Time-delay systems with application to mechanical engineering process dynamics and control

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Abstract— Time-delay systems with constant or variable delays can take the form of delay differential equations (DDEs) from a mathematical point of view. DDEs combine the continuous aspect of differential equations and sample features of difference equations. Such "mixed difference equations" go back to the astronomer's three-body problem of Condorcet in 1767. R.E. Bellman & K.L. Cooke published a seminal study on DDEs in the 1960's. Russian mathematician Komanovskii, Myshlis & Nosov (1999) developed the study of their stability and applications to the industrial problems. Mathematical software packages like Mathematica® and MATLAB® introduced solvers recently for DDEs with constant delays. DDEs are essential for modeling, forecasting, and simulation of complex real-life systems for which delays cannot be neglected. Time-delay systems are applied extensively in various domains of science and industry such as mechanical systems, electrical systems, industrial processes, and biological systems. Some delays are due to inherent properties of a system where the propagation of effects takes time. The necessity of feedback controls to stabilize a system also introduce some inevitable delays. This study introduces this modeling process and analysis using some real-life deterministic applications from mechanical engineering dynamics and control such as with the metal rolling system and machine tool chatter.

Keywords—Block diagram, delay differential equation, mechanical application, transcendental characteristic equation.

I. Introduction

THIS introduction is devoted to an initial general formulation of delay differential equations. A standard example illustrates the differences accounted between an ordinary differential equation (ODE) and a corresponding hybrid difference-differential equation (DDE). After that, we present a short history of DDEs in the literature. Finally, we mention the numerous scientific domains in which DDEs can be used such as in physics, engineering, chemistry, biology, economics for modeling complex real-life systems.

A. Formulation of Delay Differential Systems

We proposed here a general mathematical formulation. Delay differential equations belong to a class of functional differential equations between ordinary differential equations (ODEs) and partial differential equations (PDEs). More specifically, DDEs combine the continuous aspects of ODEs with sample features of difference equations with advanced or retarded delays. In the simple case of one state variable with constant delays, we can write

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$$\dot{x}(t) = f\left(t, x(t), x(t-\tau_1), \ldots, x(t-\tau_m)\right) \tag{1}$$

where $\dot{x}(t) = dx(t)/dt$. Variable delays in (1) should be such as $\tau_i = \phi_i\left(t, x(t)\right),\ i \in \{1, 2, \ldots, m\}$. In matrix form, we may have

$$\dot{\mathbf{x}}(t) = \mathbf{f}\left(t, \mathbf{x}(t), \mathbf{x}_\tau(t)\right),\ \mathbf{x} \in \mathbb{R}^n \tag{2}$$

where $\mathbf{f} : \mathbb{R} \times \mathbb{R}^n \times C^1,\ C^1 = C^1\left([0, \tau], \mathbb{R}^n\right)$. The delayed state vector in (2) is such that $\mathbf{x}_\tau(t) = \left\{\mathbf{x}(t) : t \geq \tau\right\}$.

B. Transcendental Characteristic Equation

A simple comparison between ODEs and DDEs is proposed [1] by using a simple standard example. A one-dimensional scalar linear ODE is used. The corresponding DDE delays merely the state variable by a positive fixed time delay. Let the one-dimensional scalar linear ODE

$$\dot{x}(t) = -\mu x(t),\ \mu \neq 0 \tag{3}$$

Using (3) with a fixed delay $\tau > 0$ yields the corresponding DDE

$$\dot{x}(t) = -\mu x(t-\tau),\ t \geq 0 \tag{4}$$

Solutions of ODEs are finite-dimensional and do not depend on the history. Solutions of DDEs are infinite-dimensional and strongly depend on the history. More specifically, the ODE in (3) has the algebraic characteristic equation

This work comes from a Plenary Lecture invited by AIP (American Institute of Physics) and presented in the 2nd Int. Conf. on Mathematical Methods & Computational Techniques in Science & Engineering (MMCTSE 2018) in Murray Edwards College, University of Cambridge, UK, February 16-18, 2018.

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$$\Delta_{ODE} \equiv z + \mu = 0,\ z \in \mathbb{C} \tag{5}$$

and requires an initial value. Eq. (5) is obtained by assuming the candidate function $x(t) = Ce^{zt}$ and has one eigenvalue $z = -\mu$. DDE in (4) has the transcendental characteristic equation

$$\Delta_{DDE} \equiv s + \mu e^{-\tau z} = 0 \tag{6}$$

and requires an initial continuous function on $[-\tau, 0]$. Eq. (6) has an infinite number of eigenvalues. The eigenvalues are defined by $z = \dfrac{\text{ProductLog}(-\mu\tau)}{\tau}$ where $\text{ProductLog}(z)$ gives the principal solution for w in $z = we^{w}$.

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C. History of Time-Delay Systems

A short history of delay differential equations was proposed by [2], pp. i-iv. Such mixed (or hybrid) equations were early studied by S.-D. Poisson (1781-1840) in his Memorandum “Essay on the mixed difference equations” (transl.) in 1806 [3]. We can also refer to J.B. Biot in 1806 who presented a talk “About the mixed difference equations” (transl.), in the French Institut des Sciences, Lettres et Arts par divers Savants [4]. S.-D. Poisson also published on the particular solutions of the differential and difference equations [5]. We could also quote the book S.F. Lacroix on “mixed difference equations” in 1819 [6]. However, the origin of such mixed equations was attributed to Condorcet in 1767 in the context of the astronomer’s three-body problem. This problem is to determine the motions of three point masses which attract each other according to Newton’s law (see online, the article by Chenciner [7] for a detailed numerical introduction).

The equations may not be autonomous and contain a vector of control variables. Delays may be placed on the state variables and their derivatives in the case of neutral forms. Moreover, the scalar integro-differential-difference equations (IDDEs) of the population models involve distributed time-delays. In such equations, the motion is not only explained by the position and velocity of a given physical particle but also all the history of the prior states. The motion equation will be determined not only by the initial point condition but on an integrable function defined on the pre-interval just before the motion. In the 1960’s, Bellman and Cooke published a seminal study on the DDEs [8], [9]. The Russian mathematicians Kolmanovskii, Myshkis and Nosov [10]-[12] also contributed to the study of DDEs’ stability and applications to industrial problems. More recently, the presence of algebraic constraints on the state or boundary conditions results in delay differential algebraic equations (DDAEs). Deshmukh (2010) [13] studied the stability of solutions of nonlinear DDAE with time periodic coefficients in mechanical engineering.

