Comparison of different approaches to continuous-time system identification from sampled data

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***Abstract*—This article deals with different approaches to continuous-time system identification from sampled data. Continuous-time system identification is important problem in control theory. Continuous time models provide many advantages against discrete time models because of better physical insight into the system properties. The traditional approach with least squares method with state variable filters is presented. Two alternative approaches to continuous-time identification are proposed. The generalized Laguerre functions method and the method based on least squares estimation with numerical solution of differential equation are introduced. These three different approaches to continuous-time system identification from sampled data are compared on the example. It is shown that proposed alternative methods can give better results in terms of relative root mean square error of the outputs of the identified systems than the least squares method with state variable filters.**

***Keywords*—generalized Laguerre function, continuous-time LTI system, numerical integration, identification**

I. Introduction

There has been recent interest, see [1], [2], [3], in direct identification of continuous time models from sampled discrete time data of input and output $\{u(t_k), y(t_k)\}$. Much of the system identification literature deals with the discrete time models due to their suitability for designing digital control systems. Real world described by the differential equations derived from the physical laws is naturally continuous. Continuous time models provide many advantages against discrete time models because of better physical insight into the system properties. The identified parameters of the continuous time model usually have the direct physical interpretation. One can see [4] for some motivation examples for identifying continuous time models from sampled data. The aim of this article is to present two alternative approaches to continuous-time system identification from sampled data and compare them with traditional least squares (LS) method with state variable filters (SVF). The identification method based on The Generalized Laguerre Functions (GLF) was described in [5]. The comparison between GLF and LS method was done in [6] with the emphasis on the difference between identification with simple and generalized Laguerre functions. In this article we will introduce new approach to continuous-time identification based on numerical solution of differential equations and compare it with GLF method and with traditional LS method with SVF.

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II. Identification of Continuous-time Systems From Sampled Data

Let us assume a continuous time linear dynamical system described by the linear differential equation with constant coefficients. Let $u(t)$ be the input signal and $y(t)$ the output signal.

$$A(p)y(t) = B(p)u(t), \tag{1}$$

$$y(t) = F(p)u(t) \tag{2}$$

$$F(p) = \frac{B(p)}{A(p)} = \frac{b_0p^m + b_1p^{m-1} + \ldots + b_m}{p^n + a_1p^{n-1} + \ldots + a_n}, \qquad n \geq m. \tag{3}$$

p is the time-domain differentiation operator, i.e

$$px(t) = \frac{dx(t)}{dt}, \tag{4}$$

Given this description, the identification problem is to determine a suitable model structure for (2) and then estimate the parameters that characterize this structure, based on the sampled input and output data $Z^N = \{u(t_k), y(t_k)\}_{k=1}^N$.

III. The LS Method With SVF

Many methods based on the least squares with the use of the state variable filters have been developed for identification of the continuous-time models, see [3], [4]. In this section we will describe the least squares method with SVF.

The model (2) can be written in the differential equation form

$$y^{(n)}(t) + a_1y^{(n-1)}(t) + \ldots + a_ny(t) = \\ = b_0u^{(m)}(t) + b_1u^{(m-1)} + \ldots + b_mu(t). \tag{5}$$

The state variable filter $L(p)$ can be applied on both sides of the equation in order to obtain the derivatives of input and output.

$$L(p)A(p)y(t) = L(p)B(p)u(t). \tag{6}$$

The above equation can be written in the following form

$$(L_n(p) + a_1L_{n-1}(p) + \ldots + a_nL_0(p))y(t) = \\ = (b_0L_m(p) + \ldots + b_mL_0(p))u(t), \tag{7}$$

with

$$L_k(p) = L(p)p^k, \qquad k = 0, 1, \ldots, n. \tag{8}$$

The equation (7) can be written as

$$y_f^{(n)}(t) + a_1 y_f^{(n-1)}(t) + \ldots + a_n y_f^{(0)}(t) = \\ = b_0 u_f^{(m)}(t) + b_1 u_f^{(m-1)}(t) + \ldots + b_m u_f^{(0)}(t), \tag{9}$$

$$y_f^{(k)}(t) = l_k(t) * y(t), u_f^{(k)}(t) = l_k(t) * u(t), \tag{10}$$

where $l_k(t)$ is the impulse response of the filter $L_k(p)$ and $*$ is the operator of convolution. Then we can write down the above equation (9) in the standard regression form, i.e.

