Stability and Mean Consistent of Fourier Solutions of Nonlinear Stochastic Heat Equations in 1D

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***Abstract*—Abstract. The main focus of this article is studying the stability of solutions of nonlinear stochastic heat equation and give conclusions in two cases: stability in probability and almost sure exponential stability. Also, We prove that the Fourier coefficient solution is mean consistent. The main tool is the study of related Lyapunov-type functionals. The analysis is carried out by a natural $N-$dimensional truncation in isometric Hilbert spaces and uniform estimation of moments with respect to N.**

I. Introduction

In this article we study the stability of solutions of semi-linear stochastic heat equations

$$u_t = \sigma^2 \triangle u + A(u) + B(u)\frac{dW}{dt}$$

with cubic nonlinearities $A(u)$ in one dimensions in terms of all systems parameters, i.e., with non-global Lipschitz continuous nonlinearities. Our study focusses on stability of analytic solution $u = u(x,t)$ under the geometric condition

$$\sigma^2 \frac{\pi^2}{l^2} - a_1 = \frac{\sigma^2 \pi^2 l^2 - a_1 l^2}{l^2} := \gamma > 0,$$

where $0 \le x \le l$ such that $\mathbb{D} = [0, l]$.

Many authors have treated stochastic heat equations (e.g. Chow [2] and DaPrato and Zabzcyk [4]), semi-linear stochastic heat equations (e.g. Chow [2], DaPrato and Zabzcyk [4], and Schurz [19]) or nonlinear stochastic evolution equations (e.g. Grecksch and Tudor [8] and Schurz [18]). Also, some authors study the stability of stochastic heat equations like Fournier and Printems [5] study the stability of the mild solution. Walsh [22] treats the stochastic heat equations in one dimension. Chow [2] studies that the null solution of the stochastic heat equation is stable in probability by using the definition. Recall that

$$\mathbb{L}^2(\mathbb{D}) = \{f : \mathbb{D} \to \mathbb{R} \,|\, \int_{\mathbb{D}} |f(x)|^2 \, d\mu(x) < \infty\},$$

where μ is the Lebesgue measure in one dimensions.

The paper is organized as follows. Section II states that the strong Fourier solution of equation (18) is proved. We write the solution using the finite-dimensional truncated system verifies properties of finite-dimensional Lyapunov functional. Section III discusses the stability of the strong solution of equation (18) is stable in probability and almost sure exponential stability. Eventually, Section IV summarizes the most important conclusions on the well-posedness and behaviour of the original infinite-dimensional system (18).

II. Truncated Fourier Series Solution and Finite-dimensional Lyapunov Functional

Consider the stochastic nonlinear heat equation with additive noise

$$u_t = \sigma^2 u_{xx} + (a_1 - a_2 \|u\|^2_{\mathbb{L}^2(\mathbb{D})})u + b\frac{dW}{dt} \tag{1}$$

with the initial condition $u(x, 0) = f(x)$ with $f \in \mathbb{L}^2(\mathbb{D})$ (initial position) and

$$W(x,t) = \sum_{n=1}^{\infty} \alpha_n W_n(t) e_n(x)$$

and $e_n = \sqrt{\frac{2}{l}} sin(\frac{n\pi\, x}{l})$ driven by $i.i.d.$ standard Wiener processes W_n with $\mathbb{E}[W_n(t)] = 0$, $\mathbb{E}[W_n(t)]^2 = t$. The solution of equation (18) in terms of Fourier series is proved by Schurz [19] and given by

$$u(x,t) = \sum_{n=1}^{\infty} c_n(t)\, e_n(x). \tag{2}$$

Theorem 1: Assume that $\sum_{n,m=1}^{\infty} \alpha_n^2 < \infty$, $\forall\, u \in \mathbb{L}^2(\mathbb{D}) \cap C^1(\mathbb{D} \times \mathbb{R}^1_+)$ with $u_x \in \mathbb{L}^2(\mathbb{D})$ and $W(x,t) = \sum_{n,m=1}^{\infty} \alpha_n W_n(t) e_n(x)$, then for all $t \ge 0$, $x \in \mathbb{D} = (0, l_x)$, the Fourier-series solutions (2) have Fourier coefficients c_n satisfying (a.s.)

$$\frac{d}{dt}c_n(t) = \left[-\sigma^2 \frac{\pi^2 n^2}{l^2} + a_1 - a_2 \sum_{k=1}^{\infty}(c_k)^2\right] c_n + b\alpha_n \frac{dW_n}{dt}. \tag{3}$$

Proof 1: See Schurz [19].