D. Applications of Time-Delay Systems

DDEs are used in numerous scientific domains in physics, engineering, chemistry, biology, economics for modeling, forecasting and simulating. Backer *et al*. (1999) [14], Bocharov and Rihan (2000) [15] investigated DDEs in biosciences, that is in population dynamics. Kolmanovski and Myshkis (1999) report applications in medicine [12] pp. 71-80 (e.g., arterial blood pressure regulation, cancer chemotherapy, a model of survival of red blood cells, regulation of glucose-insulin system, human respiratory system epidemiology, epidemiology, immunology, cell kinetics notably. This study is focused on mechanical engineering with applications in cutting and milling processes. In such models, we distinguish two essential pieces, the workpiece, and the cutting inserts. In cutting machines, the workpiece rotates, and the cutting inserts are fixed. On the contrary in the milling machines, the workpiece is fixed, and the cutting inserts are rotating on an axis. The time-delay corresponds to the time of one complete revolution of the workpiece or cutting inserts [16]. This substantial diffusion of applications with DDEs is also due to the inherent properties of a system and necessary feedback control to stabilize it. An inherent property of a system is that propagation of effects takes time to process. Feedback controls take into account delays between the observation of discrepancies and the introduction of effective controls.

II. TIME-DELAY SYSTEM MODELING

We provide some basic types of delays such as internal delays and control or loop delays. Different types of time-delay systems result from the comparison between the order of the derivatives and the maximal order at shifted arguments. We deduce retarded types, neutral types, and leading types. A TD system can also be characterized by time-varying coefficients and delays. There is also a distinction between time-continuous TD systems and time-discrete or digital TD systems. A system can be described as a set of inputs or control variables, a set of outputs, and a collection of information containing the history. The state-space representation consists of two equations, a state equation, and an observation equation.

A. Types of Delays

Delays can be neglected if they have no critical effects on the analysis or design of a system. Otherwise, delays have to be taken in the modeling process such as in metallurgical processing system, chemical systems, power system, transportation and communication, and environmental system.

Delays intervene between the application of input or control and the resulting output. There are two main categories of delays, internal delays and control delays. Internal delays are inherent to the system. Control or loop delays are introduced for control and correction purposes. Thus, in the steel rolling example in mechanical engineering, the thickness of metal sheets is measured at some distance from the rolls resulting in a measurement delay.

Delays may be fixed-time (or discrete time or stationary) or time-varying (or nonstationary). Delay can be certain or uncertain. In the stochastic approach, parameters may fluctuate due to the internal reasons and to a noisy environment. We recall that in cutting and milling machines, the time-delay represents the time taken for one complete revolution of the workpiece (cutting machines) or cutting inserts (milling). Parameters determine the amplitude and frequency of the delay variation.

B Types of Time-Delay Systems

Let a DDE in the general form be written in matrix form as

$$\begin{aligned}\mathbf{x}^{(n)}(t) = f\Big(&t, \mathbf{x}(t), \mathbf{x}^{(1)}(t), \ldots, \mathbf{x}^{(n-1)}(t),\\ &\mathbf{x}(t-\tau_0), \mathbf{x}^{(1)}(t-\tau_1), \ldots, \mathbf{x}^{m}(t-\tau_m)\Big)\end{aligned} \tag{7}$$

where $\mathbf{x} \in \mathbb{R}^N$ denotes the state vector of finite dimension. The expression $\mathbf{x}^{(k)}(t) = d^k\mathbf{x}/dt^k$ in (7) is the *k*th derivative evaluated at *t*. The order of the derivative on the LHS is n. The maximal order at shifted arguments is m.

A retarded type is obtained if $0 \le m < n$. A neutral type is represented for $m = n$. A leading type supposes that $m > n$.

An example can be the scalar DDE $\dot{x} = f\left(x(t), x(t-\tau)\right)$ for which $N = n = 1$ and $m = 0$. In the Hutchinson's population model (1948), the population loop is controlled by a delayed retroaction loop. Denote the intrinsic growth rate by r, the carrying capacity by K, and the time-delay by the constant τ. By Hutchinson, we have the delayed logistic equation

$$\dot{x}(t) = r\,x(t)\left(1 - \frac{x(t-\tau)}{K}\right) \tag{8}$$

This model is also known as a Verhulst-Pearl retarded logistic equation (see Keller [2] pp. 30-33 and Hutchinson (1948) [17]). The transcendental characteristic equation for (8) is

$$D(z) \equiv z + \left(1 - \frac{Ce^{t-\tau}}{K}\right) r = 0 \tag{9}$$

and the solutions of (9) are

$$z = r - \frac{\mathrm{ProductLog}\left(\dfrac{Ce^{r(t-\tau)} r(t-\tau)}{K}\right)}{t-\tau}$$

Michiels et al. (2007) [16] proposed a generalized linear time-varying delay of the form

$$\begin{aligned}&\dot{\mathbf{x}}(t) = \mathbf{A}(\omega t)\mathbf{x}(t) + \mathbf{B}(\omega t)\mathbf{x}(t - \tau(t)),\\ &\tau(t) = \tau_0 + \delta f(\Omega t)\end{aligned} \tag{10}$$

In (10), we have $\mathbf{x} \in \mathbb{R}^n$ and suppose that $\mathbf{A}, \mathbf{B} : \mathbb{R} \mapsto \mathbb{R}^{n\times n}$ are 2π- periodic functions and that $f : \mathbb{R} \mapsto [-1,1]$ is also a 2π- periodic function ns. Furthermore, we have $\delta, \tau_0, \omega, \Omega \in \mathbb{R}_{++}$ and $\delta \le \tau_0$. The two examples from mechanical engineering proposed by the authors are a model of a variable speed rotating cutting tool, and a model of an elastic column subjected to a periodic force [16].

We must also distinguish time-continuous (or analog) TD systems and time-discrete (or digital) TD system. Dynamic equations are the same except that the first-order derivative in the analog version is replaced by an advance by one sampling period, that is $\dot{\mathbf{x}}$ is replaced by $\mathbf{x}(k+1)$.

C State-Space Representation

In a system, we find a set of inputs (or control variables) $\mathbf{u} = (u_1 u_2 \ldots u_r)^T$, a set of outputs $\mathbf{y} = (y_1 y_2 \ldots y_m)^T$ and a collection of information containing the history. The inputs and the outputs are related by the dynamic equations of the system. If $r = m = 1$ the system is called SISO (single-input single-output). If $r > 1$ and/or $m > 1$, the system is called MIMO (i.e., multi-input multi-output).

