanova A.A., Shirokova Iu. V.

***Abstract—*** As is well known in many applications, approximation by local interpolating splines is preferable to approximation by interpolating polynomials or interpolating by other types of splines. Sometimes the values of integrals over net intervals besides the values of the function in the nodes are known. In this case we can use integro-differential splines. The main features of these splines are the following: the approximation is constructed separately for each grid interval (or elementary rectangular), the approximation constructed as the sum of products of the basic splines and the values of function in nodes and/or the values of its derivatives and/or the values of integrals of this function over subintervals. Basic splines are determined by using a solving system of equations which are provided by the set of functions. In this paper we present the estimation of approximation and the algorithm for constructing an interval extension of approximation when values of function in nodes, values of its first derivative in nodes, and values of its integrals over net intervals are given. The algorithm of approximation is based on the method of approximating functions using integro-differential splines. For obtaining the derivatives of the function we use the derivatives of the basic functions. For constructing the approximation of the function of two variables we use tensor product. For constructing the derivatives of the approximation of the function of two variables we use derivatives of basic functions. For constructing this interval extension, we use techniques from interval analysis. The errors of approximation are given for the approximations with the middle, left and right integro-differential polynomial splines of the fifth order. Numerical examples are given.

***Keywords—*** Integro-differential splines, Approximation, Polynomial interval extension, Tensor production

## I. Introduction

A lot of research papers have been published to date on interval mathematics. By using interval arithmetic and interval-valued functions, we can compute arbitrarily sharp upper and lower bounds on ranges of function values (see [1], [7], [9]–[13]). The problem of interpolating functions was investigated by [5]–[8], [10]–[15]. In many cases, approximation by local interpolating splines is preferable to approximation by interpolating polynomials or interpolating by other types of splines.

This paper deals with the interval extensions of integro-differential splines of the fifth order (see [2]–[4]). The value of approximation by the splines can be calculated for every point if the values of the function, the values of its first derivative, and the values of the integral of the function over the net intervals are given.

I.G. Burova is with the St. Petersburg State University, 7/9 Universitetskaya nab., St.Petersburg, 199034 Russia e-mail: i.g.burova@spbu.ru, burovaig@mail.ru.

A.A.Vartanova is with the St. Petersburg State University, St. Petersburg, Russia

Iu. V. Shirokova is with the St. Petersburg State University, St. Petersburg, Russia

## II. Approximation of the function

Suppose that $n, m$ are natural numbers, while $a$, $b$, $c$, $d$ are real numbers. Let the function $u(x)$ be such that $u \in C^5([a,b])$. We have the grid of interpolation nodes $\{x_j\}$ such that $x_0 = a$, $x_{j+1} = x_j + h_j$, $x_n = b$.

Suppose that $u(x_j)$, $u'(x_j)$, $j = 0, 1, \ldots, n$, $\int_{x_j}^{x_{j+1}} u(\xi)d\xi$, $j = 0, \ldots, n-1$, are known. We denote $\widetilde{u}(x)$ as an approximation of the function $u(x)$ in the interval $[x_j, x_{j+1}] \subset [a,b]$:

$$\widetilde{u}(x) = u(x_j)\omega_{j,0}(x) + u(x_{j+1})\omega_{j+1,0}(x) + u'(x_j)\omega_{j,1}(x) + u'(x_{j+1})\omega_{j+1,1}(x) + \int\limits_{x_j}^{x_{j+1}} u(\xi)d\xi\, \omega_j^{<0>}(x). \quad (1)$$

We obtain the basic splines $\omega_{j,0}(x)$, $\omega_{j+1,0}(x)$, $\omega_{j,1}(x)$, $\omega_{j+1,1}(x)$, $\omega_j^{<0>}(x)$ from the system:

$$\widetilde{u}(x) \equiv u(x), \quad u(x) = x^{i-1}, \ i = 1, 2, 3, 4, 5. \quad (2)$$

If $x = x_j + t\,h_j$, $t \in [0,1]$, then the basic splines can be written in the form:

$\omega_{j,0}(x_j + t\,h_j) = -(1+5t)(-1+3t)(t-1)^2$,
$\omega_{j+1,0}(x_j + t\,h_j) = -t^2(-2+3t)(-6+5t)$,
$\omega_{j,1}(x_j + t\,h_j) = -(1/2)th_j(5t-2)(t-1)^2$,
$\omega_{j+1,1}(x_j + t\,h_j) = (1/2)t^2h_j(t-1)(5t-3)$,
$\omega_j^{<0>}(x_j + t\,h_j) = 30t^2(t-1)^2/h_j$.

Let $\|u\| = \|u\|_{[x_j,x_{j+1}]} = \max\limits_{[x_j,x_{j+1}]} |u(x)|$.

*Lemma 1. Let function $u \in C^{(5)}[a,b]$. The next statement is valid:*

$$|u(x) - \widetilde{u}(x)| \le K_0 h_j^5 \|u^{(5)}\|_{[x_j,x_{j+1}]}, \ x \in [x_j, x_{j+1}],$$

$K_0 = 0.00076$.

*Proof.* Using Taylor's series expansion of $u(x_{j+1})$, $u'(x_{j+1})$, $u(t)$ about the point $x$ and relations (2), we obtain

$$\tilde{u}(x) - u(x) = I_1 + I_2 + I_3,$$

where

$$I_1 = \frac{1}{4!}\int_x^{x_j} (x_j - \nu)^4 u^{(5)}(\nu)d\nu \ \omega_{j,0}(x) +$$

$$\frac{1}{3!}\int_{x}^{x_j}(x_j-\nu)^3u^{(5)}(\nu)d\nu\ \omega_{j,1}(x),$$

$$I_2=\frac{1}{4!}\int_{x}^{x_{j+1}}(x_{j+1}-\nu)^4u^{(5)}(\nu)d\nu\ \omega_{j+1,0}(x)+$$

$$\frac{1}{3!}\int_{x}^{x_{j+1}}(x_{j+1}-\nu)^3u^{(5)}(\nu)d\nu\ \omega_{j+1,1}(x),$$

$$I_3=\frac{1}{4!}\int_{x_j}^{x_{j+1}}\int_{x}^{\xi}(\xi-\nu)^4u^{(5)}(\nu)d\nu d\xi\ \omega_j^{<0>}(x).$$

We denote $x=x_j+th_j$, $t\in[0,1]$, $\nu=x_j+\tau h_j$, $\tau\in[0,1]$, $\xi=x_j+Th_j$, $T\in[0,1]$. Now we have

$$I_1=\frac{h_j^5}{4!}\int_0^t\Psi_1(t,\tau)u^{(5)}(x_j+\tau h_j)d\tau,$$

where $\Psi_1(t,\tau)=(-1)\tau^4\omega_{j,0}(t)+4\tau^3\ \omega_{j,1}(t)/h_j$.

$$I_2=\frac{h_j^5}{4!}\int_t^1\Psi_2(t,\tau)u^{(5)}(x_j+\tau h_j)d\tau,$$

where $\Psi_2(t,\tau)=(1-\tau)^4\omega_{j+1,0}(t)+4(1-\tau)^3\ \omega_{j+1,1}(t)/h_j$.

$$I_3=\frac{h_j^5}{4!}\int_0^1\int_t^T\Psi_3(t,\tau)u^{(5)}(x_j+\tau h_j)d\tau dT,$$

where $\Psi_3(t,\tau)=(T-\tau)^4\omega_j^{<0>}(t)h_j$.

