Effects of different crack-face boundary conditions on the dynamic intensity factors in functionally graded piezoelectric solids

Michael Wünsche, Jan Sladek, Vladimir Sladek and Chuanzeng Zhang

Abstract—Piezoelectric materials have the capability to convert electric energy into mechanical energy and vice versa. Functional gradations of the material properties are getting increasing attention in advanced engineering applications. Accurate and efficient numerical methods are required for the simulation and safety analysis of such structures. In this paper, the transient dynamic analysis of functionally graded piezoelectric composites with cracks is presented. A time-domain boundary element method (BEM) is developed for this purpose. The present BEM uses the collocation method for the spatial discretization of the time-domain boundary integral equations (BIEs), while the convolution quadrature method is applied to the temporal discretization. An iterative solution algorithm is implemented to solve the non-linear semi-permeable electric crack-face boundary condition. The investigated numerical examples indicate a significant influence of the electric crack-face boundary condition, the functionally gradation and the transient dynamic loading on the normalized intensity factors.

Keywords—functionally graded piezoelectric solids, non-linear crack-face boundary condition, dynamic intensity factors.

I. Introduction

Piezoelectric materials offer advanced possibilities in modern engineering structures due to their inherent coupling effects between the mechanical and electric fields. They are widely applied in smart devices and structures like transducers, actuators and sensors. In recent years composites with a continuously change of the material properties are getting increasing attention. Such functionally graded composite materials (FGMs) can be designed to satisfy the most beneficial mechanical and electric properties. An important advantage over conventional laminates is that interfaces and stress discontinuities are avoided. Piezoelectric ceramics are very brittle and have a low fracture toughness. Therefore, the dynamic crack analysis has a considerable importance to the fracture and damage mechanics, design and optimization as well as non-destructive testing of piezoelectric structures. An important role in the crack analysis of piezoelectric materials plays the formulation of the electric crack-face boundary conditions. A more realistic non-linear electric crack-face boundary condition has been presented taking into account the electric permittivity of the medium inside the crack [9].

Since analytical solutions of the corresponding initial boundary value problem are available only for very simple crack geometries, loading configurations and linear crack-face boundary conditions, efficient numerical methods are needed to solve more general problems. The finite element method (FEM) has been used to solve various problems with continuously graded material properties by many authors [10], [11]. Meshless methods have been presented to analyze functionally graded material for example by [2], [12]. The boundary element method (BEM) has been developed for the transient dynamic crack analysis of piezoelectric solids by García-Sánchez et al. [7] and Wünsche et al. [15]. The extension of the BEM to non-homogeneous materials is rather limited since the corresponding fundamental solutions are either mathematically very complicated or not available. Fundamental solutions for heat conduction problems in exponentially graded materials have been derived by Gray et al. [8]. Chan et al. [1] presented the fundamental solutions for a two-dimensional exponentially graded elastic isotropic medium. To avoid the use of the specific fundamental solutions for functionally graded materials Gao et al. [6] have presented a boundary-domain integral equation formulation. This formulation has been used by Ekhlakov et al. [4] to develop a Laplace-domain BEM for the transient thermoelastic crack analysis in 2D functionally graded isotropic materials. Yang et al. [16] have presented the BEM for the free vibration analysis of 2D functionally graded isotropic materials by using static fundamental solutions for homogeneous isotropic materials. A boundary integral equation method (BIEM) for the analysis of time-harmonic crack problems in functionally graded piezoelectric solids have been derived by Dineva et al. [3].

In this paper, transient dynamic crack analysis in two-dimensional, functionally graded piezoelectric solids is presented. Since special fundamental solutions for such materials are not available a boundary-domain integral formulation is developed. To solve the boundary integral equations numerically a spatial collocation method and the convolution quadrature method for temporal discretization are used. This requires the Laplace transformed fundamental solutions for homogeneous piezoelectric materials. The radial integration method is implemented to compute the corresponding domain integrals. An explicit time-stepping scheme is obtained to compute the unknown boundary data. An iterative algorithm is developed to solve the non-linear semi-permeable electric crack-face boundary condition. Numerical examples will be presented and discussed to show the effects of the electric crack-face boundary conditions, the material gradation and the transient dynamic loadings on the dynamic intensity factors.