We need to truncate the infinite series (2) for practical computations. So, we have to consider finite-dimensional truncations of the form

$$u_N(x,t) = \sum_{n=1}^{N} c_n\, e_n \tag{4}$$

with Fourier coefficients c_n satisfying the naturally truncated system of stochastic differential equations (SDEs).

$$\frac{d}{dt}c_n(t) = \left[-\sigma^2 \frac{\pi^2 n^2}{l^2} + a_1 - a_2 \sum_{k=1}^{\infty}(c_k)^2\right] c_n + b\alpha_n \frac{dW_n}{dt}. \tag{5}$$

where $\lambda_n = (\frac{n\pi}{l})^2$.

Assume that $\sigma^2 \frac{\pi^2}{l^2} > a_1$. Define the Lyapunov functional V_N as follows

$$\begin{aligned} V_N(c) =& V_N((c_n)_{n=1,\ldots,N}) \\ :=& \sum_{n=1}^{N}(\sigma^2 \lambda_n - a_1)(c_n)^2 + \frac{a_2}{2}\left(\sum_{n=1}^{N}(c_n)^2\right)^2 \end{aligned} \tag{6}$$

for $N \in \mathbb{N}$.

This functional is a modification of a functional appeared in Schurz [20]. It is clear that this function is of Lyapunov-type because it is nonnegative and smooth as long as $a_2 \geq 0$, radially unbounded if additionally $\sigma^2 \pi^2 > a_1 l^2$. Equipped with Euclidean norm

$$|c|_{l^2_N} = \sqrt{\sum_{n=1}^{N} c_n^2}$$

Lemma 1: Consider the Lyapunov functional defined in equation (6), and let

$$\sigma^2 \frac{\pi^2}{l^2} - a_1 = \frac{\sigma^2 \pi^2 - a_1 l^2}{l^2} =: \gamma > 0.$$

Then $\forall\, u \in \mathbb{L}^2(\mathbb{D})$:

$$V_N(u) \geq \gamma \|u\|^2_{\mathbb{L}^2(\mathbb{D})} \tag{7}$$

Proof 2: See [9].

Lemma 2: Assume that $a_2 \geq 0$. *Then*, $\forall\, N \in \mathbb{N}$, the functional V_N is

(a) *nonnegative and positive semi-definite* if $\sigma^2\pi^2 \geq a_1 l^2$ or $a_2 \geq 0$,

(b) *positive-definite* if $\sigma^2\pi^2 > a_1 l^2$,

and

(c) satisfies the condition of radial unboundedness

$$\lim_{|c|_{l^2_N} \to +\infty} V_N(c) = +\infty, \text{ if } [\sigma^2\pi^2 - a_1 l^2]_+ + a_2 > 0.$$

Proof 3: See [9].

III. Stability of Fourier Solutions

Recall equation (5) governed by

$$\frac{d}{dt} c_n(t) = \Big[- \sigma^2 \frac{\pi^2 n^2}{l^2} + a_1 - a_2 \sum_{k=1}^{\infty} (c_k)^2 \Big] c_n + b\alpha_n \frac{dW_n}{dt} \tag{8}$$

$$= \left[-\sigma^2 \lambda_n + a_1 - a_2 \|u_N\|^2 \right] c_n + b\alpha_n \frac{dW_n}{dt}. \tag{9}$$

To simplify, let

$$f(u_N) = -\sigma^2 \lambda_n + a_1 - a_2 \|u_N\|^2$$

and

$$g(u_N) = b\alpha_n$$

Definition 1: The trivial solution of system (8) (in terms of norm $\|u\|_{\mathbb{L}^2(\mathbb{D})}$) is said to be *stochastically stable* or *stable in probability*, if for $0 < \epsilon < 1$ and $r > 0$, $\exists$ a $\delta = \delta(\epsilon, r)$ such that, $\forall\, t \geq \delta$, we have

$$\mathbb{P}\Big\{ \|u(t)\|_{\mathbb{L}^2(\mathbb{D})} < r \Big\} \geq 1 - \epsilon. \tag{10}$$

whenever $\delta > 0$.

