Nonlinear dynamics of micro-electromechanic element

Dmitrii Ju. Skubov, Alexei V. Lukin, Lev V. Shtukin

Abstract**—**Static and dynamic problems of coupled electro-elasticity are considered for electrostatic (capacitive) transducers used as sensors and actuators based on nano- and microsystem technology in various applications. Above-mentioned problems are analyzed by mathematical apparatus of nonlinear mechanics and bifurcation theory as well as modern numerical methods, including numerical continuation techniques for nonlinear boundary-value problems. Comparative analysis of analytical and numerical methods was performed for nonlinear static and dynamic boundary problems of electro-elasticity for nano- and microsystems engineering. Equilibrium forms, their stability and bifurcations were studied for afore-named elastic systems under the influence of electric fields of various configurations.

Keywords - Micro-electromechanical systems, equilibrium forms, stability of periodic motion.

I. INTRODUCTION

Recent time the achievements of micro- and nano-electromechanical systems (MEMS and NEMS) appeal the great interest of physics, biologists, electrician-engineers. The using of these devices is connected with their high sensitivity to nano- and micro-scale alterations of physical and biological parameters. For example, it's used as a determination of molecular mass, biochemical reactions, production of some new molecules, nanoparticles, and also for sensitivity for alteration of different types of external microphysical magnitudes. With help of miniature nano- and micro-sensors the supersensitive for extremely small forces and periodic excitation have been found. Reaching of this sensitivity is connected with attainment of Mega and even Giga-Hertz diapason of eigen frequency of their dynamical characteristics. Increasing of eigen frequency until 1GHz gives the opportunity to avoid the influence of some physical noise on micro-sensors, such as, for example, a temperature and humid fluctuations.

It is necessary to say that not less important the micro-sensors based on «pull-in» effect [13] have been obtained. This effect gives the opportunity to determine the micro-alteration of external or inner excitation.

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«Pull-in» instability is an inherently nonlinear effect connected with disappearance of equilibrium forms of elastic part of sensitive and actuating mechanism. This effect occurs at achievement of electro or magneto static actions the determined critical value.

II SYSTEM WITH ONE FIXED ELECTRODE

Let's consider the electromechanical model of micro-electromechanical oscillator, which consists from mass (mobile plate) attached to immobile plate of plane-parallel micro-capacitor by spring with damper (Fig. 1.1).

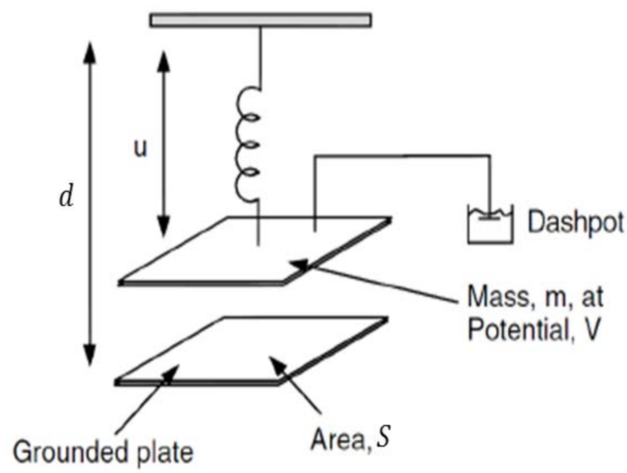


Fig.1.1 – Model of micro-electromechanical oscillator

The designations of this figure are d – distance between immobile plate and a fixing point of oscillator, m – mass of mobile plate, c – dissipation coefficient, k – stiffness of spring, l – length of unstrained spring. Taking in account the coincidence of positive direction of axe x with direction of ponderomotive force of attraction between plates, the expression of this force takes a form:

$$F_e = \frac{1}{2}\frac{\epsilon_r \epsilon_0 S V^2}{(d-u)^2}. \quad (1.1)$$

The equation of oscillator motion is written in next view:

$$m\frac{d^2u}{dt^2} + c\frac{du}{dt} + k(u-l) = \frac{1}{2}\frac{\epsilon_r \epsilon_0 S V^2}{(d-u)^2} \quad (1.2)$$

Passing to dimensionless values

$$x = \frac{u-l}{d-l},\ \tau = \sqrt{\frac{k}{m}}t = \Omega_0 t,\ \alpha = \frac{c}{\sqrt{mk}}, \tag{1.3}$$

the equation (1.2) transforms to view

$$\frac{d^2x}{d\tau^2} + \alpha\frac{dx}{d\tau} + x = \frac{\lambda}{(1-x)^2}, \tag{1.4}$$

here $\lambda = \frac{1}{2}\frac{\epsilon_r\epsilon_0 SV^2}{k(d-l)^3}$ – dimensionless parameter determinative the relation between ponderomotive and elastic force, acting in this system. Really parameter λ can be present in form

$$\lambda = \frac{1}{2}\frac{\epsilon_r\epsilon_0 SV^2}{(d-l)^2}\cdot\frac{1}{k(d-l)} = \frac{\widetilde{F_e}}{\widetilde{F_s}}, \tag{1.5}$$

where $\widetilde{F}_e = \frac{1}{2}\frac{\epsilon_r\epsilon_0 SV^2}{(d-l)^2}$ – characteristic ponderomotive force corresponding to distance between plates at unstrained spring; $\widetilde{F}_s = k(d-l)$ – characteristic elastic force, emergent in spring at coincidence of capacitor plates. In Figure 1.2 the branching diagram of equilibrium positions for equation (1.4) be absent of dissipation $\alpha = 0$ is shown. The law branch is stable and upper is unstable