M. Wünsche, J. Sladek and V. Sladek are with the Slovak Academy of Sciences, Institute of Construction and Architecture, 84503 Bratislava, Slovakia

Ch. Zhang is with University of Siegen, Faculty of Science and Technology, Chair of Structural Mechanics, D-57068 Siegen, Germany

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II. Problem statement

We consider a two-dimensional, continuously non-homogeneous and linear piezoelectric cracked solid. In the absence of body forces, free electric charges and applying the quasi-electrostatic assumption, the cracked solid satisfies the generalized equations of motion

$$\sigma_{iJ,i}(\mathbf{x},t) = \rho\delta^*_{JK}\ddot{u}_K(\mathbf{x},t),\ \ \delta^*_{JK} = \begin{cases} \delta_{jk}, & J=j;\ K=k, \\ 0, & \text{otherwise}, \end{cases} \tag{1}$$

and the generalized constitutive equations

$$\sigma_{iJ}(\mathbf{x},t) = c_{iJKl}(\mathbf{x})u_{K,l}(\mathbf{x},t), \tag{2}$$

with

$$c_{iJKl}(\mathbf{x}) = \Theta(\mathbf{x})c^0_{iJKl}, \tag{3}$$

where ρ is the mass density, δ^*_{JK} is the generalized Kronecker delta and $\Theta(\mathbf{x})$ prescribes the spatial variation of the material properties. The generalized displacements u_I, the generalized stresses σ_{iJ} and the generalized elasticity tensor c^0_{iJKl} are defined by

$$u_I = \begin{cases} u_i, & I=i \quad \text{(mechanical displacements)} \\ \varphi, & I=4 \quad \text{(electric potential)} \end{cases}, \tag{4}$$

$$\sigma_{iJ} = \begin{cases} \sigma_{ij}, & J=j \quad \text{(mechanical stresses)} \\ D_i, & J=4 \quad \text{(electric displacements)} \end{cases}, \tag{5}$$

$$c^0_{iJKl} = \begin{cases} c_{ijkl}, & J=j;\ K=k \quad \text{(elasticity tensor)} \\ e_{lij}, & J=j;\ K=4 \quad \text{(piezoelectric tensor)} \\ e_{ikl}, & J=4;\ K=k \quad \text{(piezoelectric tensor)} \\ -\kappa_{il}, & J=K=4 \quad \text{(electric permittivity tensor)} \end{cases}. \tag{6}$$

A comma after a quantity represents spatial derivatives while a dot over a quantity denotes time differentiation. Lower case Latin indices take the values 1 and 2 (elastic), while capital Latin indices take the values 1, 2 (elastic) and 4 (electric). Further the following initial conditions

$$u_I(\mathbf{x},t=0) = \dot{u}_I(\mathbf{x},t=0) = 0 \tag{7}$$

and the boundary conditions on the external boundary

$$u_I(\mathbf{x}) = \bar{u}_I(\mathbf{x}),\ \ \mathbf{x}\in\Gamma_u, \tag{8}$$

$$t_I(\mathbf{x}) = \bar{t}_I(\mathbf{x}),\ \ \mathbf{x}\in\Gamma_t \tag{9}$$

are considered. Here Γ_t and Γ_u are the external boundaries where the generalized tractions t_I and the generalized displacements u_I are prescribed.

On the upper and the lower crack-faces Γ_{c^+} and Γ_{c^-} self-equilibrated generalized tractions are considered. Three different electric crack-face boundary conditions are applied [9], [15]. Taking into account the electric permittivity $\kappa_c = \kappa_r \cdot 8.854\cdot 10^{-12}$C/(Vm) of a medium inside the crack, the semi-permeable crack-face boundary condition may be defined by

$$D_n(\mathbf{x}\in\Gamma_{c^+},t) = D_n(\mathbf{x}\in\Gamma_{c^-},t) = -\kappa_c\frac{\varphi(\mathbf{x}\in\Gamma_{c^+},t)-\varphi(\mathbf{x}\in\Gamma_{c^-},t)}{u_n(\mathbf{x}\in\Gamma_{c^+},t)-u_n(\mathbf{x}\in\Gamma_{c^-},t)}, \tag{10}$$

with κ_r being the relative electric permittivity of the medium inside the crack, u_n and D_n are the normal components of the displacements and electric displacements on the crack-faces. If both crack-faces are considered as electrically impermeable, which means $\kappa_c = 0$, the Eq. (10) simplifies to

$$D_n(\mathbf{x}\in\Gamma_{c^+},t) = D_n(\mathbf{x}\in\Gamma_{c^-},t) = 0. \tag{11}$$