The equivalent state-space representation of an invariant analog TD system consists of two matrix-equations, that is a state-equation and an observation-equation, so that

$$\begin{cases}\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t-\tau)\\ \mathbf{y}(t) = \mathbf{C}\mathbf{x}(t)\end{cases} \tag{11}$$

where $\mathbf{A}$ denotes the state matrix, $\mathbf{B}$ the input matrix, and $\mathbf{C}$ the output matrix of (11). A more general analog time-invariant TD system should be

$$\begin{cases}\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \displaystyle\sum_{i=1}^{n}\mathbf{A}_i\mathbf{x}(t-\tau_i) + \mathbf{B}\mathbf{u}(t)\\ \qquad + \displaystyle\sum_{j=1}^{r}\mathbf{B}_j\mathbf{u}(t-\theta_j)\\ \mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \displaystyle\sum_{i=1}^{n}\mathbf{C}_i\mathbf{x}(t-\tau_i) + \mathbf{D}\mathbf{u}(t)\\ \qquad + \displaystyle\sum_{j=1}^{r}\mathbf{D}_j\mathbf{u}(t-\theta_j)\end{cases} \tag{12}$$

where matrices of coefficients express direct transmissions in (12) .

III. DYNAMICS, STABILITY AND CONTROL

Problems studied by using deterministic and stochastic TD systems are common, dynamics, stability and control. Dynamics uses time and frequency representations. Stability analysis uses stability checking techniques such as the characteristic equation and usual stability criteria (e.g., root locus criterion, Nyquist criterion). Control analysis) (i.e., controllability and observability) refers to problems like H_∞ control and filtering.

A Transfer Function

Laplace Transform (LT) converts time-domains into frequency-domain functions. The consequence of this transformation is that differential equations are transformed into algebraic equations that make solving easier. The solutions of the algebraic approach are then transferred into the time domain. LT of function $f(t)$ is

$$\mathcal{L}\left[f(t)\right]=\int_{-\infty}^{\infty}e^{-st}f(t)\,dt\equiv F(s) \tag{13}$$

where s denotes the Laplace operator. Let the state space representation of a delayed SISO system be

$$\begin{cases}\dot{\mathbf{x}}(t)=\mathbf{A}\mathbf{x}(t)+\mathbf{A}_1\mathbf{x}(t-\tau)+Bu(t)\\ y(t)=\mathbf{C}\mathbf{x}(t)+Du(t)\end{cases} \tag{14}$$

Taking LT of (14) by using (13) for initial states yields

$$\begin{cases}sX(s)=\mathbf{A}X(s)+\mathbf{A}_1X(s)e^{-st}+BU(s)\\ Y(s)=\mathbf{C}X(s)+DU(s)\end{cases} \tag{15}$$

From (15) we deduce the transfer function

$$H(s)=\mathbf{C}\left(s\mathbf{I}-\mathbf{A}-\mathbf{A}_1e^{-st}\right)B+D$$

where $H(s)=Y(s)/U(s)$.

B Lambert Function-Based Method

This method is used to solve analytically the transcendental characteristic equation of DDEs as in sections I-B and II-B. Let the boundary-value problem of the Frisch-Holme prototype [18]

$$\begin{cases}\dot{y}(t)=-ay(t)-by(t-\tau),\ \tau>0\\ y(t)=\phi(t),\ t\in[0,\tau]\end{cases} \tag{16}$$

where a and b are positive coefficients. This model illustrates a cell population growth [15]. It can be shown that the lag value produces instability.

Assuming the candidate function $y(t)=Ce^{zt}$, we deduce from (16) the transcendental characteristic equation [19],[2] pp. 47-49.

$$D(z)\equiv(a+z)e^{z\tau}=-b \tag{17}$$

Multiplying both sides of (17) by $\tau e^{a\tau}$, we get

$$\tau(a+z)e^{\tau(a+z)}=-b\tau e^{a\tau} \tag{18}$$

According to the definition of a Lambert function for which ,

$W(s)e^{W(s)}=s$, we may also write

$$W\left(-b\tau e^{a\tau}\right)e^{W\left(-b\tau e^{a\tau}\right)}=-b\tau e^{a\tau} \tag{19}$$

Comparing (18) and (19), we deduce the roots

$$z=\frac{1}{\tau}W\left(-b\tau e^{a\tau}\right)-a$$

The roots of the Frisch-Holme characteristic equation are shown in Fig.1 in the complex plane.

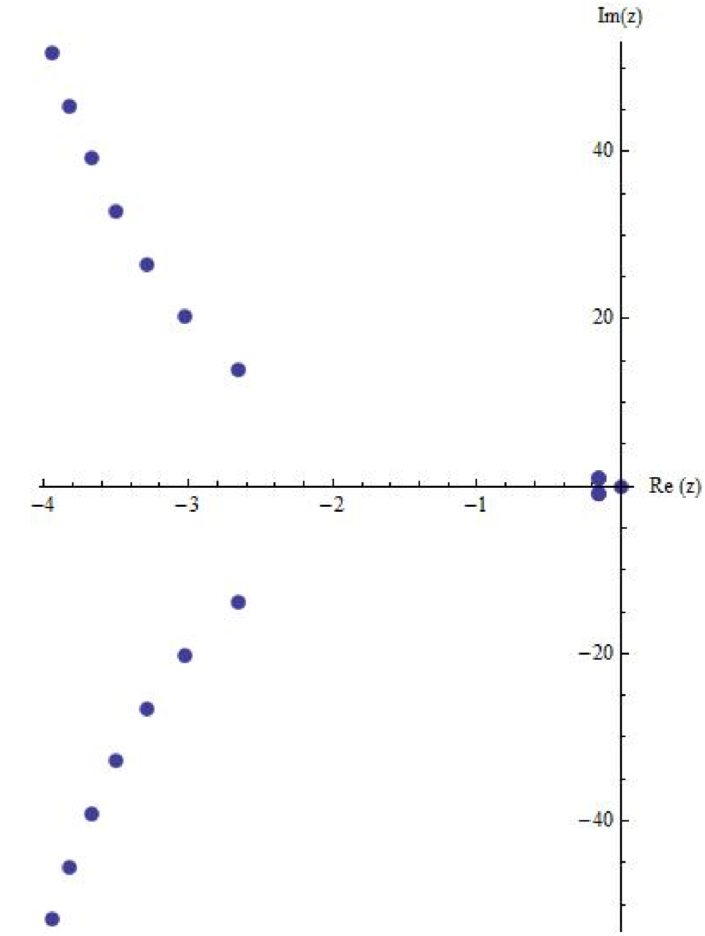

Fig.1 roots of the transcendental characteristic equation of Frisch-Holme prototype for coefficients $a=1$, $b=2$ and $\tau=1$