Lemma 3: If $\exists$ a positive-definite function $V \in \mathcal{C}^{2,1}(\mathbb{R}^d \times [0, \infty), \mathbb{R}_+)$ such that $\mathcal{L}V(x, t) \leq 0$ and $\forall\, (x, t) \in \mathbb{R}^d \times [0, \infty)$, then the trivial solution of the equation

$$dX(t) = f(x(t), t)\, dt + g(x(t), t)\, dW(t) \tag{11}$$

is *stochastically stable.*

Proof 4: See Arnold [1].

Theorem 2: Let

$$V(u(t)) = \sigma^2 \|\nabla u\|^2_{\mathbb{L}^2(\mathbb{D})} - a_1 \|u\|^2_{\mathbb{L}^2(\mathbb{D})} + \frac{a_2}{2} \|u\|^4_{\mathbb{L}^2(\mathbb{D})}.$$

If $(1 - a_2 \sum_{n=1}^{N} c_n^2) \sum_{n=1}^{N} c_n > 0$, then the trivial solution of equation (8) is stochastically stable i.e., stable in probability.

Proof 5: From Lemma 2, we know that $V_N(u(t))$ is positive-definite if $\forall\, n \in \mathbb{N}$, $\sigma^2 \lambda_n - a_1 > 0$. Define the linear operator $\mathcal{L}$ as in Schurz [19]

$$\mathcal{L} = \sum_{n=1}^{N} \left[-\sigma^2 \frac{\pi^2 n^2}{l^2} + a_1 - a_2 \sum_{k=1}^{N} (c_k)^2 \right] c_n \frac{\partial}{\partial c_n} + \frac{b^2}{2} \sum_{n=1}^{N} \alpha_n^2 \frac{\partial^2}{\partial c_n^2}.$$

The first and second partial derivative of $V_N(t)$ with respect to c_n are

$$\frac{\partial V_N}{\partial c_n} = 2 \sum_{n=1}^{N} \left[(\sigma^2 \lambda_n - a_1) + a_2 \left(\sum_{n=1}^{N} c_n^2 \right) \right] c_n$$

and

$$\frac{\partial^2 V_N}{\partial c_n^2} = 2 \sum_{n=1}^{N} (\sigma^2 \lambda_n - a_1)\, c_n + 4a_2 \left(\sum_{n=1}^{N} c_n \right)^2 + 2a_2 \sum_{n=1}^{N} c_n^2.$$

Then

$$\begin{aligned} \mathcal{L}V_N(c_n(t)) = & -2 \left(\sum_{n=1}^{N} (\sigma^2 \lambda_n - a_1)\, c_n \right)^2 \\ & - 2a_2 \sum_{n=1}^{N} (\sigma^2 \lambda_n - a_1) \left(\sum_{n=1}^{N} c_n^2 \right)^2 \\ & - 2a_2 \sum_{n=1}^{N} c_n^2 \left(1 - a_2 \sum_{n=1}^{N} c_n^2 \right) \sum_{n=1}^{N} \left[(\sigma^2 \lambda_n - a_1) \right] c_n. \end{aligned}$$

But by our assumption that

$$(1 - a_2 \sum_{n=1}^{N} c_n^2) \sum_{n=1}^{N} c_n > 0,$$

then Thus

$$\mathcal{L}V_N(c_n(t)) \leq 0.$$

So by Lemma 3, applied to truncation of (8), the trivial solution of system (8) is *stochastically stable.*

corollary 1: Let $p \geq 2$ and let V be as above. Imposing the same assumptions as in Theorem 2 with $N \to +\infty$, then we have $\forall\, 0 \leq t \leq T$,

$$\mathbb{E}\|u(t)\|^p_{\mathbb{L}^2(\mathbb{D})} \leq \frac{1}{\min(1, \gamma)} \mathbb{E}V^{\frac{p}{2}}(u(0)).$$

Proof 6: We know, from the definition of $V(u)$, and Lemma 1 that $\|u\|^2_{\mathbb{L}^2(\mathbb{D})} \leq \frac{V(u)}{\gamma}$. it is easy to show that

$$\mathbb{E}\|u(t)\|^p_{\mathbb{L}^2(\mathbb{D})} \leq \frac{1}{\min(1, \gamma)} \mathbb{E}V^{\frac{p}{2}}(u(0)).$$

corollary 2: $\forall\, p \geq 2$ and $\forall\, 0 \leq t \leq T$, with $\sigma^2 \lambda_1 - a_1 > 0$, we have $\forall\, 0 \leq t \leq T$.