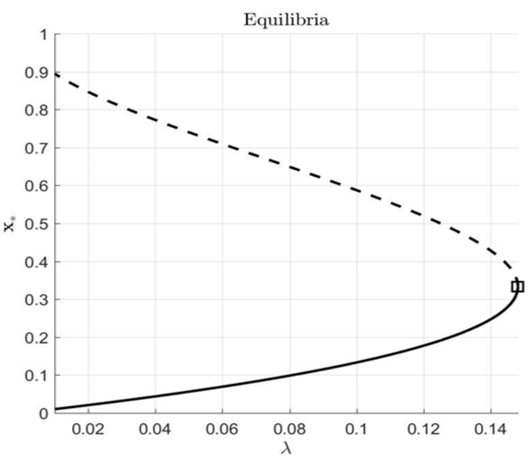


Fig.1.2 – Branching diagram for equilibrium position

Let's pass to investigation regimes of motion of microelectromechanic oscillator in field of one electrode with periodic potential. Before then consideration the task of forced oscillation let's study the quality structure of phase plane accounting dissipation determined by coefficient $\mu > 0$. The equation of motion of mobile plate in constant electric field takes a form:

$$\frac{d^2x}{d\tau^2} + \mu\frac{dx}{d\tau} + x = \frac{\lambda}{(1-x)^2}. \tag{1.6}$$

The phase portrait of system (1.6) at different parameter λ and fixed coefficient of dissipation μ is shown in Fig.1.3

As it shown from this figure, at $\lambda < \lambda_* = \frac{4}{27}$ this system has two critical points: stable focus and saddle. At $\lambda > \lambda_*$ critical points are absent, i.e. λ_* - branching point, at which these critical points conjugate.

Let's pass to task of forced oscillation of micromechanical oscillator in field of one electrode. Here we may consider two different cases of oscillation excitation. In first case the voltage between fixed plate and conductive elastic element is alternating function of time. For example voltage changes as $V_{AC}\cos\widetilde{\Omega}\tilde{t}$. At second case stationary level of voltage is added to dynamical excitation, so full voltage: $V_{DC} + V_{AC}\cos\widetilde{\Omega}\tilde{t}$.

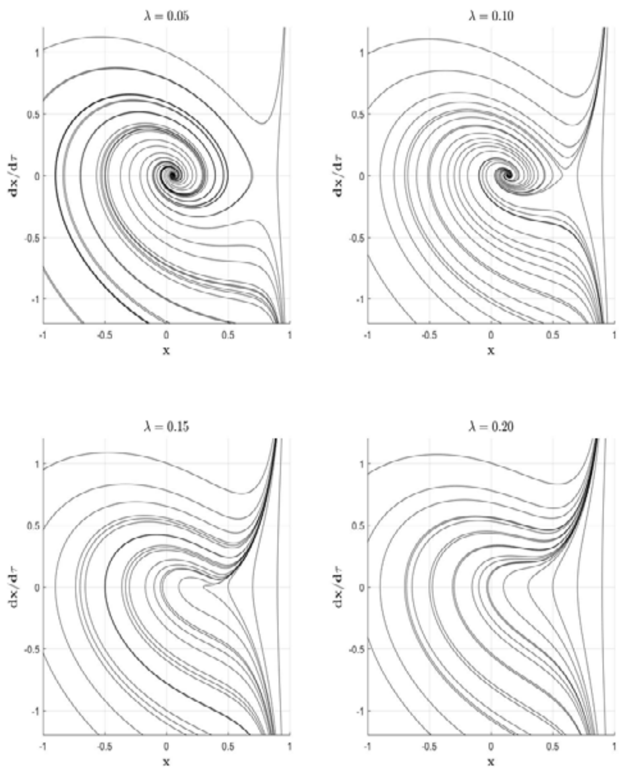


Fig.1.3 – Phase portraits MEMS – oscillator accounting damping ($\mu = 0{,}05$)

The first regime of excitation is used in high-frequency switchers and others microsystem devices, in which necessary to maximum quickly consequence of elastic element from stable equilibrium position and achieve contact with fixed plate («*pull-in*» effect). The second regime is used in works of resonators, accelerometers, pressure sensors, frequency generators.

Let's consider the first variant of oscillation excitation. The motion equation of this variant has a form:

$$\ddot{x} + \mu\dot{x} + x = \frac{\lambda\cos^2\Omega t}{(1-x)^2} \tag{1.7}$$

Here for convenience dimensionless time designate by letter t, dimensionless frequency of excitation $\Omega = \frac{\widetilde{\Omega}}{\Omega_0}$. Let's formulate analytic solution (1.7) with help asymptotic methods of the nonlinear oscillation theory [1, 2]. The small parameter ε is introduced in designation of coefficient of dissipation $\varepsilon\mu$ and amplitude of excitation $\varepsilon\lambda$ and expand nonlinear item into Taylor series in vicinity of zero:

$$\ddot{x} + \varepsilon\mu\dot{x} + x = \varepsilon\lambda\cos^2\Omega t\,(1 + 2x + 3x^2 + 4x^3 + \cdots). \tag{1.8}$$

With aim of building uniformly suitable expansion we use asymptotic manyscale method. The solution of equation (1.8) will be search in view

$$x = x_0(T_0, T_1, \ldots) + \varepsilon x_1(T_0, T_1, \ldots) + \cdots \quad (1.9)$$

where $T_k = \varepsilon^k t$ – different scale of time, $x_i,\ i = 0,1,2,\ldots$ - unknown functions.

Following [2], let's express derivatives on real time t through derivatives by T_k:

$$\frac{d}{dt} = D_0 + \varepsilon D_1 + \cdots,\ \frac{d^2}{dt^2} = D_0^2 + 2\varepsilon D_0 D_1 + \cdots, \ldots \quad (1.10)$$

where $D_k = \partial/\partial T_k$. Let's build the first approximation. Substituting (1.9), (1.10) into (1.8) and grouping terms of n order ε, we obtain next system of equations

$$D_0^2 x_0 + x_0 = 0,$$

$$D_0^2 x_1 + x_1 = -2D_0 D_1 x_0 - \mu D_0 x_0 + \lambda \cos^2 \Omega T_0\,[1 + 2x_0 + 3x_0^2 + 4x_0^3], \ldots \quad (1.11)$$

Solution of homogeneous equation for x_0 is written in complex form

$$x_0(T_0, T_1) = A(T_1)e^{iT_0} + (c.c.), \quad (1.12)$$

where shot designation for complex conjugate items is introduced. The complex amplitude is determined as

$$A(T_1) = \frac{1}{2}a(T_1)e^{i\beta(T_1)}, \quad (1.13)$$

where a and β – required function of slow time T_1 (amplitude and phase).