It can be found easily that $\Psi_1(t,\tau)$ has a root at $\tau*=2(5t-2)t/(-2t-1+15t^2)$ and if $t\in[0.4,1]$ then $\tau*\in[0,t]$. Thus if $t\in[0.4,1]$ then

$$I_1=\frac{h_j^5}{4!}\int_0^{\tau*}\Psi_1(t,\tau)u^{(5)}(x_j+\tau h_j)d\tau+$$

$$\frac{h_j^5}{4!}\int_{\tau*}^{t}\Psi_1(t,\tau)u^{(5)}(x_j+\tau h_j)d\tau.$$

We obtain

$$F_1(t)=\int_0^{\tau*}\Psi_1(t,\tau)d\tau=\frac{-8(5t-2)^5(t-1)^2t^5}{5(1+5t)^4(3t-1)^4},$$

and $\max\limits_{t\in[0.4,1]}|F_1(t)|\approx|F_1(0.7339)|=0.000316$.We obtain

$$F_2(t)=\int_{\tau*}^{t}\Psi_1(t,\tau)d\tau=\frac{9(t-1)^2t^5(-4t+1+5t^2)^2}{10(1+5t)^4(3t-1)^4}\times$$

$(6750t^6+675t^5-1740t^4-586t^3-18t^2+267t-56)$, and $\max\limits_{t\in[0.4,1]}|F_2(t)|\approx|F_2(0.81125)|=0.00555$.

Fig. 1. $F_1(t)$, $t\in[0.4,1]$ (left), $F_2(t)$, $t\in[0.4,1]$ (right)

If $t\in[0,0.4]$ then

$$F_3(t)=\int_0^t\Psi_1(t,\tau)d\tau=\frac{t^5}{10}(30t^2-29t+8)(t-1)^2,$$

Fig. 2. $F_3(t)$, $t\in[0,0.4]$ (left), $F_4(t)$, $t\in[0.6,1]$ (right)

and $\max\limits_{t\in[0,0.4]}|F_3(t)|\approx|F_3(0.4)|\approx0.00044$. Finally

$$|I_1|\leq0.0058\frac{h_j^5}{4!}\|u^{(5)}\|.$$

Similarly $\Psi_2$ has a root in $\tau^*=\frac{(5t^2-12t+6)}{(15t^2-28t+12)}$, and $\tau^*\in[t,1]$ if $t\in[0,3/5]$, and $\Psi_2$ has no roots if $t\in[3/5,1]$, so $\Psi_2$ of constant signs if $t\in[3/5,1]$.

When $t\in[3/5,1]$ then

$$F_4(t)=\int_0^t\Psi_2(t,\tau)d\tau=\frac{t^2}{10}(30t^2-31t+9)(t-1)^5.$$

Thus,

$$|I_2|\leq\frac{h_j^5}{4!}\max_{[3/5,1]}|F_4(t)|\|u^{(5)}\|\leq\frac{h_j^5}{4!}0.00044\|u^{(5)}\|.$$

If $t\in[0,3/5]$ we present $I_2$ in the form: $I_2=\int_t^{\tau^*}+\int_{\tau^*}^1$.

We obtain

$$|I_2|\leq\frac{h_j^5}{4!}(\max_{[0,3/5]}|K_1(t)|+\max_{[0,3/5]}|K_2(t)|)\|u^{(5)}\|,$$

where

$$K_1(t)=\frac{9t^2(5t^2-6t+2)^2(t-1)^5}{10(3t-2)^4(5t-6)^4}(6750t^6-41175t^5+$$

$+102885t^4-134204t^3+95784t^2-35388t+5292)$.

We find that $\max\limits_{[0,3/5]}|K_1(t)|\approx|K_1(0.1887)|\approx0.005546$,

$$K_2(t)=8\frac{t^2(5t-3)^5(t-1)^5}{5(3t-2)^4(5t-6)^4},$$

and we find that $\max\limits_{[0,3/5]}|K_2(t)|\approx|K_2(0.266)|\approx0.0003155$.

Fig. 3. $K_1(t)$, $t\in[0.6,1]$ (left), $K_2(t)$, $t\in[0.6,1]$ (right)

Thus,

$$|I_2|\leq\frac{h_j^5}{4!}(0.00552+0.00032)\|u^{(5)}\|\leq$$

$$\leq\frac{h_j^5}{4!}0.00584\|u^{(5)}\|.$$

We obtain that

$$|I_3|\leq\frac{h_j^5}{4!}\left|\max_{t\in[0,1]}t^8(t-1)^2\right|\|u^{(5)}\|\leq\frac{h_j^5}{4!}(0.0067)\|u^{(5)}\|.$$

Finally we have for $x \in [x_j, x_{j+1}]$

$$|\tilde{u}(x) - u(x)| \le 0.000763 h_j^5 \|u^{(5)}\|_{[x_j,x_{j+1}]}.$$

The proof is completed.

Table 1 shows actual and theoretical errors of approximation of functions constructed with formula (1) when $[a,b] = [-1,1]$, $h_j = 0.1$. Calculations were done in Maple with Digits=15.

TABLE 1.
ACTUAL AND THEORETICAL ERRORS WHEN
$[a,b] = [-1,1],\ h_j = 0.1$

| $u(x)$ | $\max\limits_{[-1,1]} \lvert u - \widetilde{u}\rvert$ actual errors | $\max\limits_{[-1,1]} \lvert u - \widetilde{u}\rvert$ theoretical errors |
|---|---|---|
| $\sin(3x)\cos(5x)$ | $0.12 \cdot 10^{-4}$ | $0.12 \cdot 10^{-3}$ |
| $\cos(x)$ | $0.61 \cdot 10^{-9}$ | $0.64 \cdot 10^{-8}$ |
| $\cos(2x)$ | $0.24 \cdot 10^{-7}$ | $0.24 \cdot 10^{-6}$ |
| $\sin^2(x)$ | $0.12 \cdot 10^{-7}$ | $0.12 \cdot 10^{-6}$ |
| $\sin^{16}(\pi x)$ | $0.11 \cdot 10^{-2}$ | $0.13 \cdot 10^{-1}$ |
| $\dfrac{\sin(\pi x)}{\cos(\pi x/4)}$ | $0.11 \cdot 10^{-6}$ | $0.11 \cdot 10^{-5}$ |
| $\dfrac{1}{(1+25x^2)}$ | $0.21 \cdot 10^{-3}$ | $0.24 \cdot 10^{-2}$ |

*Lemma 2. Let function $u \in C^{(5)}[a,b]$. There are points $\eta, \zeta \in [x_j, x_{j+1}]$ such that*

$$u(x) - \widetilde{u}(x) = \frac{u^{(5)}(\eta)}{5!}(x - x_j)^2 (x - x_{j+1})^2 (x - \zeta),$$

$x \in [x_j, x_{j+1}]$.

*Proof.* It can be shown that the next relations are fulfilled on approximation (1):

1) $\widetilde{u}(x_j) = u(x_j)$,
2) $\widetilde{u}(x_{j+1}) = u(x_{j+1})$,
3) $\widetilde{u}'(x_j) = u'(x_j)$,
4) $\widetilde{u}'(x_{j+1}) = u'(x_{j+1})$,
5) $\int_{x_j}^{x_{j+1}} \widetilde{u}(x)dx = \int_{x_j}^{x_{j+1}} u(x)dx$.