C Asymptotic Behavior of Time-Delay Systems

The asymptotic behavior of DDEs may involve oscillatory solutions that do not appear without delay. The delay may be destabilizing with possible stability regimes. We study the dynamics of the logistic delay prototype (8) for different values of the intrinsic growth rate (or Malthus' coefficient r and of the delay τ .By using the variable changing $x(t)=K\left(1+y\left(\frac{t}{\tau}\right)\right)$, we obtain the equivalent Wright form [20] by means of some simple algebraic manipulations

$$\dot{x}(t)=-r\tau\left(1+x(t)\right)x(t-1) \tag{20}$$

Using (20), Fig. 2 and Fig. 3 illustrate the dynamics in interactive objects from Mathematica ® software. These objects contain controls (sliders) for different parameter values. These interactive applications let explore ranges of coefficient values, delays, and initial conditions. The consequences on the results of such changes are observed immediately. Fig. 2 shows an oscillatory convergence to the origin in the phase space (x_τ, x) for an intrinsic growth rate of $r=1$ and a unit delay.

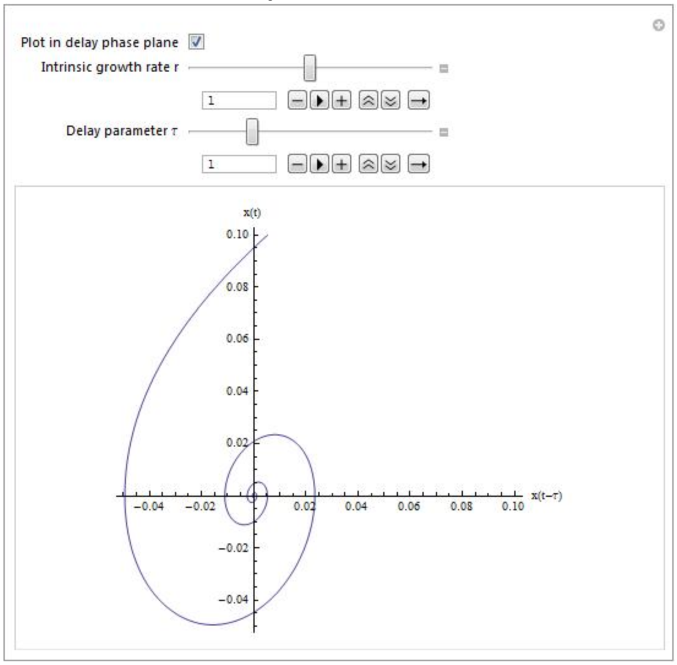

Fig. 2 logistics DDE dynamics for $r=\tau=1$.

Fig. 3 shows an oscillatory convergence to a limit cycle in the phase space (x_τ, x) for $r=1$ and $\tau=2$

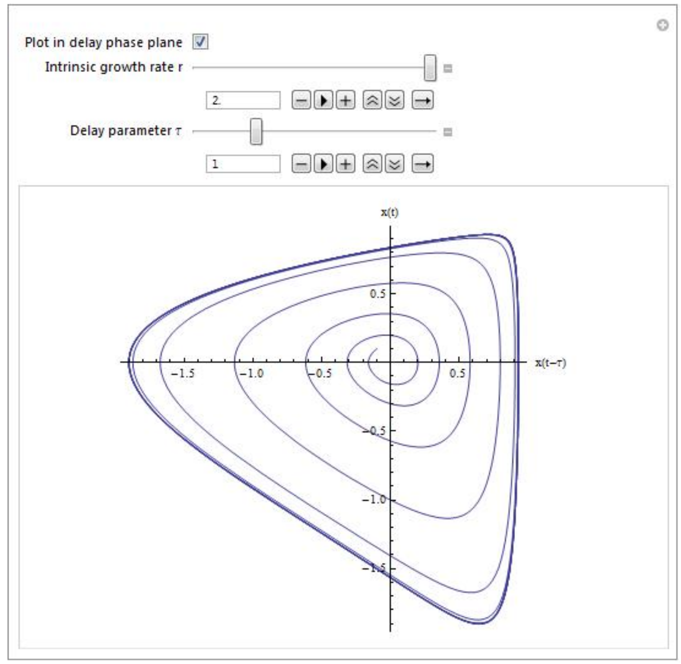

Fig. 3 logistics DDE dynamics for $r=2$ and $\tau=1$

D Stability Checking Techniques

Criteria like Ruth-Hurwitz for non-delay systems will not operate. Using the Neimar D-partition can be exhibited in the parameter space. Frequency response tools like Bode plots, Nyquist and Nichols's plots contribute to LTI system analysis.

The Neymar D-partition is retained here for this presentation [12] pp. 236-240 and [2] pp. 199-202. This technique is adequate to the study of the stability of the DDE Frisch-Holme prototype (16).

The transcendental characteristic equation for this model in (17) can be rewritten as

$$D(z) \equiv a + z + be^{z\tau} = 0 \quad , \tag{21}$$

where $z=\sigma+j\omega,\ j=\sqrt{-1}$. Putting the expression of z into (21) , separating the real and imaginary parts and solving in a and b for a constant delay τ , we get the parametric for equations defined by

$$\begin{cases} a = -\sigma - \omega\cot(\omega\tau) \\ b = \omega e^{\sigma\tau}\sin^{-1}(\omega\tau) \end{cases} \tag{22}$$

We are searching for (a,b) - values for which $D(z)=0$. Taking the pure imaginary $z=j\omega$, the boundaries of regions are formed of the straight line $a+b=0$ and curves defined parametrically by (22) with $\sigma=0$, that is $a=-\omega\cot(\omega\tau)$ and $b=\omega\sin^{-1}(\omega\tau)$. Fig. 4 shows a stability region Γ_0 in which the characteristic roots of the transcendental function are negative.