1) If $a_2 \geq 0$, then

$$\mathbb{E}\|u(t)\|^p_{\mathbb{L}^2(\mathbb{D})} \leq \frac{\mathbb{E}V^{\frac{p}{2}}(u(0))}{\left[\sigma^2 \lambda_1 - a_1 \right]^{\frac{p}{2}}}.$$

2) If $a_2 > 0$, then

$$\mathbb{E}\|u(t)\|^p_{\mathbb{L}^2(\mathbb{D})} \leq \left(\frac{2}{a_2} \right)^{\frac{p}{4}} \mathbb{E}V^{\frac{p}{4}}(u(0)).$$

Proof 7: 1) Note that we have $\Big(\sigma^2 \lambda_1 - a_1 \Big) \|u(., t)\|^2_{\mathbb{L}^2(\mathbb{D})} \leq V_N(u(t))$. Since λ_n is increasing in n,

$$\left[\sigma^2 \lambda_1 - a_1 \right] \|u(t)\|^2_{\mathbb{L}^2(\mathbb{D})} \leq V_N(u(t)).$$

So,

$$\|u(t)\|^2_{\mathbb{L}^2(\mathbb{D})} \le \frac{V_N(u(t))}{\sigma^2\lambda_1 - a_1}$$

Pull over expectation, then

$$\mathbb{E}\|u(t)\|^2_{\mathbb{L}^2(\mathbb{D})} \le \frac{\mathbb{E}V(u(t))}{\sigma^2\lambda_1 - a_1}.$$

By using Corollary 1, we have

$$\mathbb{E}\|u(t)\|^2_{\mathbb{L}^2(\mathbb{D})} \le \frac{\mathbb{E}V(u(0))}{\sigma^2\lambda_1 - a_1}.$$

2) From the definition of $V(u(t))$, it is clear that $\frac{a_2}{2}\|u(t)\|^4_{\mathbb{L}^2(\mathbb{D})} \le V(u(t))$, so

$$\left(\frac{a_2}{2}\right)\|u(t)\|^4_{\mathbb{L}^2(\mathbb{D})} \le V_N(u(0)).$$

Now, take the expectation to both sides, and we get

$$\left(\frac{a_2}{2}\right)\mathbb{E}\|u(t)\|^4_{\mathbb{L}^2(\mathbb{D})} \le \mathbb{E}V_N(u(0)),$$

i.e., $\forall\, 0 \le t \le T$,

$$\mathbb{E}\|u(t)\|^2_{\mathbb{L}^2(\mathbb{D})} \le \left(\frac{2}{a_2}\right)^{\frac{1}{2}}\mathbb{E}V^{\frac{1}{2}}(u(0)).$$

Remark: The corollary 2 means that $\forall\, t \ge 0$:

$$\mathbb{E}\|u(t)\|^2_{\mathbb{L}^2(\mathbb{D})} \le \min\Big\{\frac{\mathbb{E}V_N(u(0))}{\left[\sigma^2\lambda_1 - a_1\right]}, \left(\frac{2}{a_2}\right)^{\frac{1}{2}}\mathbb{E}V^{\frac{1}{2}}(u(0))\Big\}.$$

Definition 2: The trivial solution of system (8) is said to be $a.s.$ *exponentially stable* if

$$\theta(u_N) := \limsup_{t\to\infty}\frac{1}{t}\log\|u(t)\|_{\mathbb{L}^2(\mathbb{D})} < 0 \qquad (a.s.) \tag{12}$$

$\forall\, u(0) \in \mathbb{D}$. The quantity of the left hand side of (12) is called *the sample top Lyapunov exponent of u.*