In the opposite case both crack-faces are treated as electrically permeable, which implies $\kappa_c = \infty$, Eq. (10) reduces to

$$\begin{aligned} &D_n(\mathbf{x}\in\Gamma_{c^+},t) = D_n(\mathbf{x}\in\Gamma_{c^-},t), \\ &\varphi(\mathbf{x}\in\Gamma_{c^+},t) - \varphi(\mathbf{x}\in\Gamma_{c^-},t) = 0. \end{aligned} \tag{12}$$

The generalized crack-opening-displacements can be written as

$$\Delta u_I(\mathbf{x},t) = u_I(\mathbf{x}\in\Gamma_{c^+},t) - u_I(\mathbf{x}\in\Gamma_{c^-},t). \tag{13}$$

III. Time-domain boundary integral equations

The initial boundary value problem is formulated as BIEs in the following. Fundamental solutions for functionally graded piezoelectric materials are not available. Therefore, a boundary-domain integral formulation is derived. Using the procedure shown in Gao et al. [6] the BIEs can be expressed by

$$\begin{aligned} &c_{IJ}\tilde{u}_J(\mathbf{x},t) = \\ &\int_\Gamma \Big[u^G_{IJ}(\mathbf{x},\mathbf{y},t)*t_J(\mathbf{y},t) - t^G_{IJ}(\mathbf{x},\mathbf{y},t)*\tilde{u}_J(\mathbf{y},t)\Big]\mathrm{d}\Gamma_y \\ &+\int_\Omega h^G_{IJ}(\mathbf{x},\mathbf{y},t)*\tilde{u}_J(\mathbf{y},t)\mathrm{d}\Omega_y. \end{aligned} \tag{14}$$

In Eq. (14) an asterisk "$*$" denotes the Riemann convolution approximated by the convolution quadrature method [7] and $\tilde{u}_J(\mathbf{x},t) = \Theta(\mathbf{x})u_J(\mathbf{x},t)$ are the normalized displacements. The free term c_{IJ} is defined by $c_{IJ} = \frac{1}{2}\delta_{IJ}$ for the source point $\mathbf{x}$ on the smooth boundary Γ and $c_{IJ} = \delta_{IJ}$ for $\mathbf{x}$ inside the domain Ω. Further $u^G_{IJ}(\mathbf{x},\mathbf{y},t)$ is the generalized displacement fundamental solution and

$$t^G_{IJ}(\mathbf{x},\mathbf{y},t) = c^0_{qIKr}e_q(\mathbf{y})u^G_{KJ,r}(\mathbf{x},\mathbf{y},t), \tag{15}$$

$$h^G_{IJ}(\mathbf{x},\mathbf{y},t) = c^0_{qIKr}\Theta_{,q}(\mathbf{y})u^G_{KJ,r}(\mathbf{x},\mathbf{y},t). \tag{16}$$

The Laplace-domain fundamental solutions for homogeneous and linear piezoelectric solids can not be expressed in an explicit form [13]. By applying the Radon-transform technique they can be represented in the 2D case by a line integral over the unit-circle as

$$u^G_{IJ}(\mathbf{x},\mathbf{y},p) = \frac{1}{8\pi^2}\int_{|\mathbf{n}|=1}\sum_{m=1}^{M}\frac{P^m_{IJ}}{\rho c_m^2}\Psi\left(\frac{p}{c_m},\big|\mathbf{n}\cdot(\mathbf{y}-\mathbf{x})\big|\right)\mathrm{d}\mathbf{n}, \tag{17}$$

$$\begin{aligned} &\Psi(\xi) = \Big[\mathrm{e}^{\xi}\mathrm{Ei}\big(-\xi\big) + \mathrm{e}^{-\xi}\mathrm{Ei}\big(\xi\big)\Big], \\ &\xi = \left(\frac{p}{c_m}\big|\mathbf{n}\cdot(\mathbf{y}-\mathbf{x})\big|\right), \end{aligned} \tag{18}$$

where p, Ei, $\mathbf{n}$, c_m and P^m_{IJ} are the Laplace parameter, the complex exponential integral, the wave propagation vector, the phase velocities of the elastic waves and the projection operator as given in [13], [14]. The fundamental solutions can be divided into a static part and a dynamic part

$$u_{IJ}^{G}(\mathbf{x},\mathbf{y},p) = u_{IJ}^{S}(\mathbf{x},\mathbf{y}) + u_{IJ}^{D}(\mathbf{x},\mathbf{y},p). \tag{19}$$