Firstly, let us notice that statements 1)-2) follow from the next relations:

$\omega_{j,0}(x_j) = 1$, $\omega_{j,0}(x_{j+1}) = 0$,
$\omega_{j+1,0}(x_j) = 0$, $\omega_{j+1,0}(x_{j+1}) = 1$,
$\omega_{j,1}(x_j) = 0$, $\omega_{j,1}(x_{j+1}) = 0$,
$\omega_{j+1,1}(x_j) = 0$, $\omega_{j+1,1}(x_{j+1}) = 0$,
$\omega_j^{<0>}(x_j) = 0$, $\omega_j^{<0>}(x_{j+1}) = 0$.

Similarly, statements 3)-4) follow from the next relations:

$\omega'_{j,0}(x_j) = 0$, $\omega'_{j,0}(x_{j+1}) = 0$,
$\omega'_{j+1,0}(x_j) = 0$, $\omega'_{j+1,0}(x_{j+1}) = 1$,
$\omega'_{j,1}(x_j) = 1$, $\omega'_{j,1}(x_{j+1}) = 0$,
$\omega'^{<0>}_j(x_j) = 0$, $\omega'^{<0>}_j(x_{j+1}) = 0$.

Finally, statement 5) follows from the relations: $\int_{x_j}^{x_{j+1}} \omega_{j,0}(x)dx = 0$, $\int_{x_j}^{x_{j+1}} \omega_{j+1,0}(x)dx = 0$, $\int_{x_j}^{x_{j+1}} \omega_{j,1}(x)dx = 0$, $\int_{x_j}^{x_{j+1}} \omega_{j+1,1}(x)dx = 0$, $\int_{x_j}^{x_{j+1}} \omega_j^{<0>}(x)dx = 1$.

Further, let us notice that the next equality follows from (1)-(2) for $u(x) = C = const$:

$$C = C\omega_{j,0}(x) + C\omega_{j+1,0}(x) + Ch_j\, \omega_j^{<0>}(x).$$

Now we can find the point $\zeta$ such that $u(\zeta)h_j = \int_{x_j}^{x_{j+1}} u(\xi)d\xi$. Let us take an approximation $\widetilde{u}(x)$, $x \in [x_j, x_{j+1}]$, in the form:

$$\widetilde{u}(x) = u(x_j)\omega_{j,0}(x) + u(x_{j+1})\omega_{j+1,0}(x) + u'(x_j)\omega_{j,1}(x) + u'(x_{j+1})\omega_{j+1,1}(x) + u(\zeta)\, h_j \omega_j^{<0>}(x). \quad (3)$$

Obviously $\zeta$ is the point of interpolation. It follows from the relation:

$$\omega_{j,0}(x) + \omega_{j+1,0}(x) + h_j\omega_j^{<0>}(x) = 1.$$

Thus, from (2) we obtain:

$$\tilde{u}(\zeta) = u(\zeta)\big(\omega_{j,0}(x) + \omega_{j+1,0}(x) + h_j\omega_j^{<0>}(x)\big) = u(\zeta).$$

Approximation (3) has the next points of Hermit interpolation: $x_j$ with the second multiplicity, $x_{j+1}$ with the second multiplicity, and $\zeta$. The remainder term of Hermit interpolation in our case is as follows:

$$\frac{u^{(5)}(\eta)}{5!}(x - x_j)^2 (x - x_{j+1})^2 (x - \zeta).$$

The proof is completed.

*Corollary.* If $M = \max\limits_{x\in[a,b]} |u^{(5)}(x)|$ and we put $x = x_j + th_j$, $t \in [0,1]$, $\zeta = x_j + Th_j$, $T \in [0,1]$, then

$$|\widetilde{u}(x_j + th_j) - u(x_j + th_j)| \le \frac{Mh_j^5}{5!} t^2 (t-1)^2 |t - T|.$$

The proof is evident.

*Example.* Let us take $u(x) = \sin(3x)\cos(5x)$, $x_j = 0$, $x_{j+1} = 1$. We obtain $\int_0^1 u(t)dt \approx u(0.841) \approx -0.282$ and $\int_0^1 u(t)dt \approx u(0.378) \approx -0.282$.

Denote $\zeta_1 = 0.378$, $\zeta_2 = 0.841$. For $\zeta_1$ we have $|\widetilde{u}(x) - u(x)| \le 0.0000195$, For $\zeta_2$ we have $|\widetilde{u}(x) - u(x)| \le 0.0000349$. Thus the theoretical error is $|\widetilde{u}(x) - u(x)| \le 0.0000195$, $x \in [0,1]$. The actual error is $|\widetilde{u}(x) - u(x)| \le 0.000012$, $x \in [0,1]$.

*Lemma 3. Let function $u \in C^{(5)}[a,b]$. $M = \max\limits_{[x_j,x_{j+1}]} |u^{(5)}(x)|$ The next statement is valid:*

$$|u(x) - \widetilde{u}(x)| \le 0.0625 h_j^5 \frac{M}{5!},\ x \in [x_j, x_{j+1}]. \quad (4)$$

Proof follows from Lemma 2 and the relations

$$\max_{x\in[x_j,x_{j+1}]} |(x - x_j)^2 (x - x_{j+1})^2| \le 0.0625 h_j^4,$$

$|x - \zeta| \le h_j$ if $x, \zeta \in [x_j, x_{j+1}]$.

Table 2 shows actual and theoretical errors of approximation of functions constructed with formulae (1), (4) when $[a,b] = [-1,1]$, $h_j = 0.1$.

TABLE 2. ACTUAL AND THEORETICAL ERRORS WHEN $[a,b]=[-1,1]$, $h_j=0.1$

| $u(x)$ | $\max_{[-1,1]} \|u-\widetilde{u}\|$ ACTUAL ERRORS | $\max_{[-1,1]} \|u-\widetilde{u}\|$ THEORETICAL ERRORS |
|---|---|---|
| $\sin(3x)\cos(5x)$ | $0.12\cdot 10^{-4}$ | $0.20\cdot 10^{-4}$ |
| $\cos(x)$ | $0.61\cdot 10^{-9}$ | $0.84\cdot 10^{-9}$ |
| $\cos(2x)$ | $0.24\cdot 10^{-7}$ | $0.29\cdot 10^{-7}$ |
| $\dfrac{1}{(1+25x^2)}$ | $0.21\cdot 10^{-3}$ | $0.44\cdot 10^{-3}$ |

## III. Interval evaluation

Here we assume that the values of the function, its first derivative, and the values of the integral over the given intervals are known without rounding errors and measurement errors. Our aim is the following: to use given data to obtain the boundaries of the variation of approximation as closely as possible with no calculating approximation in points in the interval.

Approximation (1) can be written in the form:

$$\widetilde{u}^p(x_j+th_j)=C_4t^4+C_3t^3+C_2t^2+C_1t+C_0, \quad (5)$$

$t\in[0,1]$. where $C_0=u(x_j)$, $\quad C_1=u'(x_j)h_j$,
$C_2 = -18u(x_j) - 12u(x_{j+1}) - (9/2)u'(x_j)h_j + (3/2)u'(x_{j+1})h_j + 30\int\limits_{x_j}^{x_{j+1}} u(t)dt/h_j$,
$C_3 = 32u(x_j) + 28u(x_{j+1}) + 6u'(x_j)h_j - 4u'(x_{j+1})h_j - 60\int\limits_{x_j}^{x_{j+1}} u(t)dt/h_j$,
$C_4 = -15u(x_j) - 15u(x_{j+1}) - (5/2)u'(x_j)h_j + (5/2)u'(x_{j+1})h_j + 30\int\limits_{x_j}^{x_{j+1}} u(t)dt/h_j$.