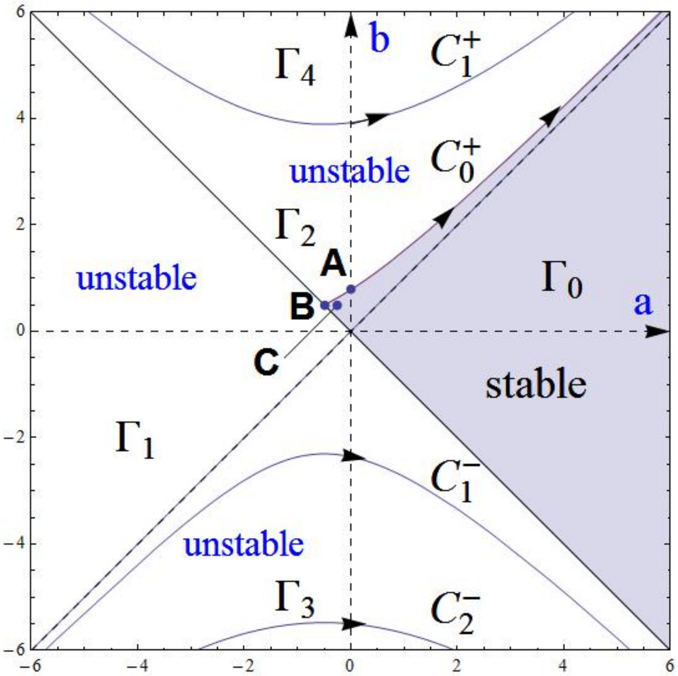

Fig. 4 D-partition of the parameter space (a,b) for a standard scalar Frisch-Holme type DDE [adapted from Keller (2011), p.220 (©LAP Lambert Academic Press)]

E Controllability and observability

The attributes controllability and observability characterize the control system. Controllability attribute characterizes the relationship between the input and the state. The problem of controllability is to know if a control $u(t)$ always exists from an initial state to any other state in a finite time. Observability attribute characterizes the relationship between the state and the output. The problem is to know if the initial state can always be identified by observing the output and the input of the system.

Let the general linear MIMO TD system

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \sum_{i=1}^{N} \mathbf{A}_i \mathbf{x}(t-\tau_i) + \mathbf{B}\mathbf{u}(t) \\ \mathbf{y} = \mathbf{C}\mathbf{x}(t) + \sum_{i=1}^{N} \mathbf{C}_i \mathbf{x}(t-\tau_i) + \mathbf{D}\mathbf{u}(t) \end{cases} \tag{23}$$

The invariant matrices of coefficients $\mathbf{A}, \mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_N, \mathbf{B}$ will have to characterize the controllability whereas the invariant matrices of coefficients $\mathbf{C}, \mathbf{C}_1, \mathbf{C}_2, \ldots, \mathbf{C}_N, \mathbf{D}$ will have to characterize the observability.

1) Controllability Criterion

Theorem. Let a state equation $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{A}_1\mathbf{x}(t-\tau) + \mathbf{B}\mathbf{u}(t)$ where $\mathbf{x} \in \mathbb{R}^N$ and $\mathbf{u} \in \mathbb{R}^m$. The system is controllable to the origin if a matrix $\mathbf{Q}$ has rank N.

Matrix $\mathbf{Q}$ is the concatenation

$$\mathbf{Q} = \left[\mathbf{Q}_1^1\mathbf{B} \mid \mathbf{Q}_1^2\mathbf{B} \mid \mathbf{Q}_2^2\mathbf{B} \mid \mathbf{Q}_1^3\mathbf{B} \mid \mathbf{Q}_2^3\mathbf{B} \mid \mathbf{Q}_3^3\mathbf{B} \right]$$

for which the calculation rules are:

(i) $\mathbf{Q}_1^1 = \mathbf{I}$,

(ii) $\mathbf{Q}_j^k = \mathbf{0}$ for $j = 0$ or $j > k$,

(iii) $\mathbf{Q}_j^{r+1} = \mathbf{A}\mathbf{Q}_j^r + \mathbf{A}_1\mathbf{Q}_{j-1}^r$.

Example (Malek-Zavarei and Jamshidi, 1987 [21], p.139). We suppose the coefficient matrices

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & 2 \\ 0 & 1 & -3 \end{pmatrix}, \mathbf{A}_1 = \begin{pmatrix} 0 & 0 & -2 \\ 0 & 0 & 1 \\ 0 & 0 & -4 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

We use the rules: (i) $\mathbf{Q}_1^1 = \mathbf{I}$, (ii), and (iii) $\mathbf{Q}_1^3 = (\mathbf{A})^2, \mathbf{Q}_2^3 = \mathbf{A}\mathbf{A}_1 + \mathbf{A}_1\mathbf{A}, \mathbf{Q}_1^3 = (\mathbf{A}_1)^2$. We deduce $\mathbf{Q} = \begin{pmatrix} 0 & -1 & -2 & 3 & 10 & 8 \\ 0 & 2 & 1 & -7 & -13 & -4 \\ 1 & -3 & -4 & 11 & 25 & 16 \end{pmatrix}$. In this example, the system is controllable to the origin since $\operatorname{rank}(\mathbf{Q}) = 3$.

2) Observability Criterion

Let a system with time-varying coefficients

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{A}_1(t)\mathbf{x}(t-1) \\ y(t) = \mathbf{C}\mathbf{x}(t) \end{cases} \tag{24}$$

where $\mathbf{A}(t) = \begin{pmatrix} 0 & 1 \\ -2 & -t \end{pmatrix}, \mathbf{A}(t) = \begin{pmatrix} 0 & 0 \\ -t & -1 \end{pmatrix}$, and $\mathbf{C} = \begin{pmatrix} 1 & 0 \end{pmatrix}$. The observability matrix of (24) is expressed by $\mathbf{P}(t) = \left[\begin{pmatrix} \mathbf{I} & \mathbf{A}^T(t) & \mathbf{A}_1^T(t) \end{pmatrix} * \mathbf{C}^T \right]$. For this example (Malek-Zavarei and Jamshidi, 1987 [21], p.148), the observability matrix takes the form $\mathbf{P}(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$.

We conclude that the TD system is observable for $t \geq 0$.

IV. APPLICATION TO MECHANICAL ENGINEERING

DDE systems have been developed in industrial areas. We will give some details about the diversity of these applications and will focus the detailed presentation on two standard well-known mechanical systems namely the metal rolling system and the machine tool chatter problems.

A. Industrial Applications of Time-Delay Systems

Industrial applications of time-delay systems are multiple and diverse. DDE systems can be found in physics, engineering (e.g., aerospace, robotics, mechanical systems, electrical systems, manufacturing, metallurgical systems), chemistry, bioscience (e.g., population dynamics), medicine (e.g., epidemiology, immunology, cell kinetics), water resources (e.g., hydraulic control), communications (e.g., traffic control, transport process), computerized systems.