$$y_f^{(n)}(t_k) = \varphi_f^T(t_k)\theta, \tag{11}$$

$$\varphi_f^T(t_k) = [-y_f^{(n-1)}(t_k), -y_f^{(n-2)}(t_k) \ldots \\ \ldots - y_f^{(0)}(t_k), u_f^{(m)}(t_k), u_f^{(m-1)}(t_k), \ldots, u_f^{(0)}(t_k)], \tag{12}$$

$$\theta = [a_1, a_2, \ldots, a_n, b_0, b_1, \ldots, b_m]^T. \tag{13}$$

The least squares SVF estimate is then given by (see [4])

$$\hat{\theta} = \left[\frac{1}{N}\sum_{k=1}^{N} \varphi_f(t_k)\varphi_f^T(t_k)\right]^{-1} \frac{1}{N}\sum_{k=1}^{N} \varphi_f(t_k) y_f^{(n)}(t_k). \tag{14}$$

The general form of the SVF is an all-pole process with a denominator $C(s)$ (see [4])

$$L(p) = \frac{1}{C(p)} = \frac{1}{p^n + c_1 p^{n+1} + c_2 p^{n-2} + \ldots + c_n}. \tag{15}$$

According to [4] it can be shown that the SVF filter is optimal in statistical terms if

$$C(p) = A(p). \tag{16}$$

For practical computation the SVF are often chosen (see [2]) as the so-called basic state variable filter

$$L(p) = \frac{1}{(p+\lambda)^n}, \tag{17}$$

where λ is chosen larger than the guessed bandwidth of the identified system, see [2] for more detailed discussion of the choice of the parameter λ.

If the noise is present in the model (5) the least squares SVF estimate will lead to the biased results. Typical solution is the use of the Instrumental variable method, where the outputs $y_f^{(k)}$ are replaced by the instruments $z_f^{(k)}$, see [7], [4].

IV. THE LS METHOD WITH NUMERICAL INTEGRATION

This section introduces an alternative approach to SVD filters, attempting to obtain least-squares estimates of the continuous-time model parameters. This methodology is applied to the identification problem represented by the following differential equation

$$y^{(1)}(t) + a_1 y(t) = b_0 u(t). \tag{18}$$

Let us suppose that the input $u(t)$ forms the sequence generated with the period $T_k \equiv t_k - t_{k-1}$, where $t_k > t_{k-1}$. The key task is to transform the sampled system (18) into the regression model (11). This transformation can be made by integrating (18) over the time interval $[t_{k-1}, t_k]$, which yields

$$y(t_k) - y(t_{k-1}) + a_1 \int_{t_{k-1}}^{t_k} y(t)dt = b_0 \int_{t_{k-1}}^{t_k} u(t)dt. \tag{19}$$

Using the discrete character of $u(t)$ allows us to write

$$\int_{t_{k-1}}^{t_k} u(t)dt = T_k u(t_{k-1}). \tag{20}$$

Since the output $y(t)$ is outside the sampling time instants unknown, the required integral shall be approximated numerically. By invoking the trapezoidal rule (chapter 2.1 in [8]), one can obtain

$$\int_{t_{k-1}}^{t_k} y(t)dt \simeq \frac{T_k}{2}\left(\frac{y(t_{k-1})}{2} + y(t_{k-1/2}) + \frac{y(t_k)}{2}\right), \tag{21}$$

where $y(t_{k-1/2}) \equiv y(t_{k-1} + T_k/2)$.

Hence, the regression vector $\varphi(t_k)$ corresponding to $\theta = [a_1, b_0]^T$ of the sampled system (18) possesses the form

$$\varphi^T(t_k) = \Big[-\frac{T_k}{2}\Big(\frac{y(t_{k-1})}{2} + y(t_{k-1/2}) + \frac{y(t_k)}{2}\Big), \\ T_k u(t_{k-1})\Big]. \tag{22}$$

V. THE GENERALIZED LAGUERRE FUNCTIONS METHOD

The history of using the Laguerre orthonormal functions in system modeling and identification since their introduction in [9], [10] and [11] is rather long, with many papers documenting the differing theoretical approaches. In [12] the Laguerre functions were applied for the identification of the finite expansion of the transfer function. The approach in [12] was further developed in [13] with the use of the Kautz functions and in [14] with the generalized orthonormal basis functions. In [15] it was proved that nth order transfer function can be expanded into the ratio of two linear combinations of n Laguerre functions and the coefficients of these combinations were identified. This was an alternative approach to the transfer function approximated by a finite sum of orthonormal basis functions in [12], [13] and [14]. In this section we will present the method for identification of the dynamical systems based on the transform of their inputs and outputs instead of the expansion of the transfer functions. The inputs and outputs will be expanded into the generalized Laguerre functions basis.