Let's consider the main resonance, at which frequency of external excitation $2\Omega \approx 1$. Let's introduced parameter of frequency mismatch σ by formula

$$\Omega = \frac{1}{2} + \varepsilon\sigma. \quad (1.14)$$

Substituting (1.12), (1.14) into right part of second equation of system (1.11) and using condition of absent of secular items we obtain next differential equation for complex amplitude

$$-2iA' - \mu iA + \lambda A + 6\lambda A^2\bar{A} + \frac{\lambda}{4}e^{2i\sigma T_1} + \frac{3\lambda}{4}A^2 e^{-2i\sigma T_1} + \frac{3\lambda}{2}A\bar{A}e^{2i\sigma T_1} = 0. \quad (1.15)$$

Pass on to exponential form of writing equation (1.15) and separating real and imaginary part of equation, we obtain next system of differential equation for slowly changing amplitude and phase of oscillation:

$$\frac{da}{dT_1} = -\frac{1}{2}\mu a + \frac{\lambda}{4}\left(1 + \frac{3}{4}a^2\right)\sin\gamma,$$

$$a\frac{d\gamma}{dT_1} = 2a\sigma + \frac{1}{2}\lambda a + \frac{3}{4}\lambda a^3 + \frac{\lambda}{4}\left(1 + \frac{9}{4}a^2\right)\cos\gamma. \quad (1.16)$$

Here input designation

$$\gamma = 2\sigma T_1 - \beta. \quad (1.17)$$

Critical points of autonomous system (1.16) correspond to stationary oscillations. These points satisfy algebraic system

$$\frac{1}{2}\mu a = \frac{\lambda}{4}\left(1 + \frac{3}{4}a^2\right)\sin\gamma,$$

$$-2a\sigma - \frac{1}{2}\lambda a - \frac{3}{4}\lambda a^3 = \frac{\lambda}{4}\left(1 + \frac{9}{4}a^2\right)\cos\gamma \quad (1.18)$$

Exclusion of variable γ leads to equation implicitly defining dependence of amplitude of stationary oscillation from frequency mismatch $a(\sigma)$:

$$\frac{\lambda^2}{16} = \frac{1}{4}\frac{\mu^2 a^2}{\left(1+\frac{3}{4}a^2\right)^2} + \left(\frac{2a\sigma + \frac{1}{2}\lambda a + \frac{3}{4}\lambda a^3}{1+\frac{9}{4}a^2}\right)^2 \quad (1.19)$$

The comparable of amplitude-frequency characteristics, obtained by three methods: according to analytic expression (1.19); direct numerical calculation of approximated system (1.8); direct numerical calculation of initial system (1.7) at different values of parameter λ are shown in Fig.1.4, Fig.1.5.

The numerical dependence $a(\sigma)$ are determined using algorithm prolongation by parameter of periodic solution which realized by software package *MATCONT* [3]. Coefficient of damper μ takes equal 0,005.

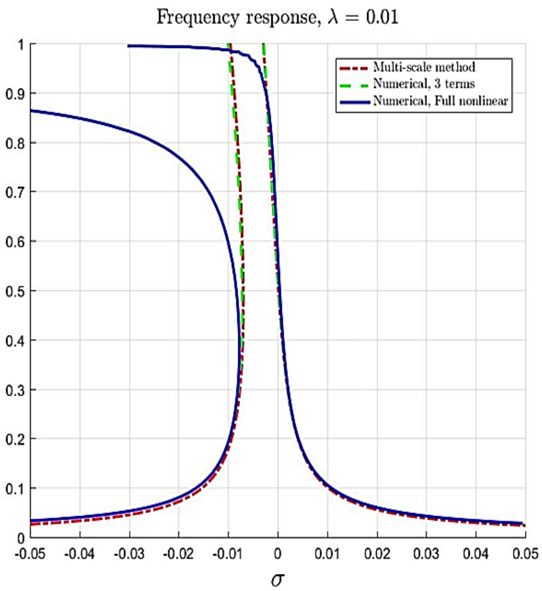


Fig.1.4 – Comparable of amplitude-frequency response $\Omega \approx \frac{1}{2}, \lambda = 0.01$

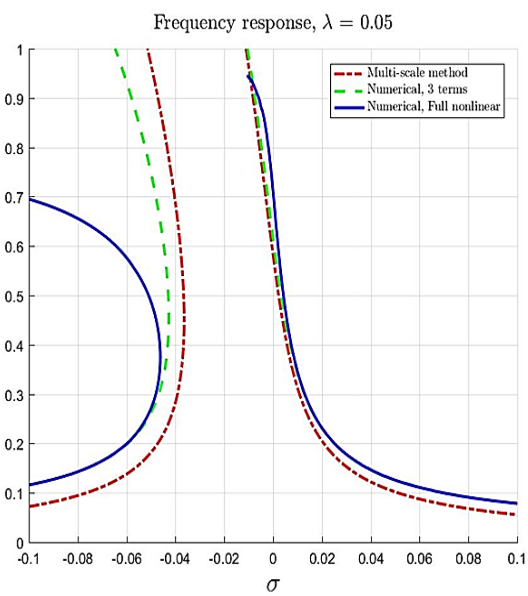


Fig.1.5–Comparable amplitude-frequency response $\Omega \approx \frac{1}{2}, \lambda = 0.05$

As it see from these Figures, at sufficiently small value λ the first approximation to solution is obtained by manyscale method practically coincide with direct numerical solution equation (1.8). The difference of approximated solution from direct numerical solution of initial nonlinear equation (1.7) consists in existence branch of unstable periodic motions at negative values of mismatch parameter σ. The practical sense of this branch consists in that it limits the field of attraction of stable periodic regime with small amplitude of oscillation. The comparable regimes of motion, which are describes by equation of systems (1.7), (1.8) in dependence of initial conditions are shown in Fig.1.6. The comparable amplitude-frequency responses are shown in Fig.1.4.