IV. Numerical solution procedure

To solve the boundary integral equations (14) numerically the boundary Γ is discretized by quadratic elements. Quarter-point elements are used at the crack-tips to describe the local square-root behavior of the crack-opening-displacements properly. The domain integrals are transformed into equivalent boundary integrals with the radial integration method [5] to avoid an additional mesh. For this purpose only internal nodes inside the domain Ω are required. A fourth order spline-type radial basis function is used.

To get a solvable system of linear algebraic equations the boundary integral equations (14) are written for all boundary nodes and internal nodes. After temporal and spatial discretizations the two systems of equations are obtained

$$\begin{aligned}\mathbf{C}\tilde{\mathbf{u}}_b^K &= \mathbf{U}^S\mathbf{t}^K - \mathbf{T}^S\tilde{\mathbf{u}}_b^K + \mathbf{H}^S\tilde{\mathbf{u}}_i^K\\ &+\sum_{k=1}^{K}\Big[\mathbf{U}^{D;K-k+1}\mathbf{t}^k - \mathbf{T}^{D;K-k+1}\tilde{\mathbf{u}}_b^k + \mathbf{H}^{D;K-k+1}\tilde{\mathbf{u}}_i^k\Big],\end{aligned} \tag{20}$$

$$\begin{aligned}\mathbf{I}\tilde{\mathbf{u}}_b^K &= \mathbf{U}^S\mathbf{t}^K - \mathbf{T}^S\tilde{\mathbf{u}}_b^K + \mathbf{H}^S\tilde{\mathbf{u}}_i^K\\ &+\sum_{k=1}^{K}\Big[\mathbf{U}^{D;K-k+1}\mathbf{t}^k - \mathbf{T}^{D;K-k+1}\tilde{\mathbf{u}}_b^k + \mathbf{H}^{D;K-k+1}\tilde{\mathbf{u}}_i^k\Big].\end{aligned} \tag{21}$$

Here, $\mathbf{u}_b$ and $\mathbf{t}$ are the vectors of the generalized displacements and tractions on the whole boundary, while $\mathbf{u}_i$ is the vector of the generalized displacements inside the domain. $\mathbf{U}$, $\mathbf{T}$ and $\mathbf{H}$ are the corresponding matrices of the Laplace-domain fundamental solutions arising in Eq. (14). The superscripts S and D indicate the static and the dynamic parts. The diagonal matrices $\mathbf{C}$ and $\mathbf{I}$ result from the free term on the left side. Both equations can be summarized into a common system of linear algebraic equations. By invoking the boundary conditions (8) and (9), the following explicit time-stepping scheme is obtained

$$\mathbf{x}^K = (\mathbf{G}^1)^{-1}\left[\mathbf{F}^1\mathbf{y}^K + \sum_{k=1}^{K-1}\Big(\mathbf{B}^{K-k+1}\mathbf{t}^k - \mathbf{A}^{K-k+1}\mathbf{u}^k\Big)\right], \tag{22}$$

where $\mathbf{y}^K$ defines the vector of the prescribed boundary data and $\mathbf{x}^K$ is the vector of the unknown boundary and interior data.

Since special crack-tip shape functions are implemented in the present time-domain BEM to describe the local square-root behavior of the generalized crack-opening-displacements at the crack-tips properly, the dynamic intensity factors are obtained in a direct and accurate manner without special techniques from the numerically computed generalized crack-opening-displacements at the closest node to the crack-tip using

$$\begin{Bmatrix} K_{II}(t)\\ K_I(t)\\ K_{IV}(t)\end{Bmatrix} = \sqrt{\frac{2\pi}{l_e}}\mathbf{H}\begin{Bmatrix}\Delta u_1(l_e,t)\\ \Delta u_2(l_e,t)\\ \Delta\varphi(l_e,t)\end{Bmatrix}. \tag{23}$$

In the Eq. (23), l_e is the distance between the crack-tip and the closest node, K_I and K_{II} are the mode-I and mode-II stress intensity factors respectively, K_{IV} is the electric displacement intensity factor and the matrix $\mathbf{H}$ is defined in [14].