Lemma 4: [Øksendal] [16]. Let $v(t)$ be a nonnegative integrable function such that

$$v(t) \le C + A\int_0^t v(s)\,ds, \qquad 0 \le t \le T \tag{13}$$

for some constants C, A. Then $C \ge 0$ and

$$v(t) \le C\exp(A\,t), \qquad 0 \le t \le T. \tag{14}$$

Theorem 3: Let $V(u(t))$ as in Theorem 2. If $(1 - a_2\sum_{n=1}^{N} c_n^2)\sum_{n=1}^{N} c_n > 0$, then the norm of the trivial solution of $N-$dimensional system (8)is *a.s. exponentially stable* with sample top Lyapunov exponent

$$\theta(u_N) \le 0.$$

Proof 8: Return to the analysis of finite $N-$dimensional equation (5). Recall that

$$\begin{aligned} V(u_N(t)) &= \sigma^2\|\nabla u_N\|^2_{\mathbb{L}^2(\mathbb{D})} - a_1\|u_N\|^2_{\mathbb{L}^2(\mathbb{D})} + \frac{a_2}{2}\|u_N\|^4_{\mathbb{L}^2(\mathbb{D})} \\ &= \sum_{n=1}^{N}\left[\sigma^2\lambda_n - a_1\right](c_n(t))^2 + \frac{a_2}{2}\Big(\sum_{n=1}^{N}(c_n(t))^2\Big)^2 \end{aligned}$$

where $V(u_N) = V_N(c)$ and from Theorem 2 we know that

$$\begin{aligned} \mathcal{L}V_N(c_n(t)) = &-2\left(\sum_{n=1}^{N}\left(\sigma^2\lambda_n - a_1\right)c_n\right)^2 \\ &- 2a_2\sum_{n=1}^{N}\left(\sigma^2\lambda_n - a_1\right)\left(\sum_{n=1}^{N} c_n^2\right)^2 \\ &- 2a_2\sum_{n=1}^{N} c_n^2\left(1 - a_2\sum_{n=1}^{N} c_n^2\right)\sum_{n=1}^{N}\left[(\sigma^2\lambda_n - a_1)\right]c_n. \end{aligned}$$

But by our assumption that

$$(1 - a_2\sum_{n=1}^{N} c_n^2)\sum_{n=1}^{N} c_n > 0,$$

so

$$\mathcal{L}V_N(c_n(t)) \le -k,$$

where $k \ge 0$.

Using Dynkin's formula, we find that

$$\begin{aligned} \mathbb{E}V_N(c(t)) &= \mathbb{E}V_N(c(0)) + \mathbb{E}\int_0^t \mathcal{L}V_N(c_n(s))ds \\ &\le \mathbb{E}V_N(c,v)(0) - k\,t \end{aligned}$$

so

$$\mathbb{E}\|u_N(t)\|^2_{\mathbb{L}^2(\mathbb{D})} \le \mathbb{E}V_N(c(t)) \le \mathbb{E}V_N(c(0)) - k\,t$$

using extended Gronwall lemma, Lemma 4, gives us

$$\mathbb{E}\|u_N(t)\|^2_{\mathbb{L}^2(\mathbb{D})} \le \mathbb{E}V_N(c(0))\,e^{-k\,t}$$

hence

$$\log\mathbb{E}\|u_N(t)\|^2_{\mathbb{L}^2(\mathbb{D})} \le \log\mathbb{E}V_N(c(0)) - k\,t$$

thus

$$\begin{aligned} \limsup_{t\to\infty}\frac{log\mathbb{E}\|u_N(t)\|^2_{\mathbb{L}^2(\mathbb{D})}}{t} &\le \limsup_{t\to\infty}\frac{log\mathbb{E}V_N(c(0))}{t} - k \\ &\le b_2^2\,\|\alpha\|^2_{l^2_{4N\times N}} - 2\kappa \le -k. \end{aligned} \tag{15}$$

If $(1 - a_2\sum_{n=1}^{N} c_n^2)\sum_{n=1}^{N} c_n > 0$ then the left side of identity (12) is negative and the trivial solution of the velocity v of $N-$dimensional system (8) is $a.s.$ exponential stable.