We shall use techniques from interval analysis. Suppose $a_1$, $a_2$, $b_1$, $b_2$ are real numbers. The result of the operations between the intervals $A=[a_1,a_2]$ and $B=[b_1,b_2]$ can be obtained with the next formulae (see [9])

1. $A+B=[a_1+b_1,a_2+b_2]$,
2. $A-B=[a_1-b_2,a_2-b_1]=A+[-1,-1]\cdot B$,
3. $A\cdot B = [\min\{a_1b_1,a_1b_2,a_2b_1,a_2b_2\}, \max\{a_1b_1,a_1b_2,a_2b_1,a_2b_2\}]$,
4. $A:B=[a_1,a_2]\cdot[1/b_2,1/b_1]$.

We can put $T=[0,1]$ instead of $t$, thus we have $X_j=x_j+Th_j=[x_j,x_{j+1}]$ in:

$$\widetilde{u}^p(x_j+Th_j)=C_4T^4+C_3T^3+C_2T^2+C_1T+C_0. \quad (6)$$

Using operations 1–4 between the intervals mentioned above and the rule:
$T^k=[\min_{t\in T}T^k, \max_{t\in T}T^k]$, $k=1,2,3,4$, we receive the interval $Z_j^p=\widetilde{u}^p(x_j+Th_j)$, that contains the result of the approximation of the function $u$ on the interval $X_j$.

Besides scheme (5) we can also use the Horner scheme to represent the approximation function errors. If $x=x_j+th_j$, $t\in[0,1]$, we can transform (5) to the form:

$$\widetilde{u}^H(x_j+th_j)=(C_0+t(C_1+t(C_2+t(C_3+t(C_4))))). \quad (7)$$

Now we can put $T=[0,1]$ instead of $t$ in (7). Therefore the result of the approximation of the function $u(x)$, $x\in[x_j,x_{j+1}]$ is contained in the evaluation interval $Z_j^H=\widetilde{u}^H(X_j)$:

$$\widetilde{u}^H(X_j)=(C_0+T(C_1+T(C_2+T(C_3+T(C_4))))). \quad (8)$$

Now we will construct one more expression through which the values of $u$ are estimated. Our aim is to find interval extension with the widths as narrow as possible. We use the result from [1] about possibility representation of the polynomial when interval extension equals to the range of a function. Our aim is to find the expression of a function $\widetilde{u}(x_j+th_j)$ that provides equality the interval evaluation of some function to its range.

Suppose the hypothesis of Lemma 1 are fulfilled. Suppose $C_4\neq 0$. We put $N_P=A_0((A_1+t)^2+A_3)^2+A_4$ where $A_0,A_1,A_3,A_4$ we have to determine. We can write $N_P$ as the following $N_P=K_0+K_1t+K_2t^2+K_3t^3+K_4t^4$, where
$K_0=A_0(A_1^2+A_3)^2+A_4$, $K_1=4A_0(A_1^2+A_3)A_1$,
$K_2=A_0(6A_1^2+2A_3)$, $K_3=4A_0A_1$, $K_4=A_0$.

Solving the system of equations: $K_4=C_4$, $K_3=C_3, K_2=C_2$, $K_0=C_0$,

$$A_0=C_4,\ A_1=C_3/(4C_4), \quad (9)$$

$$A_3=(-3C_3^2+8C_2C_4)/(16C_4^2), \quad (10)$$

$$A_4=\frac{(-C_3^4+8C_3^2C_2C_4-16C_2^2C_4^2+64C_0C_4^3)}{(64C_4^3)}. \quad (11)$$

From $R_S=N_P-P_P$, where
$P_P=\widetilde{u}(x_j+th_j)=C_0+C_1t+C_2t^2+C_3t^3+C_4t^4$, we obtain $R_S=-t(C_3^3-4C_3C_4C_2+8C_1C_4^2)/(8C_4^2)$. Finally the polynomial $\widetilde{u}(x_j+th_j)$ is transformed to the form:

$$\widetilde{u}^N(x_j+th_j)=A_0((A_1+t)^2+A_3)^2+A_4+ t(C_3^3-4C_3C_4C_2+8C_1C_4^2)/(8C_4^2),$$

where we find from (9)-(11). Here the powers appearing in the expression are evaluated as $X^k=[\min_{x\in X}x^k, \max_{x\in X}x^k]$. The evaluation interval of $\widetilde{u}(x_j+th_j)$, $t\in[0,1]=T$ is the following:

$$\widetilde{u}^N(x_j+Th_j)=A_0((A_1+T)^2+A_3)^2+A_4+ T(C_3^3-4C_3C_4C_2+8C_1C_4^2)/(8C_4^2). \quad (12)$$

Let us examine each of the methods through several functions and see the difference between the width of the intervals which include the values of the approximation of the function $u$. The following calculations show that for the functions which were examined we receive the relation: $\widetilde{u}^H(X_j)=Z_j^H\subset\widetilde{u}^p(X_j)$.

*Example 1.* First let us take $u(x)=\cos(x)$, $X_j=[x_j,x_{j+1}]=[-0.1,0.2]$. Figure 4 (left) shows us the interval $Z_j^p=\tilde{u}^p(X_j)=[0.94978,1.02529]$, when we use method (6) and Figure 4 (right) shows us the interval $Z_j^H=u^H(X_j)=[0.97973,1.024954]$, when we use method (8).

*Example 2.* Now let us take $u(x)=x^4-x^2$, $X_j=[x_j,x_{j+1}]=[-0.1,0.2]$. Figure 5 (left) shows us the interval $Z_j^p=\tilde{u}^p(X_j)=[-0.1053,0.057]$, when we use method (6) and Figure 5 (right) shows us the interval $Z_j^H=\tilde{u}^H(X_j)=$

Fig. 4. Plots of the function $u(x) = \cos(x)$, $X_j = [-0.1, 0.2]$, obtained with method (6) (left), and obtained with method (8) (right)

$[-0.0465, 0.0489]$, when we use method (8). Method (8) for both functions gives us a better result: the width of the interval $Z_j^H$ is less if we use the Horner scheme instead of polynomial scheme (6).

We have $Zj^H \subset Z_j^p$. But there may exist functions for which we can receive $Z_j^p \subset Zj^H$.

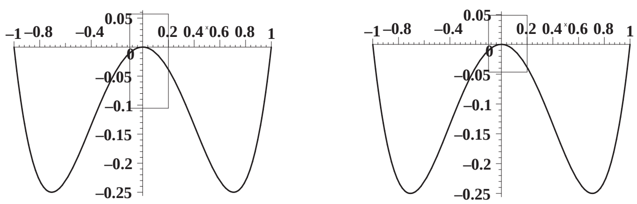

Fig. 5. Plots of the function $u(x) = x^4 - x^2$, $X_j = [-0.1, 0.2]$, and the interval evaluation : obtained with method (6) (left), and obtained with method (8) (right)

The following examples show the results of the interval evaluation of derivatives of functions.