V. Numerical examples

To show the effects of the material gradation, the transient dynamic loading and the crack-face boundary conditions on the dynamic intensity factors, a numerical example is investigated. For the convenience of the presentation the dynamic intensity factors are normalized by

$$\begin{aligned} K_I^*(t) &= \frac{K_I(t)}{K_0},\quad K_{II}^*(t) = \frac{K_{II}(t)}{K_0},\\ K_{IV}^*(t) &= \frac{e_{22}}{\kappa_{22}}\frac{K_{IV}(t)}{K_0},\end{aligned} \tag{24}$$

with $K_0 = \sigma_0\sqrt{\pi a}$ and a being the half-length of an internal crack. To measure the intensity of the electric impact, the following loading parameter is introduced

$$\chi = \frac{e_{22}}{\kappa_{22}}\frac{D_0}{\sigma_0}, \tag{25}$$

where σ_0 and D_0 are the loading amplitudes. Plain strain condition is assumed in all computations.

As example let us consider a rectangular plate with a central crack of length $2a$ subjected to a combined impact tensile loading $\sigma(t) = \sigma_0\mathrm{H}(t)$ and impact electric loading $D(t) = D_0\mathrm{H}(t)$ on the left and the right boundary, as shown in the Fig. 1. $\mathrm{H}(t)$ denotes the Heaviside step function.

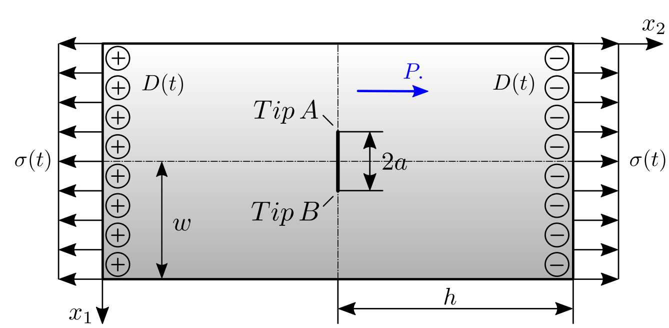

Fig. 1. A rectangular plate with a central crack and functionally gradation in the x_1-direction

The geometry is determined by $h = 20.0mm$, $2w = h$ and $2a = 4.8mm$. As material a piezoelectric Zirconate Titanate (PZT-5H) is chosen, which has the linear material constants

$$\begin{aligned} &c_{11}^0 = 126.0\,\mathrm{GPa},\ \ c_{12}^0 = 84.1\,\mathrm{GPa},\\ &c_{22}^0 = 117.0\,\mathrm{GPa},\ \ c_{66}^0 = 23.0\,\mathrm{GPa},\\ &e_{21}^0 = -6.5\,\mathrm{C/m^2},\ \ e_{22}^0 = 23.3\,\mathrm{C/m^2},\\ &e_{16}^0 = 17.0\,\mathrm{C/m^2},\\ &\kappa_{11}^0 = 15.04\,\mathrm{C/(GVm)},\ \ \kappa_{22}^0 = 13.0\,\mathrm{C/(GVm)}\end{aligned} \tag{26}$$

and the mass density $\rho = 7500\,\mathrm{kg/m^3}$. The gradation of the material in the x_1-direction is defined by the exponential law

$$c_{IJ}(\mathbf{x}) = c^0_{IJ}\mathrm{e}^{\beta x_1}, \ \ \beta = \frac{1}{2w}\ln[\alpha], \ \ \alpha = \frac{c_{IJ}(x_1 = 2w)}{c^0_{IJ}(x_1 = 0)}. \tag{27}$$

The spatial discretization of the external boundary is done by an element-length of 2.0mm and the crack is divided into 6 elements. Inside the domain uniform distributed nodes with the distance of 2.0mm are used. A normalized time-step of $c_L\Delta t/h = 0.04$ is used in all computations, where $c_L = \sqrt{c^0_{22}/\rho}$ is the quasi-longitudinal wave velocity. The numerical results of the developed TDBEM obtained for the electromechanical loading $\chi = 0.5$ and different material gradations α are shown in Figs. 2 and 3.