Finally, we observe that all the previous estimates are uniformly bounded as $N \to \infty$. Hence, we arrive at

$$\limsup_{t\to\infty}\frac{log\mathbb{E}\|u(t)\|^2_{\mathbb{L}^2(\mathbb{D})}}{t} < 0. \tag{16}$$

corollary 3: Let $V(u(t))$ as in Theorem 2. If $\left(1 - a_2\sum_{n=1}^{N} c_n^2\right)$, then the norm of the v-component of the trivial solution of infinite-dimensional system (18) is *a.s. exponentially stable* with sample top Lyapunov exponent

$$\theta(v) \le -k < 0.$$

Proof 9: Return to the proof of previous theorem, theorem 3 and take the limit N to $+\infty$ after the estimation process (16) in the sample Lyapunov exponent $\theta(v_N)$.

IV. Explicit Representation

Define the linear-implicit Euler method as the following:

$$c_n(t_{k+1}) = c_n(t_k) + h_k\, f_n(c(t_k))c_n(t_{k+1}) + b\alpha_n \triangle_k W_n \quad (17)$$

where $f_n(c(t_k)) = -\sigma^2 \frac{n^2\pi^2}{l^2} + a_1 - a_2 \sum_{n=1}^{N} c_n^2$.

Theorem 4: Assume that $\left|c_{n,m}^{i,j}(t_k)\right| < \infty$, and

$$a_2 \geq 0,\, and \forall\, n \in \mathbb{N} : \Big[-\sigma^2 \left(\lambda_n + \beta_m\right) + a_1 \Big] h_k^2 < 1.$$

Then the method (LIEM) governed by equation (17) has the non-exploding explicit representation

$$c_n(t_{k+1}) = \frac{c_n(t_k) + b\alpha_n \triangle_k Wn}{1 - h_k\, f_n(c(t_k))} \quad (18)$$

and where $f_n(c(t_k))$ as above, $\triangle_k W_{n,m}^{i,j} = W_n(t_{k+1}) - W_n(t_k)$ and $h_k = t_{k+1} - t_k$.

Proof 10: See Schurz [19].

V. Local Mean Consistency On Numerical Solutions

To do numerical solution of the Fourier coefficient of the heat equation, it is better to start with mean consistent. In [11], we proved that the solution of the nonlinear stochastic wave equation is mean consistent. Also, in [12], we proved that solution is mean square consistent.

Definition 3: A numerical approximation $\hat{c_n}$ of the $n-$th Fourier coefficient c_n along the partition of $[0,T]$ is said be **mean consistent** with rate $r_0 > 0\, on\, \mathbb{D}\, iff\, \exists\, K_0^c =$ consistency constant $\forall\, 0 \leq t \leq t+h \leq T$, where h is sufficiently small, i.e., $0 < h \leq \delta \leq 1$, and $\exists$ a positive function $V(c(t))$ such that

$$\|\mathbb{E}\left(c(t+h|t,\mathcal{F}_t) - \hat{c}(t+h|t,\mathcal{F}_t)\right)\|_H \leq K_0^c\, \left(V(c(t))\right)\, h^{r_0}$$

provided $\hat{c} = (c_n)_{n=1,\dots,N}\ \forall\, u(t) \in H$, where $H := \left\{u \in \mathbb{L}^2(\mathbb{D})\right\}$, and $\|u\|_H = \|c\|^2_{\mathbb{L}^2(\mathbb{D})} : (\mathcal{F}_t, \mathcal{B}(\mathbb{L}^2))-$measurable, then

$$\|\mathbb{E}\left(c(t+h|t,\mathcal{F}_t) - \hat{c}(t+h|t,\mathcal{F}_t)\right)\|_H \leq K_0^c\, \left(V(c(t))\right)\, h^{r_0}$$

where $\hat{c}(t+h|t,c(t)) = \hat{c}(t) + \int_t^{t+h} f(c(s))\, ds + \int_t^{t+h} b\alpha_n\, dW(s)\, ds$.