Let us assume that the input and output signals are square-integrable in the Lebesgue sense, i.e.,

$$u(t), y(t) \in L_2[0, \infty). \tag{23}$$

Thus, we can expand the input and output signals into the generalized Laguerre function series $L_n^{(\alpha)}(t, p)$

$$L_n^{(\alpha)}(t,p) = \sqrt{\frac{2p\Gamma(n+1)}{\Gamma(n+\alpha+1)}} e^{-pt}(2pt)^{\alpha/2} l_n^{\alpha}(t,p), \tag{24}$$

with the time-scale parameter p and the generalization parameter α

$$u(t) = \sum_{n=0}^{\infty} U_n(\alpha_1, p_1) L_n^{(\alpha_1)}(t, p_1), \tag{25}$$

$$y(t) = \sum_{n=0}^{\infty} Y_n(\alpha_2, p_2) L_n^{(\alpha_2)}(t, p_2). \tag{26}$$

The generalized Laguerre functions are orthonormal in the $[0, \infty)$, and therefore the coefficients $U_n(\alpha_1, p_1), Y_n(\alpha_2, p_2)$ can be expressed as

$$U_n(\alpha_1, p_1) = \int_0^{\infty} u(t) L_n^{(\alpha_1)}(t, p_1) dt, \tag{27}$$

$$Y_n(\alpha_2, p_2) = \int_0^{\infty} y(t) L_n^{(\alpha_2)}(t, p_2) dt. \tag{28}$$

For the Laplace images of the input and output of the system (2), we have the equation

$$\mathcal{L}\{y(t)\} = \frac{B(s)}{A(s)} \mathcal{L}\{u(t)\}, \tag{29}$$

where $\mathcal{L}$ is the symbol of the Laplace transform. With additional computation we will get

$$\sum_{n=0}^{\infty} Y_n(\alpha_2, p_2) \mathcal{L}\{L_n^{(\alpha_2)}(t, p_2)\} = = \frac{B(s)}{A(s)} \sum_{n=0}^{\infty} U_n(\alpha_1, p_1) \mathcal{L}\{L_n^{(\alpha_1)}(t, p_1)\}. \tag{30}$$

For the Laplace transform of the generalized Laguerre function, we have the identity

$$\mathcal{L}\{L_n^{(\alpha)}(t, p)\} = \Phi(n, \alpha, p) \frac{P_n^{(\alpha)}(s)}{(s + p)^{n+1+\alpha/2}}, \tag{31}$$

$$\Phi(n, \alpha, p) = \sqrt{(2p)^{\alpha+1} \Gamma(n+1) \Gamma(n+\alpha+1)}, \tag{32}$$

$$P_n^{(\alpha)}(s) = \sum_{m=0}^{n} A_{n,m}^{(\alpha)} \sum_{i=0}^{n-m} \binom{n-m}{i} s^{n-m-i} p^i, \tag{33}$$

$$A_{n,m}^{(\alpha)} = \frac{(-1)^m (2p)^m \Gamma(m + \alpha/2 + 1)}{m!(n-m)! \Gamma(m + \alpha + 1)}. \tag{34}$$

The above expression for the Laplace transform of the generalized Laguerre functions is another form of the identity derived in [16]. By additional editing of the previously introduced formula (30) we obtain

$$\sum_{n=0}^{\infty} Y_n(\alpha_2, p_2) \Phi(n, \alpha_2, p_2) \frac{P_n^{(\alpha_2)}(s)}{(s + p_2)^{n+1+\alpha_2/2}} = = \frac{B(s)}{A(s)} \sum_{n=0}^{\infty} U_n(\alpha_1, p_1) \Phi(n, \alpha_1, p_1) \frac{P_n^{(\alpha_1)}(s)}{(s + p_1)^{n+1+\alpha_1/2}}. \tag{35}$$

This expression comprises only the powers of s; thus, it is sufficient to equate the coefficients of the same powers of s in order to get the coefficients of the polynomials $A(s), B(s)$, namely to obtain the unknown transfer function $F(s)$ as a fraction of two polynomials from the time progression of the input and output signals.