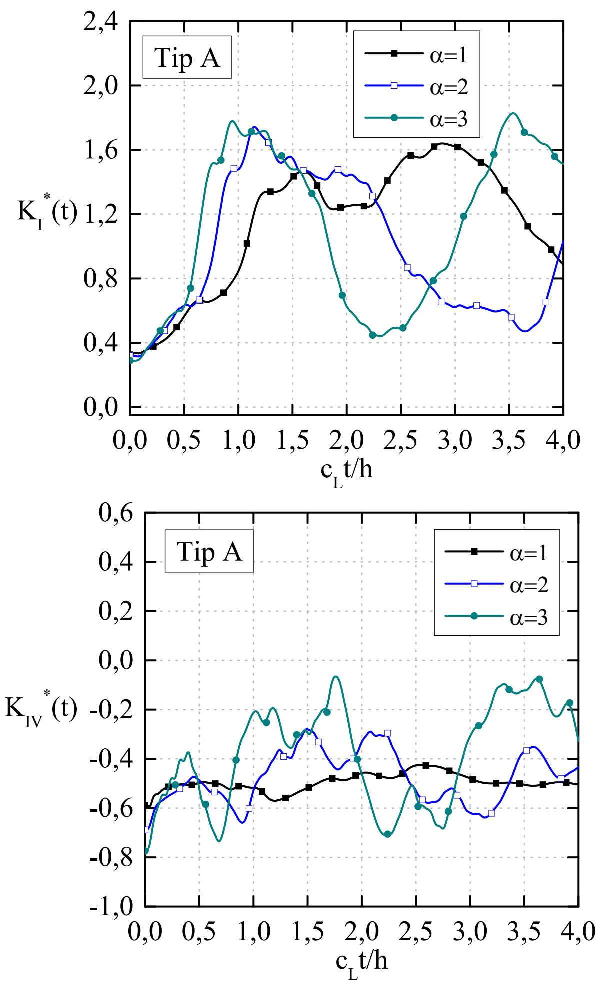

Fig. 2. Normalized dynamic intensity factors of Tip A for the electromechanical loading $\chi = 0.5$ and different gradations α

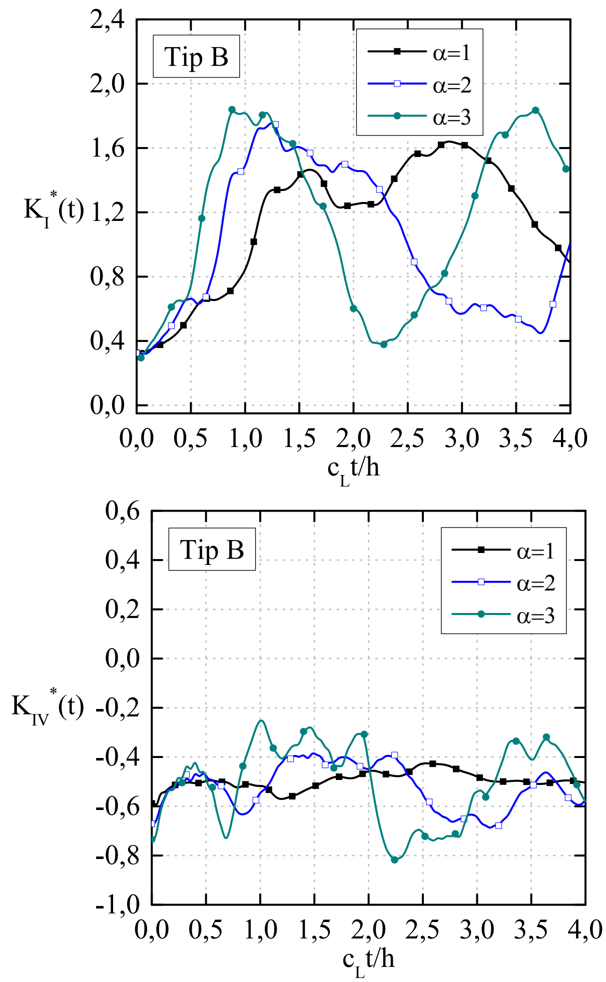

Fig. 3. Normalized dynamic intensity factors of Tip B for the electromechanical loading $\chi = 0.5$ and different gradations α

Due to the quasi-static assumptions of the electric field, which implies that the cracked plate is immediately subjected to an electric impact, the normalized dynamic intensity factors start from a non-zero value for the investigated combined loading. The elastic waves induced by the mechanical impact need some time to reach the crack and after that the dynamic intensity factors increase rapidly until their maximum peak values. The dynamic mode-II intensity factor vanishes, since no shear stress components are induced by the investigated loading. Both Figs. 2 and 3 indicate a significant influence of the functionally gradation on the normalized dynamic mode-I and mode-IV intensity factors. The dynamic mode-I stress intensity factor for the lower crack-tip is larger than that for the upper crack-tip in the case of the functionally graded material. In contrast, the mode-I stress intensity factors are identical at both crack-tips for the homogeneous material due to the symmetry conditions.