Recall that, in the Fourier space $\|u\|^2_{\mathbb{L}^2(\mathbb{D})} = \|c\|^2_{l^2}$. Now,

$$\begin{aligned} c_n(t+h) &= \frac{c_n(t_k) + b\,\alpha_n \triangle_k W_n}{1 - h_k\, f_n(c(t_k))} \\ &= \frac{c_n(t_k) - h_k\, f_n(c(t_k))c_n(t_k) + h_k\, f_n(c(t_k))c_n(t_k)}{1 - h_k\, f_n(c(t_k))} \\ &+ \frac{b\,\alpha_n \triangle_k W_n}{1 - h_k\, f_n(c(t_k))} \\ &= \frac{\left[1 - h_k\, f_n(c(t_k))\right] c_n(t_k)}{1 - h_k\, f_n(c(t_k))} + h_k \frac{f_n(c(t_k))}{1 - h_k\, f_n(c(t_k))} \\ &+ \frac{b\,\alpha_n}{1 - h_k\, f_n(c(t_k))} \triangle_k W_n \end{aligned}$$

write the above equation

$$\hat{c}_n(t_k + h_k) = c_n(t_k) + h_k\, \hat{f}_n(c(t_k))c_n(t_k) + \hat{g}_n(c(t_k))\triangle_k W_n.$$

Since

$$f - \hat{f} = -h\frac{f^2}{1 - hf}$$

and

$$g - \hat{g} = -h\frac{b\alpha_n f}{1 - hf}$$

So,

$$\begin{aligned} c_(t_k + h_k) - \hat{c}_n(t_k + h_k) &= c_n(t_k) - \hat{c}_n(t_k) \\ &+ h_k \left[f_n(c(t_k)) - \hat{f}_n(c(t_k))\right] c_n(t_k) \\ &+ \left[g_n(c(t_k)) - \hat{g}_n(c(t_k))\right] \triangle_k W_n(t_k) \end{aligned}$$

but $c_n(t_k) = \hat{c}_n(t_k)$ and $\triangle_k W_n(t_k) = \int_{t_k}^{t_{k+1}} dW(s)$ and

$$\int_{t_k}^{t_{k+1}} \left[g_n(c(t_k)) - \hat{g}_n(c(t_k))\right] dW(s) = 0, \quad (martingale),$$

pulling the expectation, we get

$$\begin{aligned} &\|\mathbb{E}\left[c_n(t_k + h_k) - \hat{c}_n(t_k + h_k)\right]\|_N \\ &= h_k \|\mathbb{E}\left(\left[f_n(c(t_k)) - \hat{f}_n(c(t_k))\right] c_n(t_k)\right)\|_N \\ &= h_k^2 \left\|\mathbb{E}\left[\frac{f_n^2(c(t_k))}{1 - h_k\, f_n(c(t_k))}\, c_n(t_k)\right]\right\|_N \\ &= h_k^2 \left\|\mathbb{E}\left[\frac{(-\sigma^2\,\lambda_n + a_1)^2}{1 - h_k\, f_n(c(t_k))}\right]\right\|_N \\ &- 2\, a_2\, h_k^2 \|\mathbb{E}[\frac{(-\sigma^2\,\lambda_n + a_1)\sum_{n=1}^{N} c_n^2(t_k)}{1 - h_k\, f_n(c(t_k))}]\|_N \\ &+ h_k^2 a_2^2 \|\mathbb{E}[\frac{(\sum_{n=1}^{N} c_n^2(t_k))^2}{1 - h_k\, f_n(c(t_k))}]\|_N \\ &\leq K_0^c \left(V_N(c(t_k))\right) h_k^2. \end{aligned} \quad (19)$$

where

$$\begin{aligned} V_N &= (-\sigma^2\,\lambda_n + a_1)^2 \\ &+ 2a_2(\sigma^2\,\lambda_n - a_1)\|u\|^2_{\mathbb{L}^2(\mathbb{D})} + a_2^2\, \|u\|^4_{\mathbb{L}^2(\mathbb{D})}. \end{aligned} \quad (20)$$

$\lambda_n = \frac{n^2\,\phi^2}{l}$, and $\mathbb{D} = [0,l]$. Thus the approximation solution $\hat{c}$ is mean consistent with rate $r_0 = 2$.

VI. Conclusion

By analyzing appropriate N-dimensional truncations of the original semi-linear heat equations (18), we can verify the asymptotic stability of random Fourier series solutions with strongly unique, Markovian, continuous time Fourier coefficients under the presence of cubic nonlinearities. For this purpose, we introduced and studied an appropriate Lyapunov. The analysis is basicly relying on the fact that all estimations of moments of Lyapunov functional are made independent of dimensions N of their finite-dimensional truncations. Thus, the techniques of our proof are finite-dimensional in character, however the conclusions can be drawn to the original infinite-dimensional semi-linear equation. Also, in this article, we showed that the approximation solution is mean consistent with rate $r_0 = 2$.