*Example 1a.* Here we take the function from example 1 and calculate the evaluation interval for its derivative. We have $u'(x) = -\sin(x)$, $X_j = [x_j, x_{j+1}] = [-0.1, 0.2]$.

Figure 6 (left) shows us the interval $Z_j^p = \tilde{u}'^p(X_j) = [-0.2032, 0.1043]$, when we use method (6) and Figure 6 (right) shows us the interval $Z_j^H = \tilde{u}'^H(X_j) = [-0.204, 0.0998]$, when we use method (8).

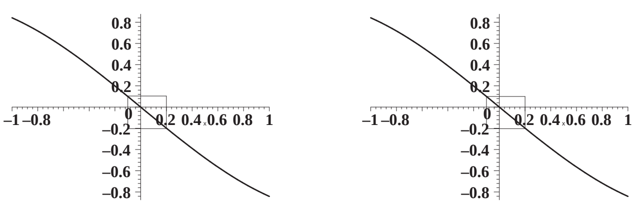

Fig. 6. Plots of the function $u'(x) = -sin(x)$, $X_j = [-0.1, 0.2]$, and the interval evaluation: obtained with method (6) (left), and obtained with method (8) (right)

*Example 2a.* Here we take the function from example 2 $u(x) = x^4 - x^2$, $X_j = [-0.1, 0.2]$.

Figure 7 (left) shows us the interval $Z_j^p = \tilde{u}'^p(X_j) = [-0.476, 0.304]$, when we use method (6) and Figure 7 (right) shows us the interval $Z_j^H = \tilde{u}'^H(X_j) = [-0.4868, 0.196]$, when we use method (8).

*Example 3.* Here we take the function from Example 1 and Example 2: $u(x) = x^4 - x^2$, $u(x) = \cos(x)$, $X_j = [x_j, x_{j+1}] = [-0.1, 0.2]$.

Figure 8 (left) shows us the evaluation interval $\widetilde{u}^N(x_j + Th_j) = P_P(X_j) = [0.98005, 0.99999904]$, when we use method (12) for $u(x) = \cos(x)$ and Figure 8 (right) shows

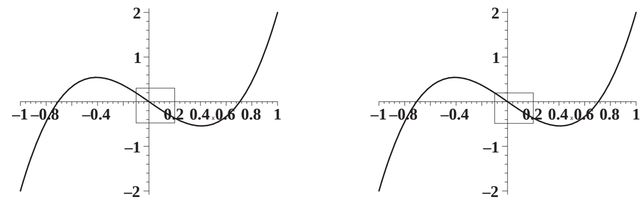

Fig. 7. Plots of the function $u'(x)$, $u = x^4 - x^2$, $X_j = [-0.1, 0.2]$, and the interval evaluation: obtained with method (6) (left), and obtained with method (8) (right)

us the evaluation interval $\widetilde{u}^N(x_j + Th_j) = P_P(X_j) = [-0.0383999, 0]$, when we use method (12), $u(x) = x^4 - x^2$.

We should notice that we have the range $W(u, X)$ of the function $u = \cos(x)$, $x \in [-0.1, 0.2] = X_j$ as $W(\cos, X_j) = [\cos(-0.1), \cos(0.2)] \approx [0.9950, 0.98007] \subset [0.98005, 0.99999904] = \widetilde{u}^N(X_j)$, and we have the range of the function $u(x) = x^4 - x^2$ as the following $W(x^4 - x^2, X_j) = [u(-0.1), u(0.2)] = [-0.0099, -0.0384] \subset [-0.0383999, 0.0] = \widetilde{u}^N(X_j)$.

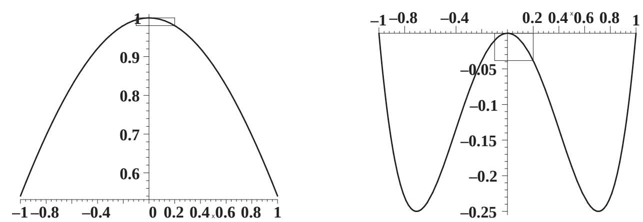

Fig. 8. Plots of the function $u(x)$, and the interval evaluation obtained with method (**??**), $X_j = [-0.1, 0.2]$: $u = \cos(x)$ (left), and $u = x^4 - x^2$ (right)

*Example 4.* Here we take the function from [1] $u(x) = x^2(x^2/3 + \sqrt{2}\sin(x)) - \sqrt{3}/19$, $X_j = [x_j, x_{j+1}] = [-0.1, 0.2]$. The range of the function $u(x)$, $x \in [-0.1, 0.2]$ is the next: $W(u, X_j) = [u(0.2), u(-0.1)] = [-0.0793888010, -0.09253909320]$.

When we use method (12), the evaluation interval $\widetilde{u}^N(x_j + Th_j)$ is the following: $[-0.07938880, -0.0925390932] = W(u, X_j)$. It is equals to the range of the function. When we use the Horner method the evaluation interval is the following: $[-0.081613529, -0.0925390932]$ that is a little wider then the range of the function.

The last two examples provide an opportunity to hypothesize that scheme (12) gives the best interval evaluation if $C_4 \neq 0$.

## IV. Interval evaluation of approximation of function of two variables

On every line parallel to axis $y$, we can construct the approximation in the form:

$$\widetilde{u}(y) = u(y_k)\omega_{k,0}(y) + u(y_{k+1})\omega_{k+1,0}(y) +$$

$$u'(y_j)\omega_{k,1}(y) + u'(y_{k+1})\omega_{k+1,1}(y) +$$

$$+ \int_{y_k}^{y_{k+1}} u(t)dt\, \omega_k^{<0>}(y), \quad y \in [y_k, y_{k+1}].$$

Now we have the following formulae for $y = y_k + t_1 h$, $t_1 \in [0,1]$, $h = y_{k+1} - y_k$:

$$\omega_{k,0}(y_k + t_1 h) = -18\,t_1^2 + 32\,t_1^3 - 15\,t_1^4 + 1,$$

$$\omega_{k+1,0}(y_k + t_1 h) = -12\,t_1^2 + 28\,t_1^3 - 15\,t_1^4,$$

$$\omega_{k,1}(y_k+t_1 h) = -(9/2)\,h\,t_1^2 + 6\,h\,t_1^3 - (5/2)\,h\,t_1^4 + t_1\,h,$$

$$\omega_{k+1,1}(y_k + t_1 h) = (3/2)\,h\,t_1^2 - 4\,h\,t_1^3 + (5/2)\,h\,t_1^4,$$

$$\omega_k^{<0>}(y_k + t_1 h) = (30\,t_1^2 - 60\,t_1^3 + 30\,t_1^4)/h.$$

If $(x, y) \in \Omega_{j,k}$ then we get the next expression using the tensor product:

$$\tilde{u}(x,y) = \sum_{i=0}^{1}\sum_{p=0}^{1} u(x_{j+i}, y_{k+p})\ \omega_{j+i,0}(x)\ \omega_{k+p,0}(y) +$$

$$+\sum_{i=0}^{1}\ \sum_{p=0}^{1} u'_y(x_{j+i}, y_{k+p})\ \omega_{j+i,0}(x)\ \omega_{k+p,1}(y) +$$

$$\sum_{i=0}^{1}\ \int_{y_k}^{y_{k+1}} u(x_{j+i}, t)dtdy\ \omega_{j+i,0}(x)\omega_k^{<0>}(y) +$$