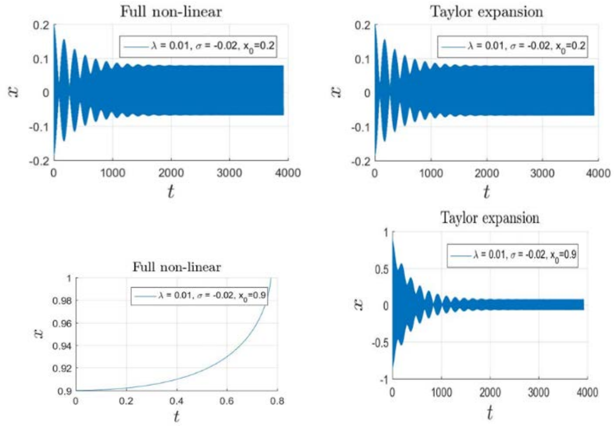


Fig.1.6 – Comparing regimes of motion for systems (1.7), (1.8)

As it see from this Figure, the expanding of right part of equation (1.7) in Taylor series keeping items with order not higher third can leads to wrong conclusion about stability of stationary oscillation motion at large initial perturbation.

More detail let's consider the question of determination stability of founding periodic regime of oscillation. According to common statements of theory stability of motion [4-8], for determination stability of stationary regime it's need to build the equation in variation in vicinity of investigation regime. In case of analysis of periodic oscillation the equation in variation present itself the linear equation with periodic coefficients [9]. The condition of asymptotic stability of periodic motion is location of all multipliers (eigen values of monodromy matrix) inside of unit circle in complex plane. The existence of multiplier with module equal unit corresponds to limited (in common case nonperiodical) motion. Availability of multiplier outside of unit circle presents the criterion of instability of investigated periodic motion.

Using in numerical calculation software package *MATCONT* gives possibility to calculate multiplier in vicinity of every obtained stationary periodic regime of oscillation. This possibility gives us the condition of confirmation the loss of stability of periodic motion of system (1.7) in regular extremal point on the left branch of amplitude-frequency response (Fig. 1.4). The dependence maximum module of multiplier from parameter of mismatch σ is shown in Fig.1.7. Two connected curves in point of intersection corresponds to loss of stability (in this Figure $\lambda = 0{,}05$).

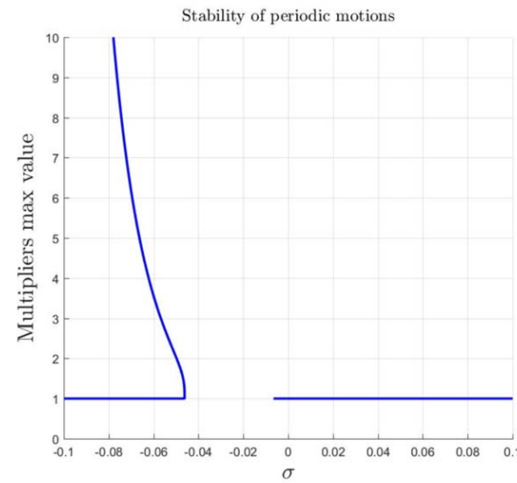


Fig. 1.7 – The dependence of multiplier maximum module from frequency mismatch σ

As it see from this picture disaffiliation of multiplier with unit circle of complex plane take place at the same mismatch that corresponds to inflection of left branch of amplitude-frequency response in Figure 1.5. Independent from σ availability in system multiplier with module equal unit causes by peculiarity of numerical realization in *MATCONT* task of prolongation on parameter periodic motion. Exactly, that corresponding program from *MATCONT* appropriate for investigation of autonomous dynamical system. In connection with it the necessary of build autonomous system equivalent to (1.7) in sense of description of micro-electromechanical oscillator motion. For this non-autonomous item $\cos\Omega t$ in equation (1.7) changed by fixed solution (limit cycle) for next dynamical system [10]:

$$\begin{aligned} \dot{x} &= x + \Omega y - x(x^2 + y^2), \\ \dot{y} &= -\Omega x + y - y(x^2 + y^2), \end{aligned} \tag{1.20}$$

possess asymptotically stable solution

$$x = \sin\Omega t\,, y = \cos\Omega t. \tag{1.21}$$

Thus, current system equation in *MATCONT*, written down in standard form, has a form

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= -\mu x_2 - x_1 - \frac{\lambda x_4^2}{(1-x_1)^2}, \\ \dot{x}_3 &= x_3 + \Omega x_4 - x_3(x_3^2 + x_4^2), \\ \dot{x}_4 &= -\Omega x_3 + x_4 - x_4(x_3^2 + x_4^2) \end{aligned} \tag{1.22}$$

Linear system differential equation with periodic coefficients, obtained from (1.22) by way of build equation in variation relatively stationary periodic solution, has multiplier equal unity, that allows to interpret Fig. 1.7.