References

[1] L. Arnold, *Stochastic Differential Equations: Theory and Applicants*, John Wiley and Sons, Inc., New York, 1974.

[2] P.L. Chow, *Stochastic Partial Differential Equations*. Chapman and Hall/CRC, Boca Raton, 2007.

[3] R. Dalang and N.E. Frangos, *The stochastic wave equation in two spatial dimensions.* Ann. Probab. *26* (1998), 187-212.

[4] G. Da Prato and J. Zabzcyk, *Stochastic Equations in Infinite dimensions*, Cambridge University Press, Cambridge, 1992.

[5] N. FOURNIER AND J. PRINTEMS, *Stability of the Stochastic Heat Equation in* $L^1([0,1])$, Elect. Comm. in Probab. 16 (2011), pp. 337–352.

[6] A. Friedman, *Stochastic Differential Equations and Applications*, Dover Publications, Inc. Mineola, New York, 1975.

[7] T. C. Gard, *Introduction to Stochastic Differential Equations*, Marcel Dekker, Basel, 1988.

[8] W. Grecksch and C. Tudor, *Stochastic Evolutions: A Hilbert Space Approach*, Akademie-Verlag, Berlin, 1995.

[9] Schurz, H., and Hazaimeh, H. M. (2014). Existence, uniqueness, and stability of stochastic wave equation with cubic nonlinearities in two dimensions. Journal of Mathematical Analysis and Applications, 418(2), 775-795.

[10] Hazaimeh, H. M. (2012). Stochastic Wave Equations With Cubic Nonlinearities in Two Dimensions.

[11] Hazaimeh, H. M. (2017, June). Local mean consistency on numerical solutions of stochastic wave equation with cubic nonlinearities on 2D rectangles. In AIP Conference Proceedings (Vol. 1836, No. 1, p. 020021). AIP Publishing.

[12] Hazaimeh, H. M. (2017). Mean Square Consistency On Numerical Solutions of Stochastic Wave Equation with Cubic Nonlinearities on 2D Rectangles. In MATEC Web of Conferences (Vol. 125, p. 05020). EDP Sciences.

[13] Desmond Higham, Mean?square and asymptotic stability of the stochastic theta method, SIAM J. Numer. Anal. , 38 (2000), 753?769

[14] Higham, D.J., Mao, X., Stuart, A.M.: Strong convergence of Euler-like methods for nonlinear stochastic differential equations. SIAM J. Numer. Anal. 40, 1041?1063 (2002)

[15] I. Karatzas and S.E. Schreve, *Brownian Motion and Stochastic Calculus*, Second Edition. Springer-Verlag, New York, 1991.

[16] B. *O*ksendal, *Stochastic Differential Equations: An Introduction With applications*, Six Edition, Springer-Verlag, Berlin, 2003.

[17] A. Millet and M. Sanz-Sol*é*, *A stochastic wave equation in two space dimension: Smootheness of the law,* Ann. Probab. *27* 1999, 803-844.

[18] H. SCHURZ, *Existence and uniqueness of solutions of semilinear stochastic infinite-dimensional differential systems with H-Regular noise*, J. Math. Anal. Appl. 332 (1) (2007), pp. 334–345.

[19] H. Schurz, Nonlinear Stochastic heat equations with cubic nonlinearity and Additive Q-regular Noise in $\mathbb{R}^1$, *Electronic Journal of Differential Equations*, Conf. 19 (2010), 221-233.

[20] H. Schurz, Stochastic wave equations with cubic nonlinearity and Q-regular Additive Noise in $\mathbb{R}^2$, *Discrete and Continuous Dynamical Systems*, Suppl. Vol. 2011 (2011), 1299-1308.

[21] A. N. Shiryaev, *Probability* (2nd edition), Springer-Verlag, Berlin, 1996.

[22] J. B. Walsh, *An introduction to stochastic partial differential equations*, École d'été de probabilités de Saint-Flour, XIV-1984, 265-439, Lecture Notes in Math. 1180, Springer-Verlag, Berlin, 1986.