$$\sum_{i=0}^{1}\Bigg(\int_{x_j}^{x_{j+1}} u(t, y_{k+i})dt\omega_j^{<0>}(x)\omega_{k+i,0}(y) +$$

$$\int_{x_j}^{x_{j+1}} u'_y(t, y_{k+i})dt\omega_j^{<0>}(x)\omega_{k+i,1}(y)\Bigg) +$$

$$\int_{y_k}^{y_{k+1}}\int_{x_j}^{x_{j+1}} u(x,y)dxdy\omega_k^{<0>}(y)\omega_j^{<0>}(x) +$$

$$\sum_{i=0}^{1} u'_x(x_j, y_{k+i})dt\omega_{j,0}(x)\omega_{k+i,0}(y) +$$

$$\sum_{i=0}^{1} u''_{xy}(x_j, y_{k+i})\,dt\ \omega_{j,0}(x)\ \omega_{k+i,1}(y) +$$

$$\int_{y_k}^{y_{k+1}} u'_x(x_j, t)dt\,\omega_{j,1}(x)\,\omega_k^{<0>}(y). \qquad (13)$$

For obtaining the lower and the upper boundary we shall use the Horner method for variable $x$ and after that for variable $y$. Let us take $x_j = y_k = 0$, $x_{j+1} = y_{k+1} = 0.2$. Figure 9 and 10 show the plots of the function and the intervals evaluation from different angles.

For obtaining the lower and the upper boundary of the partial derivatives of the functions of two variables we we shall use the Horner method for variable $x$ and after that for variable $y$. We use formula (13) where we replace $\omega$ with their derivatives.

Here we the next formulae are useful:
$\omega'_{j,0}(x_j + th_j) = (-36t + 96t^2 - 60t^3)/h_j$,
$\omega'_{j+1,0}(x_j + th_j) = (-24t + 84t^2 - 60t^3)/h_j$,
$\omega'_{j,1}(x_j + th_j) = (h_j - 9th_j + 18h_jt^2 - 10h_jt^3)/h_j$,

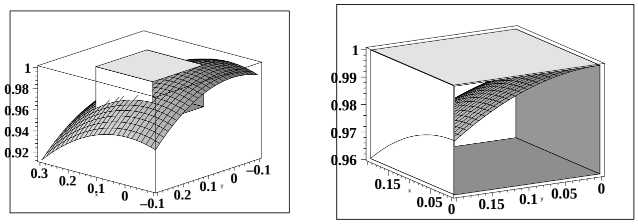

Fig. 9. Plots of the function $\cos(x)\cos(y)$ and the evaluation interval, when $X_j = Y_k = [0, 0.2]$

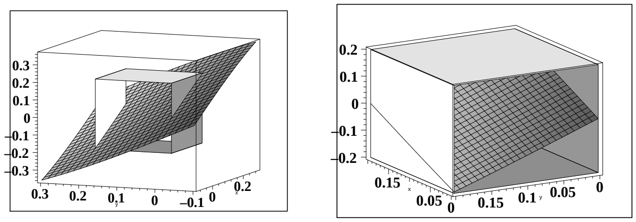

Fig. 10. Plots of the function $\cos(x-y)\sin(x-y)$ and the evaluation interval, when $X_j = Y_k = [0, 0.2]$

$\omega'_{j+1,1}(x_j + th_j) = (3th_j - 12h_jt^2 + 10h_jt^3)/h_j$,
$\omega_j'^{<0>}(x_j + th_j) = (60t - 180t^2 + 120t^3)/{h_j}^2$.

For example let us take $u(x,y) = \sin(x-y)\cos(x-y)$, $u'_x(x,y) = \cos(x-y)^2 - \sin(x-y)^2$ and $u(x,y) = 1/(1 + (x+y)^2)$, $u'_x(x,y) = -2(x+y)/(1+(x+y)^2)^2$.

Figure 11 shows the plots of the function $u'_x(x,y) = \cos(x-y)^2 - \sin(x-y)^2$ and the intervals evaluation (left) and also $u'_x(x,y) = -2(x+y)/(1+(x+y)^2)^2$ and the intervals evaluation (right).

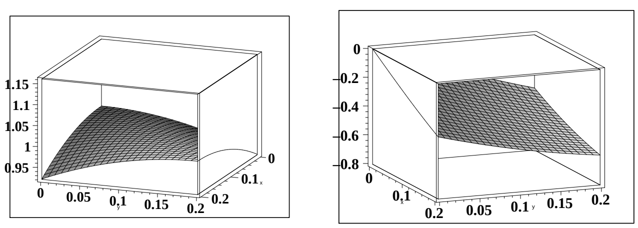

Fig. 11. Plots of the function $\cos(x-y)^2 - \sin(x-y)^2$ and the evaluation interval (left), $u'_x(x,y) = -2(x+y)/(1+(x+y)^2)^2$ and the intervals evaluation (right) when $X_j = Y_k = [0, 0.2]$

## V. About left integro-differential splines

Let the function $u(x)$ be such that $u \in C^5([a-h, b])$. We have the grid of interpolation nodes $\{x_j\}$ such that $x_{-1} = a - h$, $x_0 = a$, $x_{j+1} = x_j + h$, $x_n = b$.

Suppose that $u(x_j)$, $u'(x_j)$, $j = 0, 1, \ldots, n$, $\int_{x_j}^{x_{j+1}} u(\xi)d\xi$, $j = -1, \ldots, n-1$, are known. We denote $\widetilde{u}(x)$ as an approximation of the function $u(x)$ in the interval $[x_j, x_{j+1}] \subset [a, b]$:

$$\widetilde{u}(x) = u(x_j)\omega_{j,0}(x) + u(x_{j+1})\omega_{j+1,0}(x) +$$

$$u'(x_j)\omega_{j,1}(x) + u'(x_{j+1})\omega_{j+1,1}(x) +$$

$$+\int_{x_{j-1}}^{x_j} u(\xi)d\xi\,\omega_j^{<-1>}(x). \qquad (14)$$

We obtain the basic splines $\omega_{j,0}(x)$, $\omega_{j+1,0}(x)$, $\omega_{j,1}(x)$, $\omega_{j+1,1}(x)$, $\omega_j^{<-1>}(x)$ from the system:

$$\widetilde{u}(x) \equiv u(x), \quad u(x) = x^{i-1}, \ i = 1, 2, 3, 4, 5. \tag{15}$$

If $x = x_j + t\,h$, $t \in [0, 1]$, then the basic splines can be written in the form:

$\omega_{j,0}(x_j + t\,h) = \frac{1}{31}(15t^2 + 62t + 31)(t-1)^2$,
$\omega_{j+1,0}(x_j + t\,h) = -\frac{1}{31}t^2(-48 - 28t + 45t^2)$,
$\omega_{j,1}(x_j + t\,h) = \frac{1}{62}th(85t + 62)(t-1)^2$,
$\omega_{j+1,1}(x_j + t\,h) = \frac{1}{62}ht^2(35t + 27)(t-1)$,
$\omega_j^{<-1>}(x_j + t\,h) = \frac{30}{31}t^2(t-1)^2/h$.