Now we investigate the dependence of amplitude fixed oscillation from parameter of excitation λ at given frequency in vicinity of main resonance $\Omega \approx \frac{1}{2}$. The comparing of direct numerical solution of system (1.7) with approximated analytic solution of (1.8) and also numerical solution of corresponded system (1.19) is shown in Fig.1,8

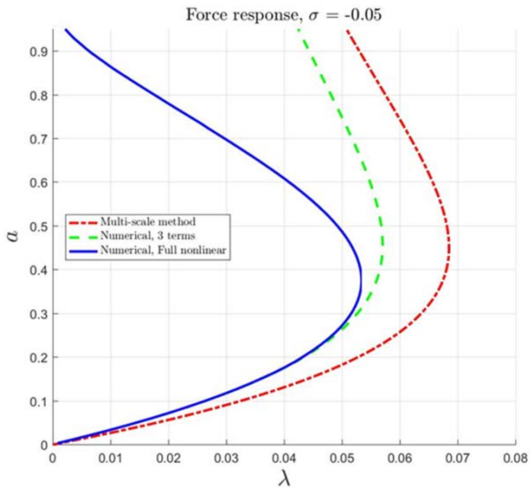


Fig.1.8 – Dependence of oscillation amplitude from parameter λ, $\sigma = -0{,}05$

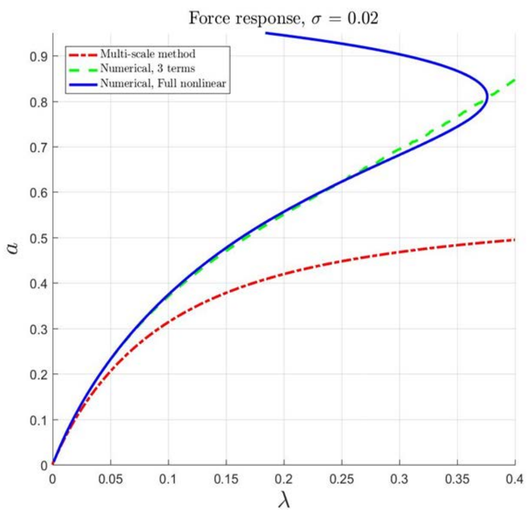


Fig.1.9 – Dependence of oscillation amplitude from parameter λ, $\sigma = 0{,}02$

As it see from Fig.1.8, at frequency smaller resonance ($\sigma < 0$), the dependence amplitude of oscillation from parameter λ in form coincide with obtained early diagram of branching equilibrium positions shown in Figure 1.2 and characterized of existence critical value of parameter λ, which limit to upper the field of existence stationary regimes of oscillation. The maximum amplitude of established periodic motion don't exceed value 0,5.

According to Fig.1.9 in superresonance zone ($\sigma > 0$) the character of dependence amplitude a from parameter λ is another: critical value essentially increase $\lambda \cong 0{,}37$, also maximum amplitude of superresonance zone increase $a \cong 0{,}8$. For comparing critical value of parameter λ at static analysis (*«static pull-in»*), according Fig.1.2 in static case $\lambda_* \cong 0{,}148$ at corresponding value of static displacement $a_* \cong 0{,}32$. Thus in superresonance zone establish oscillation with amplitude greatly exceeding possible amplitude of stable equilibrium positions.

As it see from Fig.1.8, Fig.1.9 approximated analytical solution coincide with numerical at small values of parameter λ, that corresponds to initial assumption of applying manyscale method.

Let's pass to investigation of second method excitation, in which to alternating component of voltage V_{AC} the stationary level V_{DC} is added. Corresponded dynamical system has a view

$$\ddot{x} + \mu\dot{x} + x = \lambda \frac{(1+\upsilon\cos\Omega t)^2}{(1-x)^2} \tag{1.23}$$

This equation is solved numerically That as early with software package *MATCONT*. The amplitude-frequency responses for this system for different values λ are shown in Fig. 1., Fig.1.11 presented for values of mismatch σ in vicinity of main resonance $\Omega = 1 + \sigma$. Here the resonance frequency $\Omega_* = 1$ in difference of first case then resonance frequency $\Omega_* = 1/2$. Dimensionless amplitude of alternating component of voltage $\upsilon = 0{,}01$; coefficient of dissipation $\mu = 0{,}005$.

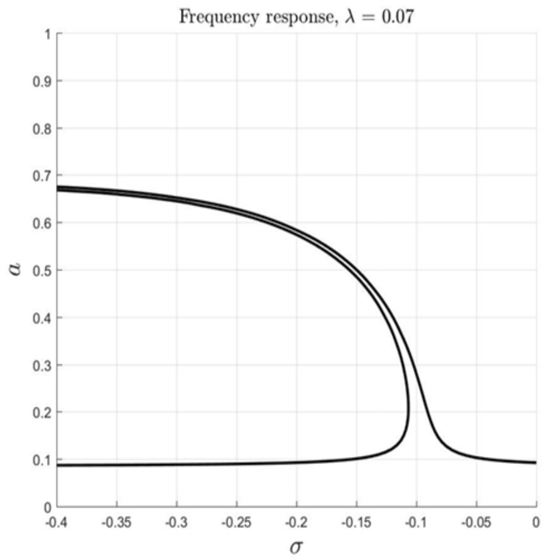
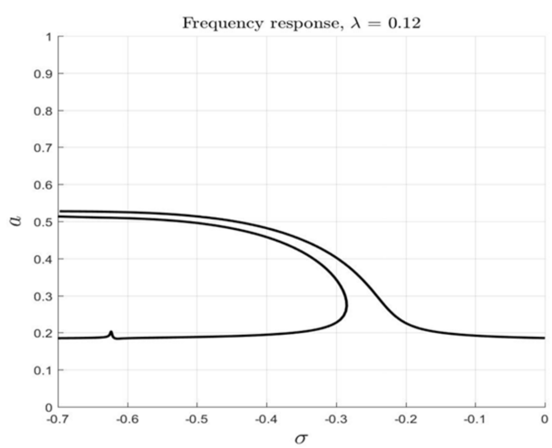


Fig.1.10 – amplitude-frequency response in vicinity of main resonance $\lambda = 0{,}07$



Fig,1.11 – amplitude-frequency response in vicinity of main resonance $\lambda = 0{,}12$

As evident from these Figures, at given values of parameter in system not observe the opportunity going out from regime of stationary oscillations. Small resonance peac in Figure 1.11 at $\Omega \cong \frac{1}{3}$ corresponds to second resonance, produced by presence cubical item of expansion into Tailor series the right part equation.