This presentation is devoted to the stability and control of mechanical systems. Mechanical engineering can be represented by fixed or time-varying nonlinear TD systems. We may also have algebraic boundary conditions as with time periodic DDAEs (i.e., delay-differential-algebraic equations). Boukas and Liu (2002) [22] studied three applications: (1) A rolling system, 2) a heating system, and (3) a controlled feeding of the production system. Michiels *et al*. [16] proposed two mechanical applications: (1) a variable speed rotating cutting tool and (2) an elastic column subjected to a periodic force.

This presentation focuses on the description and analysis of two mechanical engineering applications with milling and cutting machines, i.e., (1) Metal rolling system and (2) Machine tool chatter. We describe the model for each application, show the dynamic equation, the equivalent representations. Elements of stability analysis are shown in application 2.

B. Mechanical Application 1: Metal Rolling System

In metal rolling system, ribbons of hot metal sheets run though rollers as in Fig. 5. The goal is to produce metal sheets of a given thickness by controlling the upper roller, whereas the lower one is fixed. A drive motor is used to operate this adjustment. The sensor thickness is placed downstream at a distance d of the rollers. The ribbon is fed at speed v into the system [22] pp. 4-6, [23].

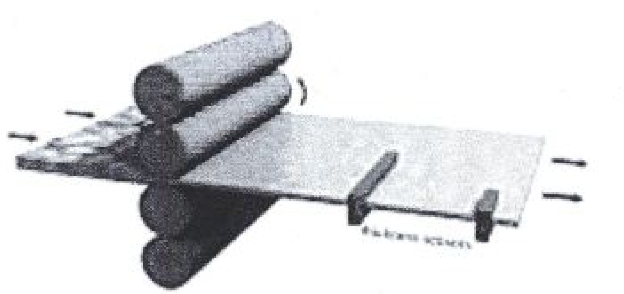

Fig. 5 metal rolling system [adapted from Mirkin [23]]

The delay depends on both distance and speed, such that $\tau = d / v$. The measured thickness is $y(t-\tau)$, the real thickness $y(t)$, and desired thickness be denoted by $y_d(t)$. We obtain the DDE

$$\dot{y}(t) + k_p k_m y(t-\tau) = k_p k_m y_d(t) \tag{25}$$

where k_p and k_m are gains of proportional controller in (25) . The state-space representation of the metal rolling system is

$$\begin{cases} \dot{x}(t) = A_d x(t-\tau) + Bu(t), \;\; x(0) = x_0 \\ y(t) = x(t) \end{cases} \tag{26}$$

where k_p and k_m are gains of proportional controllers.

The block diagram of this metal rolling system in (26) is represented by Fig. 6. Suppose that the transfer function between the upper roller position and the input voltage of the driver motor is $G(s) = k_m / s$. The transfer function of the used controller is $C(s) = k_p / s$. The transfer function of the censor is $H(s) = e^{-\tau s}$. The effect of the external disturbances on the system is denoted by $P(s)$.Suppose that $G(s)$ is given by

$$G(s) = \frac{k_m}{s(1+Ts)}$$

We get the following second-order DDE [22] p.6

$$\ddot{y}(t) + \dot{y}(t) + k_p k_m y(t-\tau) = k_p k_m y_d(t).$$

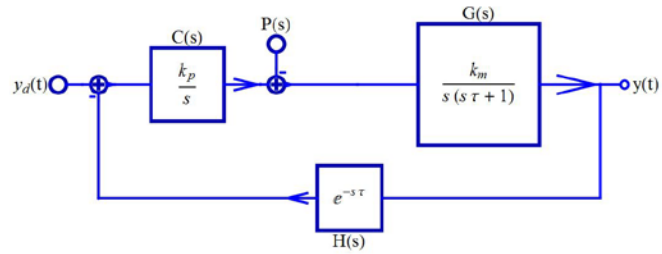

Fig. 6 block diagram of the metel rolling system [adapted from Boukas and Liu (2002) Figure 1.2, p. 6]

C. Mechanical Application 2: Machine Tool Chatter

Machining processes like turning, milling, and drilling can be disturbed by vibrations produced by coupling between a machine tool and the cutting forces. Vibrations result in waves on the machine surface. In this study, a machine tool removing a chip is deviated from its steady-state cutting by a small vibration $x(t)$ between the machine tool and a workpiece as in Fig. 7. Dynamic forces $P(t)$ act between the machine tool and the workpiece, as in Fig. 7. The goal is to analyze the machine tool stability by using a method based on the feedback control theory [24]. Fig. 7(a) shows the positions of the workpiece, the tool,, and chip. It also indicates the fee direction. Fig. 7(b) shows how forces are acting in the system.

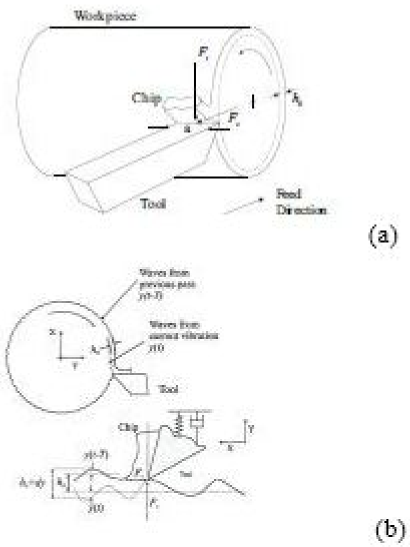

Fig. 7 Regenerative chatter in metal cutting [taken fron Mirkin (2012) [23]]

The machining process model shows a closed loop interaction between two blocks, a cutting process sub-model, and a structural dynamics sub-model. Deflections $\mathbf{x}(t)$ result from the action of cutting forces $F(t)$ on the structural dynamics submodel. These deflections, in turn, modulate the cutting forces [16][24]-[25]. .

The dynamic equation results from the combination of the cutting process and the structural dynamics. A cutting force equation can be expressed by

$$F(t)=k\Big(f+x\big(t-\phi(t)\big)-x(t)\Big)d \tag{27}$$

where k denotes a cutting force coefficient, f a nominal feed rate, d a nominal depth of cut, and where the colored expression is a regeneration term which modulates the cutting force. The model of structural dynamics computes the deflection of the machine tool structure resulting from the cutting forces. Using the transfer function approach, we have

$$G(s)=\frac{X(s)}{F(s)}=\frac{1/m}{s^2+2\xi\omega s+\omega^2} \tag{28}$$

The dynamic equation results from (27)-(28). We have the typical second-order DDE

$$\ddot{x}(t)+2\xi\omega\dot{x}(t)+\omega^2 x(t)=\frac{k}{m}\Big(x\big(t-\phi(t)-x(t)\big)\Big),$$

where ξ denotes the damping ratio, ω the frequency, and m the modal mass.