*Lemma 4. Let function $u \in C^{(5)}[a-h, b]$. There are points $\eta \in [x_{j-1}, x_{j+1}]$, $\zeta \in [x_{j-1}, x_j]$ such that*

$$u(x) - \widetilde{u}(x) = \frac{u^{(5)}(\eta)}{5!}(x - x_j)^2(x - x_{j+1})^2(x - \zeta),$$

$x \in [x_j, x_{j+1}]$.

*Proof.* We have approximation (14). It can be shown that the next relations are fulfilled:

1) $\widetilde{u}(x_j) = u(x_j)$,
2) $\widetilde{u}(x_{j+1}) = u(x_{j+1})$,
3) $\widetilde{u}'(x_j) = u'(x_j)$,
4) $\widetilde{u}'(x_{j+1}) = u'(x_{j+1})$,
5) $\int_{x_{j-1}}^{x_j} \widetilde{u}(x)dx = \int_{x_{j-1}}^{x_j} u(x)dx$.

The proof is similar to Lemma 2.

Further, let us notice that the next equality follows from (14)- (15) for $u(x) = C = const$:

$$C = C\omega_{j,0}(x) + C\omega_{j+1,0}(x) + Ch\,\omega_j^{<-1>}(x).$$

Now we can find the point $\zeta$ such that $u(\zeta)h = \int_{x_{j-1}}^{x_j} u(\xi)d\xi$. Let us take an approximation $\widetilde{u}(x)$, $x \in [x_j, x_{j+1}]$, in the form:

$$\widetilde{u}(x) = u(x_j)\omega_{j,0}(x) + u(x_{j+1})\omega_{j+1,0}(x) + u'(x_j)\omega_{j,1}(x) + u'(x_{j+1})\omega_{j+1,1}(x) + u(\zeta)\,h\omega_j^{<-1>}(x). \tag{16}$$

Obviously $\zeta$ is the point of interpolation. It follows from the relation:

$$\omega_{j,0}(x) + \omega_{j+1,0}(x) + h_j\omega_j^{<-1>}(x) = 1.$$

Thus, from (14)-(15) we obtain:

$$\tilde{u}(\zeta) = u(\zeta)\big(\omega_{j,0}(x) + \omega_{j+1,0}(x) + h\omega_j^{<-1>}(x)\big) = u(\zeta).$$

Approximation (16) has the next points of Hermit interpolation: $x_j$ with the second multiplicity, $x_{j+1}$ with the second multiplicity, and $\zeta$. The remainder term of Hermit interpolation in our case is as follows:

$$\frac{u^{(5)}(\eta)}{5!}(x - x_j)^2(x - x_{j+1})^2(x - \zeta).$$

The proof is completed.

*Corollary.* If $M = \max_{x \in [a-h,b]} |u^{(5)}(x)|$ and we put $x = x_j + th$, $t \in [0, 1]$, $\zeta = x_{j-1} + Th$, $T \in [0, 1]$, then

$$|\widetilde{u}(x_j + th) - u(x_j + th)| \le \frac{Mh^5}{5!}t^2(t-1)^2|t - T + 1|.$$

*Lemma 5. Let function $u \in C^{(5)}[a-h, b]$. $M = \max_{[x_j-h, x_{j+1}]} |u^{(5)}(x)|$. The next statement is valid:*

$$|u(x) - \widetilde{u}(x)| \le 2 \cdot 0.0625h^5\frac{M}{5!}, \tag{17}$$

$x \in [x_j, x_{j+1}]$.

Proof follows from Lemma 4 and the relations

$$\max_{x \in [x_j, x_{j+1}]} |(x - x_j)^2(x - x_{j+1})^2| \le 0.0625h^4,$$

$|x - \zeta| \le 2h$ if $x \in [x_j, x_{j+1}]$, $\zeta \in [x_{j-1}, x_j]$.

Table 3 shows actual and theoretical errors of approximation of functions constructed with formulae (14), (17) when $[a, b] = [-1, 1]$, $h = 0.1$.

TABLE 3.
ACTUAL AND THEORETICAL ERRORS $[a, b]=[-1, 1]$, $h$=0.1

| $u(x)$ | $\max_{[-1,1]} \|u-\widetilde{u}\|$ ACTUAL ERRORS | $\max_{[-1,1]} \|u-\widetilde{u}\|$ THEORETICAL ERRORS |
|---|---|---|
| $\sin(3x)\cos(5x)$ | $0.109 \cdot 10^{-3}$ | $0.17 \cdot 10^{-3}$ |
| $\cos(x)$ | $0.565 \cdot 10^{-8}$ | $0.88 \cdot 10^{-8}$ |
| $\cos(2x)$ | $0.218 \cdot 10^{-6}$ | $0.33 \cdot 10^{-6}$ |
| $\frac{1}{(1+25x^2)}$ | $0.141 \cdot 10^{-2}$ | $0.33 \cdot 10^{-2}$ |

## VI. ABOUT RIGHT INTEGRO-DIFFERENTIAL SPLINES

Let the function $u(x)$ be such that $u \in C^5([a, b+h])$. We have the grid of interpolation nodes $\{x_j\}$ such that $x_0 = a$, $x_{j+1} = x_j + h$, $x_n = b$, $x_{n+1} = b + h$.

Suppose that $u(x_j)$, $u'(x_j)$, $j = 0, 1, \ldots, n$, $\int_{x_j}^{x_{j+1}} u(\xi)d\xi$, $j = 0, \ldots, n$, are known. We denote $\widetilde{u}(x)$ as an approximation of the function $u(x)$ in the interval $[x_j, x_{j+1}] \subset [a, b]$:

$$\widetilde{u}(x) = u(x_j)\omega_{j,0}(x) + u(x_{j+1})\omega_{j+1,0}(x) + u'(x_j)\omega_{j,1}(x) + u'(x_{j+1})\omega_{j+1,1}(x) + \int_{x_{j+1}}^{x_{j+2}} u(\xi)d\xi\,\omega_j^{<1>}(x). \tag{18}$$

We obtain the basic splines $\omega_{j,0}(x)$, $\omega_{j+1,0}(x)$, $\omega_{j,1}(x)$, $\omega_{j+1,1}(x)$, $\omega_j^{<1>}(x)$ from the system:

$$\widetilde{u}(x) \equiv u(x), \quad u(x) = x^{i-1}, \ i = 1, 2, 3, 4, 5. \tag{19}$$

If $x = x_j + t\,h$, $t \in [0, 1]$, then the basic splines can be written in the form:

$\omega_{j,0}(x_j + t\,h) = -(45t^2 - 62t - 31)(t-1)^2/31$,
$\omega_{j+1,0}(x_j + t\,h) = t^2(15t^2 + 108 - 92t)/31$,
$\omega_{j,1}(x_j + t\,h) = -th(35t - 62)(t-1)^2/62$,
$\omega_{j+1,1}(x_j + t\,h) = -ht^2(t-1)(85t - 147)/62$,
$\omega_j^{<-1>}(x_j + t\,h) = \frac{30}{31h}t^2(t-1)^2$.

*Lemma 6. Let function $u \in C^{(5)}[a, b+h]$. There are points $\eta \in [x_j, x_{j+2}]$, $\zeta \in [x_{j+1}, x_{j+2}]$ such that*

$$u(x) - \widetilde{u}(x) = \frac{u^{(5)}(\eta)}{5!}(x - x_j)^2(x - x_{j+1})^2(x - \zeta),$$

*when* $x \in [x_j, x_{j+1}]$.