The dependence of amplitude stationary oscillation from amplitude of alternating external excitation υ at different values of frequency mismatch σ is shown in Fig. 1.12, 1.13

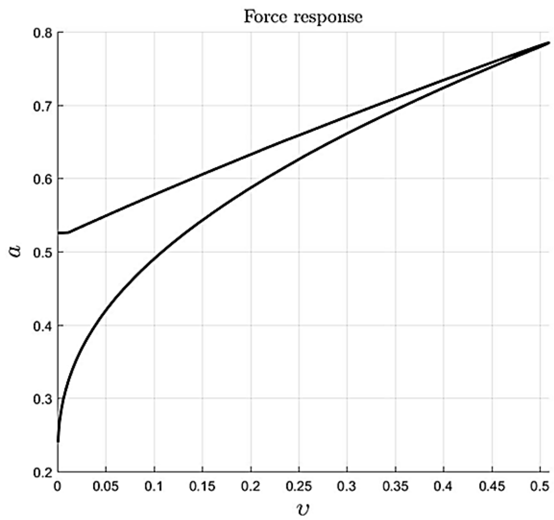


Fig.1.12–Dependence $a(\upsilon), \lambda = 0{,}12, \mu = 0{,}005,$
$\sigma = -0{,}25$

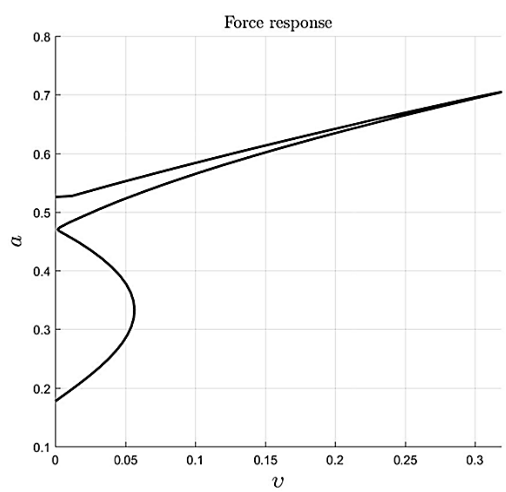


Fig.1.13 – Dependence $a(\upsilon), \lambda = 0{,}12, \mu = 0{,}005,$
$\sigma = -0{,}4$

III System with two fixed electrodes

Let's proceed to consideration of micro-electromechanical oscillator in field of two electrodes. To mobile plate (point mass of oscillator) and fixed electrodes disclose identical potential difference V (Fig.2.1). In assumption of symmetry system the length of unstrained spring l is believed equal zero.

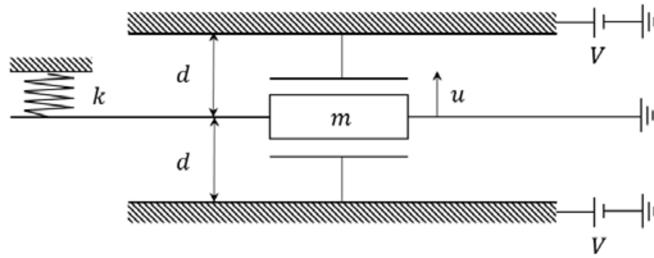


Fig.2.1 – Micromechanical oscillator in field of two electrodes

Equation of oscillator motion is written in next form:

$$m\frac{d^2u}{dt^2} + c\frac{du}{dt} + ku = \frac{1}{2}\frac{\epsilon_r \epsilon_0 S V^2}{(d-u)^2} - \frac{1}{2}\frac{\epsilon_r \epsilon_0 S V^2}{(d+u)^2} \qquad (2.1)$$

or, in dimensionless form

$$\frac{d^2x}{d\tau^2} + \mu\frac{dx}{d\tau} + x = \lambda\left[\frac{1}{(1-x)^2} - \frac{1}{(1+x)^2}\right], \qquad (2.2)$$

Let's begin the investigation of micromechanical oscillator motion in field of two electrodes. Before then consideration task of forced oscillation let's study quality structure of phase plate at accounting of dissipation, which determine by coefficient $\mu > 0$. The phase portraits of this system at different values of parameter λ and fixed μ are shown in Fig.2 2.

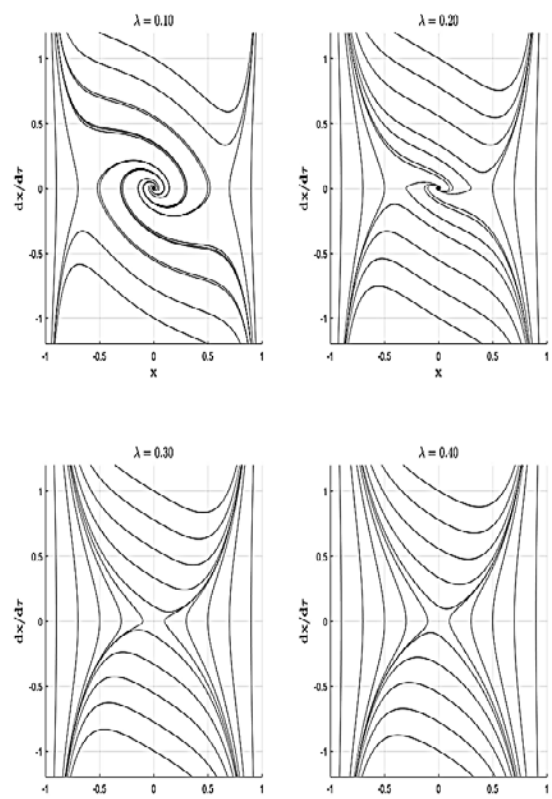


Fig.2.2 – Phase portraits MEMS oscillator accounting damper ($\mu = 0{,}05$)

As it see from this pictures, at $\lambda < 0{,}25$ system has three critical points: stable focus in point of origin and two saddles, situated symmetrically relatively origin point (0,0). At $\lambda = 0{,}25$ critical points flow together and produce one saddle in origin coordinates.