For practical computation, we can obtain only a finite number of terms N_1, N_2 in the Fourier expansion series for the input and output signals:

$$u_{N_1}(t) = \sum_{n=0}^{N_1} U_n(\alpha_1, p_1) L_n^{(\alpha_1)}(t, p_1), \tag{36}$$

$$y_{N_2}(t) = \sum_{n=0}^{N_2} Y_n(\alpha_2, p_2) L_n^{(\alpha_2)}(t, p_2). \tag{37}$$

It is also possible to measure the input and output signals only for the finite time T:

$$U_n^T(\alpha_1, p_1) = \int_0^{T} u(t) L_n^{(\alpha_1)}(t, p_1) dt, \tag{38}$$

$$Y_n^T(\alpha_2, p_2) = \int_0^{T} y(t) L_n^{(\alpha_2)}(t, p_2) dt. \tag{39}$$

We can write the above equation (35) in the form

$$\sum_{n=0}^{N_2} Y_n^T(\alpha_2, p_2) \Phi(n, \alpha_2, p_2) \frac{P_n^{(\alpha_2)}(s)}{(s + p_2)^{n+1+\alpha_2/2}} \approx \approx \frac{B(s)}{A(s)} \sum_{n=0}^{N_1} U_n^T(\alpha_1, p_1) \Phi(n, \alpha_1, p_1) \frac{P_n^{(\alpha_1)}(s)}{(s + p_1)^{n+1+\alpha_1/2}}. \tag{40}$$

After multiplying both sides of the equation by the term

$$(s + p_2)^{N_2+1+\alpha_2/2} (s + p_1)^{N_1+1+\alpha_1/2} \tag{41}$$

and performing some computation, we obtain the following approximation of the transfer function:

$$\tilde{F}(s, N_1, N_2, T) \approx \frac{(s + p_1)^{N_1+1+\alpha_1/2}}{(s + p_2)^{N_2+1+\alpha_2/2}} * * \frac{\sum_{n=0}^{N_2} Y_n^T(\alpha_2, p_2) \Phi(n, \alpha_2, p_2) P_n^{(\alpha_2)}(s) (s + p_2)^{N_2 - n}}{\sum_{n=0}^{N_1} U_n^T(\alpha_1, p_1) \Phi(n, \alpha_1, p_1) P_n^{(\alpha_1)}(s) (s + p_1)^{N_1 - n}}. \tag{42}$$

The following limit holds

$$\lim_{N_1 \to \infty} \lim_{N_2 \to \infty} \lim_{T \to \infty} \tilde{F}(s, N_1, N_2, T) = F(s). \tag{43}$$

The order of the above-approximated system (42) is

$$N = N_1 + N_2 + 1 + \max(\alpha_1/2, \alpha_2/2). \tag{44}$$

The quality of the approximation depends on the numbers N_1, N_2 in the truncated expansions of the input and output signals, on the time scale parameters p_1, p_2, and on the choice of the generalization parameters α_1 and α_2. The difference between the order of the numerator and the denominator in the transfer function approximation (42) is $\frac{\alpha_2 - \alpha_1}{2}$; this can lead us to the non-integer transfer function approximation when the difference $\alpha_2 - \alpha_1$ is not an even integer. Study [16] presents the relationship between the choice of the optimal parameter α and the time scale parameter p in the case of approximating the given signal $x(t) \in L_2[0, \infty)$ by the truncated series of the GLF.

$$x_N(t) = \sum_{n=0}^{N} X_n L_n^{(\alpha)}(t, p) \tag{45}$$

At this point, let us define the following moments:

$$m_{-1} = (x(t), \frac{1}{t}x(t)), \qquad m_0 = (x(t), x(t)), \tag{46}$$

$$m_1 = (x(t), tx(t)), \qquad m_2 = (x'(t), tx'(t)). \tag{47}$$

The optimal parameter p with the given generalization parameter α for truncated expansion of $x(t)$ to the GLF can be computed as follows:

$$p(\alpha) = \frac{\sqrt{\frac{\alpha^2 m_{-1} + 4m_2}{m_1}}}{2} \tag{48}$$

The optimal parameters p, α for truncated expansion of $x(t)$ to GLF can be computed

$$p = \sqrt{\frac{m_{-1}m_2}{|m_1 m_{-1} - m_0^2|}} \tag{49}$$

$$\alpha = \frac{m_0}{m_{-1}} 2p \tag{50}$$

This choice of parameters minimizes the ISE $\zeta(\alpha, p)$:

$$\zeta(\alpha, p) = (e(t), e(t)) = \sum_{n=N+1}^{\infty} X_n^2(\alpha, p), \tag{51}$$

$$e(t) = x(t) - x_N(t). \tag{52}$$

In paper [17], the above results are generalized for more classes of orthogonal functions; in this connection, it should also be mentioned that the result (48) for the SLF ($\alpha = 0$) was first derived in [18]. For simplification purposes, we used the nearest even approximation of the computed α in order not to deal with the fractional order system approximation of (42) in the example below.