The system consists of three blocks: the machine tool structure, the chip thickness modulation, and the cutting process. Each of them represents a transfer function with inputs and outputs. The input of the machine tool block is the force F and the output is the relative displacement x . The transfer function is $G(s)$ or the corresponding frequency function $G(j\omega)=Me^{j\phi}$ in complex polar form with usual notations. The chip thickness modulation is expressed by $\delta(t)=x(t)-\mu x(t-T)$. The response function is $H_1(j\omega)=1-\mu e^{-j\omega T}$. The frequency response function of the cutting process is $H_2(j\omega)=Ke^{j\omega}$. The block diagram of the metal cutting system is shown in Fig. 8.

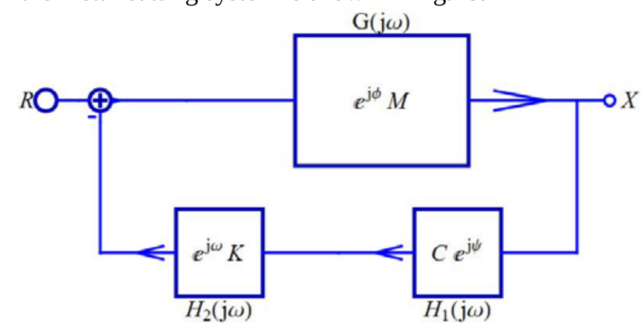

Fig. 8 block diagram of the metal cutting system [adapted from Nigm [24] Fig.2, p. 253]

The open-loop frequency response function of the system is

$$T(j\omega)=G(j\omega)\times H_1(j\omega)\times H_2(j\omega) \tag{29}$$

Eq.(29) is equivalent to $T(j\omega)=Ae^{j\alpha}$ with gain $A=MCK$ and phase $\alpha=\phi+\psi+\theta$.

Finally, we apply a stability criterion for the frequency at which the open-loop phase is such as $\alpha=\pi$. The system is stable if the gain is $A<1$. The system is unstable if $A>1$, and on the threshold of stability if. $A=1$ If $A=1$, then $K=1/MC$. The system is stable if $K<1/MC$ and unstable if K>1/M.

V. Conclusion

The challenge of this article was to be an introduction to the vast domain of the DDE modeling. Interest for these hybrid systems goes back to an ancient times as we showed it in the introduction. Block diagram techniques have contributed significantly to these developments and the industrial applications. The approach is mostly introductory on the whole subject without addressing particular points of concern. This article is limited to some illustrative techniques without addressing them all of course. The field of mechanics is privileged, but the presentation is limited to two significant examples such as rolling sheet steel and the vibrations of a machine cutting tool.

Other examples relating to mechanics are presented in the book by Boukas [26] pp. 14-28 on “Control systems engineering” (transl.). These examples are the speed control of a conveyor, the flushing system, the positioning servo of a parabolic antenna, the control of a ship's rudder, the control of the angular position of an airplane rudder. Other specifications on circuit analysis are proposed in different studies of the author such as other system representations (e.g., phasor representation [27]) , solving methods (Method of Step Algorithm MSA, Differential Transform Methods DTM [28] with application to the retarded Kalecki's business cycle model, Solow vintage capital growth model with variable delay [29]), stability criteria besides the root locus criterion (e.g., BIBO stability, Lyapunov stability criterion, Nyquist stability criterion) with applications to environmental pollution systems and fishery environmental systems [27].

Other important technical aspects of dynamics and control can be addressed such as the assumption of a stochastic environment, PID control and fuzzy control. A noisy environment can be adapted to describe real-life modeling processes of populations. In the papers [29]-[30], populations interact, fight or cooperate, in real life conditions with constant time-delays in a noisy environment. A typical small delay stochastic process can be

$$dx(t) = f\left(x(t), x(t-\tau)\right)dt + \sigma g\left(x(t)\right)dW(t),$$

where f, g are known functions, τ is a constant time-delay, σ scales the noise amplitude, and $W(t)$ is a Wiener process. PID controllers involve three possible coefficients: a proportional gain (P) in reaction to the current error, an integral gain (I) reacting on the sum of past errors, and a derivative gain (D) in reaction to the rate of change of the error. In this paper PID controls are applied to a continuous multiplier-accelerator model (the Phillip's macroeconomic business cycle model) and to nonlinear Goodwin 's macroeconomic growth model [31]. A fuzzy logic controller operates is similar to an artificial decision-maker that operates in a closed-loop system in real time. Description and applications to dynamic macroeconomic models can be found in [31]-[32]. Compared to a PID control, the simulation results with fuzzy controllers show more efficient stabilization of an economy.

In our studies, specialized MATHEMATICA® packages (e.g., Control System Professional, Polynomial Control System, and SchematicSolver) are used for analyzing and solving the systems symbolically and numerically [33]-[35]. These studies also used MATLAB® specialized packages (e.g., Simullink and ControlSystem, Fuzzy Logic Tool 2) [36].

REFERENCES

[1] M.S. Fofana, "Delay dynamical systems and applications to nonlinear machine-tool chatter," *Chaos Solitons Fractals*, vol 17, no 4, pp. 731-747, 2003.

[2] A.A. Keller, *Time-Delay Systems with Applications to Economic Dynamics and Control.* Saarbrücken, DE, Lambert Academic Press LAP, 2011.

[3] S.D. Poisson, "Mémoire sur les équations aux différences mêlées [in French], " *J. Ecole Polytech., JEP*, vol 13, no. 6, pp. 126-147, 1806.

[4] J.B. Biot, "Sur les équations aux différences mêlées [in French], " Essay of the Institut des Sciences, Lettres et Arts par divers Savants, Paris, France, vol., pp. 296-327, 1806.

[5] S.D. Poisson, "Mémoire sur les solutions particulières des équations différentielles et des équations aux différences [in French], " *J. Ecole Polytech., JEP*, vol 13, no. 6, pp. 60-125, 1806.