In absent of dissipation in equation (2.2) $\mu = 0$ diagram of branching of equilibrium positions takes a form:

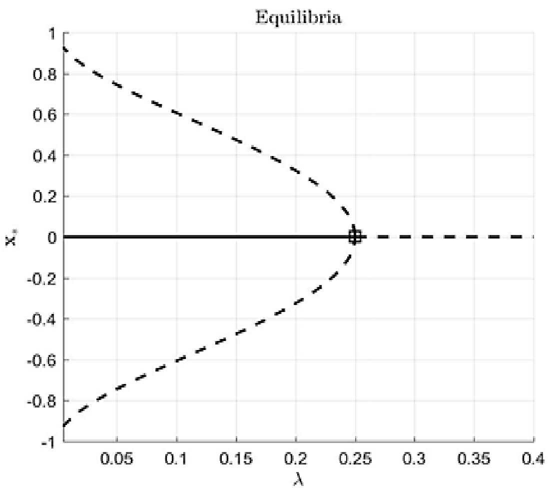


Fig.2.3 – Diagram of branching of equilibrium position

As it see from this Figure, value λ_* corresponds one-sided subcritical bifurcation with disappearance of stable equilibrium positions [11, 12].

Also one main step, premonitory the investigation of dynamical task, present the building branching diagram of equilibrium position for system with two electrodes in the presence of cubic item in expression of elastic force:

$$\frac{d^2x}{d\tau^2} + x + \beta x^3 = \lambda\left[\frac{1}{(1-x)^2} - \frac{1}{(1+x)^2}\right]. \quad (2.3)$$

Accounting of cubic item gives the quality changing of bifurcation diagram at sufficiently large value of parameter β, characterizing relation between gap of capacitor and width of plate. Namely, as it see from (2.3) at increasing β the type of bifurcation is changed from subcritical to supercritical, that gives the branching from zero solution the stable nontrivial equilibrium forms, obtained with help software package *MATCONT* at different values of parameter β, as it shown in Fig. 2.4

As it see from Fig.2.4, the accounting cubic item in expression for nonlinear recreate force allow to quality describe possibility nontrivial equilibrium positions of elastic elements of microsystem technic in field of two electrodes..

Let's pass to investigation micro-electromechanical oscillator under the action of periodic excitation. Taking in account foresaid equation of motion:

$$\ddot{x} + \varepsilon\mu\dot{x} + [1 - 4\varepsilon\lambda\cos^2\Omega t]x + \varepsilon[\beta - 8\lambda\cos^2\Omega t]x^3 = 0 \quad (2.4)$$

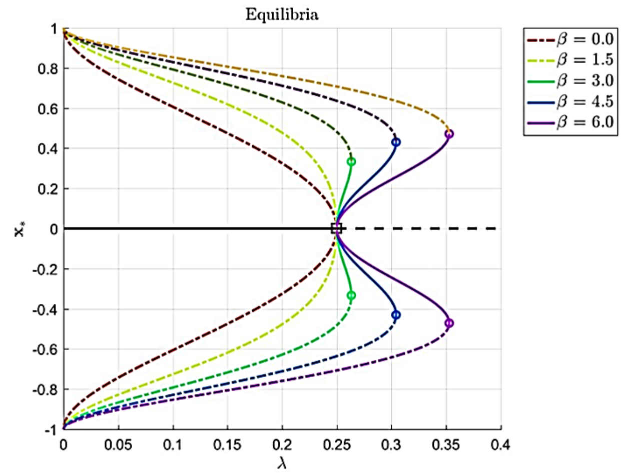


Fig. 2.4 – Diagram of branching equilibrium positions

Thus in contrast to system with one electrode here excitation has pure parametric character. The solution of this equation with periodic coefficients will be search with help asymptotic manyscale method. The solution of equation is written in form

$$x = x_0(T_0, T_1, \dots) + \varepsilon x_1(T_0, T_1, \dots) + \cdots, \quad (2.6)$$

where $T_k = \varepsilon^k t$ – different scale of time, $x_i, i = 0,1,2, \dots$ - sought functions.

Let's build the first approximation. Substituting (2.6) into (2.5) and grouping members on degree of ε, we obtain next system of equation:

$$D_0^2 x_0 + x_0 = 0,$$
$$D_0^2 x_1 + x_1 = -2D_0 D_1 x_0 - \mu D_0 x_0 + 4\lambda\cos^2\Omega T_0\,[x_0 + 2x_0^3\ ] - \beta x_0^3, \dots \quad (2.7)$$

The solution of homogeneous equation for x_0 is written in complex form

$$x_0(T_0, T_1) = A(T_1)e^{iT_0} + (complex\ conjugate), \quad (2.8)$$

Complex amplitude of oscillation is determined as

$$A(T_1) = \frac{1}{2}a(T_1)e^{i\varphi(T_1)}, \quad (2.9)$$

where a и φ – sought functions of slow time T_1.