The number of dominant Hankel singular values of system (42) can help us to find the order of the original system and to appropriately choose the order K of the reduced system. The balanced truncation of the system based on omitting the part of the system corresponding to the $N - K$ smallest Hankel singular values in the Singular value decomposition of the approximated systems will be used in the next section. More detailed description of the procedure is available in, for example, paper [19]. The MATLAB implementation "balred" of this approximation will be used in the next chapter.

VI. Example Of System Identification

The experiments with the system identification were done in MATLAB. The first order dynamical system with one real pole was chosen for comparison.

$$F(p) = \frac{5}{p+4}. \tag{53}$$

The input signal for the identification was chosen as $u(t) = e^{-t}$. Sampling periods were chosen as $Ts = 0.1s, 0.05s, 0.01s$. The step input responses of the reduced approximated systems of the order $K = 2$ (reduced from order $N = 8$, see (44)) with the GLF method, with LS method with SVF (LSSVF) given by $L(p) = \frac{1}{(p+\lambda)^n}$ and with LS method with trapezodial integration (LSTRAPZ) are presented in the Fig. 1 with sampling period $Ts = 0.1s$. The corresponding relative RMS errors (rRMSE) are shown

$$rRMSE = \frac{RMS(y(t) - \hat{y}(t))}{RMS(y(t))} * 100\%, \tag{54}$$

where $y(t)$ is step input response of the original system and $\hat{y}(t)$ is step input response of the approximated systems. The number n in the definition of state variable filter $L(p)$ is given by the order of the original system, λ is chosen larger than the guessed bandwidth, see [2] for more detailed discussion of the choice of the parameter λ in LSSVF. The graph of the relative approximation errors

$$\frac{y(t_k) - \hat{y}(t_k)}{y(t_k)} \tag{55}$$

is displayed in the Fig. 2 with sampling period $Ts = 0.1s$. The comparison of relative RMS errors for different sampling periods is in the TABLE I.

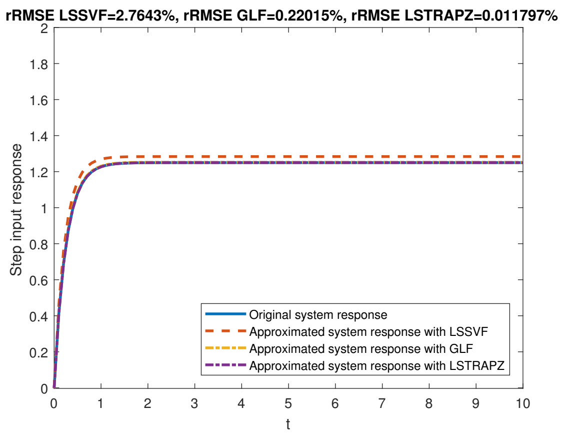

Fig. 1. Step input response of the approximated systems, $Ts = 0.1s$

VII. Conclusion

New method for continuous-time system identification from sampled data based on least squares estimation with numerical solution of differential equation was introduced. It was compared with traditional least squares method with state

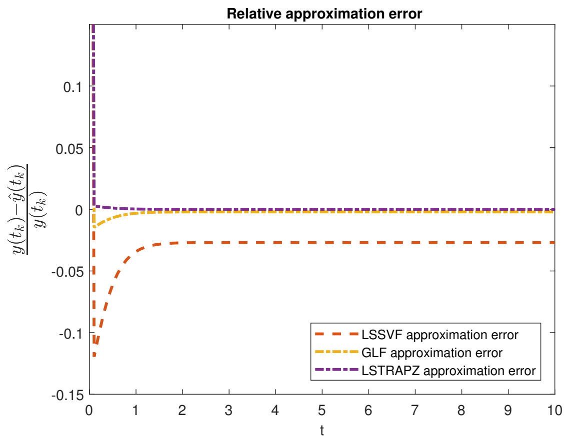

Fig. 2. Relative approximation error of the approximated systems, $Ts = 0.1s$

TABLE I
RELATIVE RMS ERRORS FOR DIFFERENT METHODS OF IDENTIFICATION FOR DIFFERENT SAMPLING PERIODS

$rRMSE$	LSSVF method	GLF method	LSTRAPZ method
$Ts = 0.1s$	2.7643%	0.22015%	0.011797%
$Ts = 0.05s$	1.8955%	0.066312%	0.0029554%
$Ts = 0.01s$	0.022859%	0.015304%	0.00011829%

variable filters and with the method based on expansion of the input and output signals into generalized Laguerre functions. It was shown that we can obtain better results in terms of relative RMS error with proposed LSTRAPZ method than with LSSVF and GLF methods for the identification of the first order dynamical system. In the future work we will take a look on the generalization of the least squares with numerical integration for the systems of higher orders.

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