*Proof.* It can be shown that the next relations are fulfilled on approximation (18):
1) $\widetilde{u}(x_j) = u(x_j)$,
2) $\widetilde{u}(x_{j+1}) = u(x_{j+1})$,
3) $\widetilde{u}'(x_j) = u'(x_j)$,
4) $\widetilde{u}'(x_{j+1}) = u'(x_{j+1})$,
5) $\int_{x_{j+1}}^{x_{j+2}} \widetilde{u}(x)dx = \int_{x_{j+1}}^{x_{j+2}} u(x)dx$.

The proof is similar to Lemma 2. Further, let us notice that the next equality follows from (18)- (19) for $u(x) = C = const$:

$$C = C\omega_{j,0}(x) + C\omega_{j+1,0}(x) + Ch\,\omega_j^{<1>}(x).$$

Now we can find the point $\zeta$ such that $u(\zeta)h = \int_{x_{j-1}}^{x_j} u(\xi)d\xi$. Let us take an approximation $\widetilde{u}(x)$, $x \in [x_j, x_{j+1}]$, in the form:

$$\widetilde{u}(x) = u(x_j)\omega_{j,0}(x) + u(x_{j+1})\omega_{j+1,0}(x)+ u'(x_j)\omega_{j,1}(x) + u'(x_{j+1})\omega_{j+1,1}(x)+ u(\zeta)\,h\omega_j^{<1>}(x). \quad (20)$$

Obviously $\zeta$ is the point of interpolation. It follows from the relation:

$$\omega_{j,0}(x) + \omega_{j+1,0}(x) + h\omega_j^{<1>}(x) = 1.$$

Thus, from (18) we obtain:

$$\tilde{u}(\zeta) = u(\zeta)\big(\omega_{j,0}(x) + \omega_{j+1,0}(x) + h\omega_j^{<1>}(x)\big) = u(\zeta).$$

Approximation (20) has the next points of Hermit interpolation: $x_j$ with the second multiplicity, $x_{j+1}$ with the second multiplicity, and $\zeta$. The remainder term of Hermit interpolation in our case is as follows:

$$\frac{u^{(5)}(\eta)}{5!}(x - x_j)^2(x - x_{j+1})^2(x - \zeta).$$

The proof is completed.

*Corollary*. If $M = \max\limits_{x\in[a,b+h]} |u^{(5)}(x)|$ and we put $x = x_j + th$, $t \in [0, 1]$, $\zeta = x_{j+1} + Th$, $T \in [0, 1]$, then

$$|\widetilde{u}(x_j + th) - u(x_j + th)| \le \frac{Mh^5}{5!} t^2(t - 1)^2|t - T - 1|.$$

The proof is evident.

*Lemma 7. Let function* $u \in C^{(5)}[a, b + h]$. $M = \max\limits_{[x_j,x_{j+2}]} |u^{(5)}(x)|$. *The next statement is valid:*

$$|u(x) - \widetilde{u}(x)| \le 2 \cdot 0.0625h^5 \frac{M}{5!}, x \in [x_j, x_{j+1}]. \quad (21)$$

Proof follows from Lemma 6 and the relations

$$\max_{x\in[x_j,x_{j+1}]} |(x - x_j)^2(x - x_{j+1})^2| \le 0.0625h^4,$$

$|x - \zeta| \le 2h$, if $x \in [x_j, x_{j+1}]$ and $\zeta \in [x_{j+1}, x_{j+2}]$.

Table 4 shows actual and theoretical errors of approximation of functions constructed with formulae (18), (21) when $[a, b] = [-1, 1]$, $h = 0.1$.

TABLE 4.
ACTUAL AND THEORETICAL ERRORS WHEN $[a, b] = [-1, 1]$, $h = 0.1$

| $u(x)$ | $\max\limits_{[-1,1]} \lvert u - \widetilde{u} \rvert$ ACTUAL ERRORS | $\max\limits_{[-1,1]} \lvert u - \widetilde{u} \rvert$ THEORETICAL ERRORS |
|---|---|---|
| $\sin(3x)\cos(5x)$ | $0.109 \cdot 10^{-3}$ | $0.17 \cdot 10^{-3}$ |
| $\cos(x)$ | $0.565 \cdot 10^{-8}$ | $0.88 \cdot 10^{-8}$ |
| $\cos(2x)$ | $0.218 \cdot 10^{-6}$ | $0.33 \cdot 10^{-6}$ |
| $\frac{1}{(1+25x^2)}$ | $0.141 \cdot 10^{-2}$ | $0.33 \cdot 10^{-2}$ |

## VII. CONCLUSION

Here we construct the intervals evaluation which include the values of the function of one variable and parallelepipeds which include the values of the functions of two variables when we know the values of the function, the value of its first derivative in the nodes and the value of the integrals over the net intervals. So we don't need to calculate the approximation of the function at every point. Using techniques from interval analysis, we construct the two-sided estimations of approximation of the functions with integro-differential polynomial splines. Before constructing the interval extension it is recommended to check the error of approximation using the results given in Lemmas.

## REFERENCES

[1] G. Alefeld, J. Herzberger, *Introduction to interval computations*, Academic Press, New York, 1983.
[2] I. G. Burova, "On left integro-differential splines and Cauchy problem", *International Journal of Mathematical Models and Methods in Applied Sciences*, 9, pp. 683-690, 2015.
[3] I. G. Burova, S. V. Poluyanov, "Construction of meansquare approximation with integro-differential splines of fifth order and first level", *Vestn. St. Petersbg Univ., Math.*, 47, No 2, pp. 57-63, 2014.
[4] I. G. Burova, S. V. Poluyanov, "On Approximations by polynomial and trigonometrical integro-differential splines", *International Journal of Mathematical Models and Methods in Applied Sciences*, 10, pp. 190-199, 2016.
[5] M. Fiedler, J. Nedoma, J. Ramik, J. Rohn, K. Zimmermann, *Linear optimization problems with inexact data*, Springer Science+Business Media, New York, 1996.
[6] R. Hammer, M. Hocks, U. Kulisch, D. Ratz, *Numerical toolbox for verified computing I: Basic numerical problems*, Springer, Berlin-Heidelberg, 1993.
[7] E. Hansen, G. W. Walster, *Global optimization using interval analysis*, Marcel Dekker, New York, 2004.
[8] R. B. Kearfott, *Rigorous global search: continuos problems*, Kluwer, Dordrecht, 1996.
[9] R. E. Moore, *Methods and applications of interval analysis*, SIAM, Philadelphia, 1979.
[10] R. E. Moore, R. B. Kearfott, M. J. Cloud, *Introduction to interval analysis*, SIAM, Philadelphia, 2009.
[11] A. Neumaier, *Interval methods for systems of equations*, Cambridge University Press, Cambridge, 1990.
[12] H. Ratschek, J. Rokne, *Computer methods for the range of functions*, Chichester: Ellis Horwood, New York, 1984.
[13] H. Ratschek, J. Rokne, *New computer methods for global optimization*, Chichester: Ellis Horwood, New York, 1988.
[14] J. Rohn, *A handbook of results on interval linear problems*, Institute of Computer Science, Academy of Sciences of the Czech Republic, Prague, 2005.
[15] J. Stolfi, L. H. de Figueiredo, *Self-validated numerical methods and applications*, Brazilian Mathematics Colloquium monograph, Rio de Janeiro, 1997.