The normalized dynamic intensity factors obtained by the present time-domain BEM for the impermeable (ip.), permeable (p.) and semi-permeable (sp.) crack-face boundary conditions are presented in Fig. 4. Vacuum is assumed inside the internal crack for the computations using the semi-permeable crack-face boundary condition.

The electric permittivity has a significant influence on the normalized dynamic mode-IV. By applying the permeable crack-face boundary condition the crack does not exist for the electric field and therefore the curve of the mode-IV intensity factor has a similar behavior as that for the mode-I. In contrast, the mode-IV intensity factor depends only weakly on the time for the impermeable crack-face boundary condition. In a static

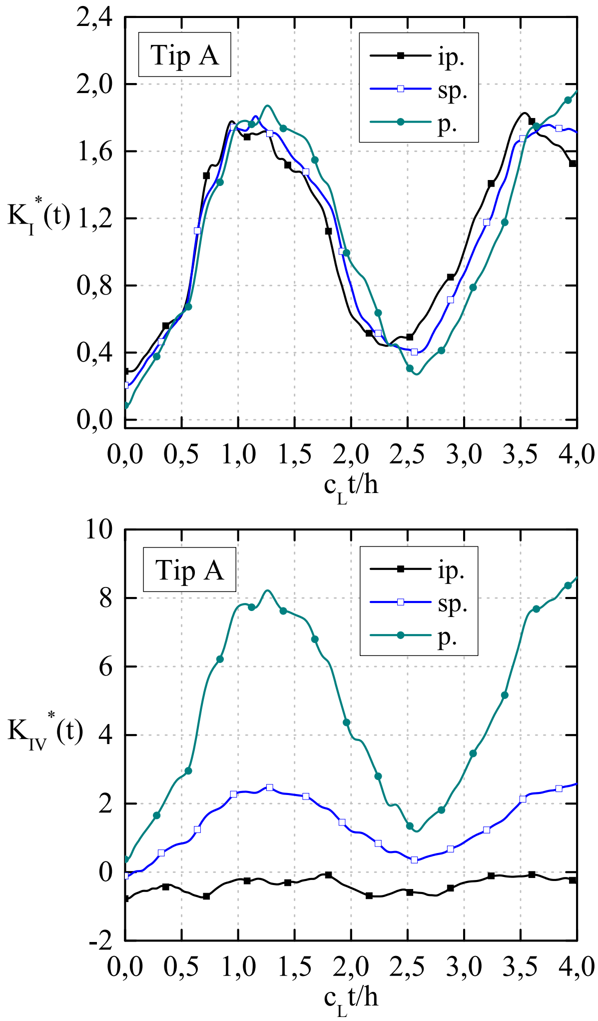

Fig. 4. Comparison of the normalized dynamic intensity factors for different electric crack-face boundary conditions

loading case, the impermeable crack-face boundary condition leads to the strongest possible electric crack-tip field, while an opposite tendency could be induced by a dynamic loading. As well expected, the results of the semi-permeable crack are between the bounds given by the impermeable and the permeable crack-face boundary conditions.

VI. Conclusions

An improved 2D transient dynamic crack analysis in functionally graded piezoelectric solids is presented in this paper. A time-domain BEM is developed for this purpose. The spatial discretization is performed by the collocation method while the convolution quadrature method is adopted for the temporal discretization. The domain integrals are transformed into equivalent boundary integrals by the radial integration method. An iterative algorithm is implemented to solve the non-linear electrically semi-permeable crack-face boundary condition. Numerical examples demonstrate the suitability of the present time-domain BEM for the transient dynamic crack analysis. The results indicate the necessity of the non-linear electric crack-face boundary conditions for a more realistic crack analysis.

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