Let's consider the main parametric resonance, at which frequency of external excitation $\Omega \approx 1$. Parameter of mismatch frequency σ is used by formula

$$\Omega = 1 + \varepsilon\sigma. \quad (2.10)$$

Substituting into right part of second equation of system and writing condition of absent the secular items we obtain next differential equation for complex amplitude:

$$-2iA' - \mu iA + 2\lambda A + \lambda\bar{A}e^{2i\sigma T_1} + 3A^2\bar{A}(4\lambda - \beta) + 2\lambda A^3 e^{-2i\sigma T_1} + 3A\bar{A}^2 e^{2i\sigma T_1} = 0. \quad (2.11)$$

Pass on to exponential form of writing equation and separating real and imaginary part of equation, we obtain next system of differential equations for slowly changing amplitude and phase of oscillation

$$2\frac{da}{dT_1} = -\mu a + \left[\lambda a\left(1 - \frac{a^2}{2}\right) + \frac{3a^2}{4}\right]\sin\gamma,$$
$$a\frac{d\gamma}{dT_1} = 2a\sigma + 2\lambda a + \frac{3a^3}{4}(4\lambda - \beta) + \left[\lambda a\left(1 + \frac{a^2}{2}\right) + \frac{3a^3}{4}\right]\cos\gamma. \quad (2.12)$$

Here

$$\gamma = 2\sigma T_1 - 2\varphi.$$

Critical points of autonomous system (2.12) corresponds to stationary oscillations. These points satisfy algebraic system

$$\mu a = \left[\lambda a\left(1 - \frac{a^2}{2}\right) + \frac{3a^2}{4}\right]\sin\gamma,$$
$$-2a\sigma - 2\lambda a - \frac{3a^3}{4}(4\lambda - \beta) = \left[\lambda a\left(1 + \frac{a^2}{2}\right) + \frac{3a^3}{4}\right]\cos\gamma \quad (2.13)$$

Exclusion of variable γ leads to equation implicitly defining dependence of amplitude of stationary oscillation from frequency mismatch $a(\sigma)$:

$$\left[\frac{\mu a}{\lambda a\left(1-\frac{a^2}{2}\right)+\frac{3a^3}{4}}\right]^2 + \left[\frac{2a\sigma + 2\lambda a + \frac{3a^3}{4}(4\lambda - \beta)}{\lambda a\left(1+\frac{a^2}{2}\right)+\frac{3a^3}{4}}\right]^2 = 1. \quad (2.14)$$

The amplitude-frequency response calculated by formula (2.14) for different values of geometric nonlinear parameter β at fixed values $\lambda = 0{,}1$, $\mu = 0{,}005$, $\varepsilon = 1$ is shown in Fig. 2.5

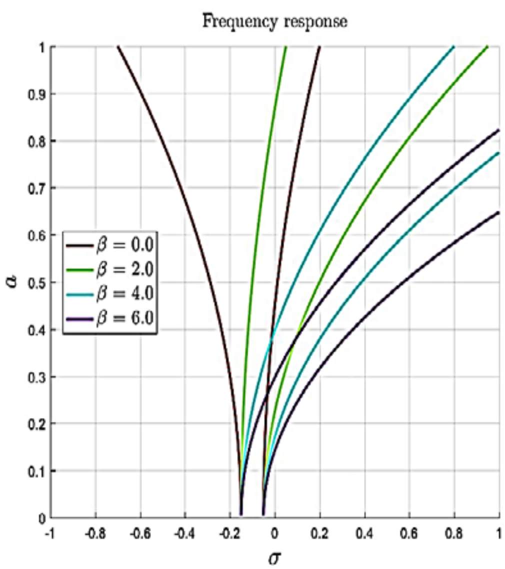


Fig. 2.5 – Amplitude-frequency response in vicinity of main parametric resonance

The observing character of dependence amplitude of establish oscillation a from frequency mismatch σ corresponds to common properties of parametric oscillation [9]: outside «zone of synchronization» oscillations caused perturbation in initial condition damp from presence of dissipation forces. Inside of synchronization zone the zero equilibrium position became unstable and the character of motion is determined by nonlinear items in equation.. In contrast to linear system with periodic coefficient, for which according to Floquet-Lyapunov theory periodic oscillations only separate the field of limited and unlimited solutions, for considerate nonlinear system the zone of synchronization can present the field of existence of stationary periodic motion.

The regimes of initial nonlinear system (2.4) at different values of frequency mismatch σ and parameter of geometrical nonlinearity β is shown in Fig. 2.6. The another parameters are fixed: $\lambda = 0.1, \mu = 0.005, \varepsilon = 1$.

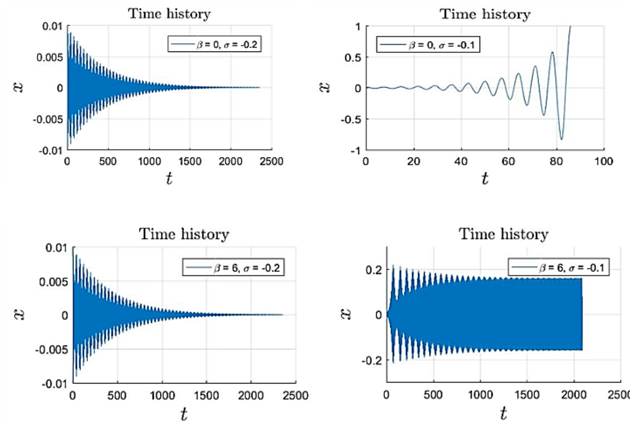


Fig. 2.6 – Regimes of motions system

As it see from these pictures, at $\beta = 0$ and $\sigma = -0{,}1$ amplitude of oscillations increases without limit, that quality corresponded to parametric resonance in linear system. At $\beta = 6$ in zone of synchronization ($\sigma = -0{,}1$) stationary oscillation with amplitude $a \cong 0{,}17$ is observe; outside of this zone ($\sigma = -0{,}2$) oscillation is damper. Thus the character of motions initial nonlinear system corresponds approximated analytic results, obtained by manyscale methods (see Fig.2.6). According to obtained results the factor of geometrical nonlinearity is determined possibility or impossibility of collapsing plates of capacitor (*“pull-in”*) at parametric excitation of elastic element MEMS.

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