[6] S.F. Lacroix, " Des équations aux differences mêlées [in French]," in *Traité du Calcul Différentiel et du Calcul Intégral* , vol. 3, Ch.8, Paris, France, 1819, pp.575-600.

[7] A. Chenciner (2007). Availablet http://www.scholarpedia.org/article/Three body problem .

[8] R. E. Bellman and K.L. Cooke, "Asymptotic behavior of solutions of differential-difference equations," Tech. Report 35, American Mathematical Society, New York, 1959.

[9] R. E. Bellman and K.L. Cooke, "Differential-difference equations," Tech. Report R-374-PR , The Rand Corporation, Santa Monica, CA, USA, 1963.

[10] V.B. Kolmanovskii .and V.R. Nosov, "Stability of functional differential equations," in *Mathematics and Its Applications*, vol. 463, Academic Press, London New York, 1986.

[11] V.B. Kolmanovskii .and A. Myshkis, "Applied theory of functional differential equation," in *Mathematics and Its Applications* vol. 85, Kluwer Academic Press, Dordrecht New York, 1992.

[12] V.B. Kolmanovskii .and A. Myshkis, "Introduction to the theory and applications of functional differential equations," in *Mathematics and Its Applications* vol. 463, Kluwer Academic Press, Dordrecht New York, 1999.

[13] V. Deshmukh, "Approximate stability analysis and computation of solutions of nonlinear delay differential algebraic equations with time periodic coefficients," *J. Vib. Control*, vol. 16, no. 7-8, pp. 1235-1260, 2010.

[14] C.T.H. Backer, G.A. Bocharov and F.A. Rihan, "A report on the use of delay differential equations in numerical modelling in the biosciences," Tech. Report no 343, University of Manchester, UK, 1999.

[15] G.A. Bocharov and F.A. Rihan, "Numerical modelling in biosciences using delay differential equation," *J. Comput. Appl. Math*., vol. 125, no. 1-2, pp. 183-199, 2000.

[16] W. Michiels, K. Verheyden and S.-I., Niculescu, "Mathematical and computational tools for the stability analysis of time-varying delay systems and applications in Mechanical Engineering", in J. Chiasson and J.J. Loiseau (Eds.), *Appl. of Time-Delay Systems*, NCIS, vol. 352, 2000, pp. 199-216.

[17] G.E. Hutchinson, "Circular causal systems in ecology," *Ann. N.Y. Acad. Sci.*, vol. 50, pp. 221-246, 1948.

[18] R. Frisch and H. Holme, "The characteristic solutions of mixed difference and differential equations occurring in economic dynamics," *Econometrica*, vol. 3, no. 2, pp. 225-239, 1935.

[19] F.M. Asl and A.G. Ulsoy, "Analysis of a system of linear delay differential equations by using differential transform method," *Appl. Math. Comput*., vol. 181, pp. 215-223, 2003.

[20] E.M. Wright, "A non-linear difference-differential equation," J. Reine Angew. Math., vol. 1955, no. 194, 2009.

[21] M. Malek-Zavarei and M. Jamshidi, *Time-Delay Systems: Analysis, Optimization and Applications*, Amsterdam, NL, North-Holland, 1987.

[22] E.-K. Boukas and Z.-K. Liu, *Deterministic and Stochastic Time Delay Systems*, Boston, MA, USA, Birkhäuser, 2002.

[23] L. Mirkin, Introduction to Time-Delay Systems, Lectures of Automatic Control–Doctorate Program, Lund Univ.(2012).See http://www.control.lth.se/media/Education/DoctorateProgram/2012/Lectures

[24] M.M. Nigm, "A method for the analysis of machine tool chatter," *Int. J. Mach. Tool Des. Res.* **21** (3) 251-261, 1981.

[25] S. Jayaram, S.G. Kapoor and R.E. DeVor,"Estimation of the specific cutting pressure for mechanistic cutting force models," *ASME J. Manuf. Sci. Eng.* **122** (3), 391-397 (2000).

[26] E-K. Boukas, *Systèmes asservis* [in French], Montréal, CN, Editions Ecole Polytech. Montréal, 1995.

[27] A.A. Keller, "Circuit analysis to environmental economic dynamics and control," in Proc. Int. Conf. Development, Energy, Environment, Economics (DEEE), Puerto De La Cruz, Tenerife, ES, WSEAS Press, 2010, pp. 133-141, Available http://www.wseas.org/multimedia/books/2010/Tenerife/DEEE.pdf

[28] A.A. Keller, "Generalized delay differential equations to economic dynamics and control," in Recent Advances in Applied Mathematics, *Proc. American Conf. Applied Mathematics* (AMERICAN-MATH'10),

University of Harvard, Cambridge, MA, USA, 2010, pp. 278-286. Available http://www.wseas.org/multimedia/books/2010/Harvard/MATH.pdf

[29] A.A. Keller, "Stochastic delay Lotka-Volterra system to interactive population dynamics, ", in Recent Research in Applied Mathematics, Simulation and Modelling, *Proc. 5th Int. Conf. Applied Mathematics, Simulation, Modelling* (ASM'11), Corfu Island, GR, 2011, pp. 191-196

[30] A.A. Keller, "Population biology models with time-delay in a noisy environment," WSEAS Trans. Biology Biomedicine, vol. 8, no. 4, October 2011, pp. 113-134. Available http://wseas.us/e-library/transactions/biology/2011/54-076.pdf

[31] A.A. Keller, "Optimal economic stabilization policy under uncertainty," in *Advanced Technologies*, K. Jayanthakumaran, Vukavar, HR: In-The, 2009, pp. 441-470.

[32] A.A. Keller,"Fuzzy control of macroeconomic models," *Proc. World Academy of Science, Engineering and Technology*, Prague, CZ, vol. 31, July 2008, pp. 145-154.

[33] M.D. Ludovac and D.V. Tošić, SchematicSolver version 2.2, MATHEMATICA® Application: Symbolic Signal Processing, 2009. Available kondor.etf.bg.ac.rs/lutovac

[34] M.D. Ludovac, D.V. Tošić and B.L. Evans, *Filter Design for Signal Processing Using MATLAB® and MATHEMATICA®*, Upper Saddle River, NJ, USA: Prentice Hall, 2001

[35] M.S. Stachowicz and L. Beal, *MATHEMAICA Fuzzy Logic*, Wolfram Research. Available http://www.wolfram.com

[36] The MathWorks, *User's Guide*: MATLAB 7, Simulink 7, Simulink Control Design, Control System Design, Fuzzy Logic Toolbox 2", Natick, MA, USA, 2008

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