ARMA Model Parameter Estimation of Non-Gaussian Processes using QR Factorization

Adnan M. Al-Smadi and Othman M. Alsmadi

Abstract—This paper presents an approach to time series analysis of a general Autoregressive moving average (ARMA) model parameters estimation. The proposed technique is based on the QR factorization of a covariance matrix of a third order cumulants from only the output sequence. The system is driven by an independent and identically distributed (i.i.d.) non-Gaussian sequence. Simulation results are presented which demonstrate the performance of the proposed method compared to others.

Keywords—QR factorization, ARMA model, cumulant, parameters estimation.

I. Introduction

A TIME series is a sequence of taken at successive equally spaced points in time order. Time series analysis is about the study of data collected to extract meaningful statistics of the data. The Autoregressive moving average (ARMA) models have been widely used to predict time series data [1]. The problem of estimating ARMA parameters has attracted much attention because of its broad applications to many fields such as signal modeling, system identification, adaptive control, and biomedical signal processing.

Various methods have been developed for estimating the parameters of a general ARMA model [2, 3]. Most of these algorithms are based on second-order statistics. The use of second-order statistical information is motivated by the implicit assumption that the processes are Gaussian. However, most real world signals are non-Gaussian. Hence, second-order based techniques often have serious difficulties in practice [4]. That is, second-order spectral estimators are inherently phase blind and sensitive to additive noise.

Signal processing with higher order statistics (HOS), known as cumulants, has attracted considerable attention in the literature [4, 5, 6, 7]. In non-Gaussian process identification, the corrupted cumulants can be used as important information because cumulants are generally asymmetric functions of their arguments, as such carry phase information about the ARMA transfer function. Therefore cumulant statistics can capture the non-minimum phase information in the available signal.

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QR factorization (also called QR decomposition) is often used to solve the linear least squares problems and generalized linear regression problems [8]. The idea of the QR factorization is the decomposition of a matrix into two factors: Q and R in which the first one is orthogonal matrix and the second is upper right triangular matrix (with nonzero diagonal elements). The good point about the QR factorization is that it always exits; i.e., whatever matrix one has, the matrix can be decomposed into Q and R factored.

In this paper, we present an approach to estimate the parameters of a general ARMA(p,q) process using the QR factorization of a special covariance matrix formed from the third order cumulants of the output sequence only. Section 2 presents the formulation of the problem. Simulation examples are discussed in Section 3. Section 4 draws some conclusion remarks.

II. Problem Formulation

Let $f(n)$ denote the input sequence to a discrete-time linear system, $y(n)$ denote the observed output, then the input/output relationship between two sequences $f(n)$ and $y(n)$ in the form of the difference equation is as follows:

$$y(n) = -a_1 y(n-1) - a_2 y(n-2) - \cdots - a_p y(n-p) + b_0 f(n) + b_1 f(n-1) + \cdots + b_q f(n-q) \quad (1)$$

The system in Eq. (1) produces an ARMA (p, q) process. The a_i's are the autoregressive (AR) parameters and the b_i's are the moving average (MA) parameters; p is the order of AR part while q is the order of the MA part. The relation in (1) can be written as

$$\sum_{i=0}^{p} a_i y(n-i) = \sum_{i=0}^{q} b_i f(n-i) \ ; a_0 = 1 \quad (2)$$

Now, the signal The signal $y(n)$ is observed in additive Gaussian noise, $w(n)$:

$$y_0(n) = y(n) + w(n) \tag{3}$$

Assuming that $y(k)$ is stationary, the third-order cumulant of the noisy output $y_0(n)$ is given by

$$C_{y0}^3(n,m) =$$

$$E[y_0(k)y_0(k+n)y_0(k+m)] = C_y^3(n,m) \tag{4}$$

and the cross-cumulant between y(k) and f(k) is given by

$$C_{fyy}^3(n,m) = E[f(k)y(k+n)y(k+m)] \tag{5}$$

Multiplying both sides of Eq. (1) by $y(k+n)y(k+m)$ and taking expectation results in

$$c_{3y}(n,m) + a_1 c_{3y}(n+1,m+1) + \cdots + a_p c_{3y}(n+p,m+p) =$$

$$c_{fyy}(n,m) + b_1 c_{fyy}(n+1,m+1) + \cdots + b_q c_{fyy}(n+q,m+q) \tag{6}$$

By stacking Equation (6) for several of n and m of the third order cumulants to be used, the system in (6) can be expressed in matrix format as follows

$$\boldsymbol{c} = -C_{3y}\boldsymbol{a}_p + C_{fyy}\boldsymbol{b}_q \tag{7}$$

The vector $\boldsymbol{c}$ contains third order cumulants at $n = m = 0$, the vectors $\boldsymbol{a}_p$ and $\boldsymbol{b}_q$ contain the parameters for the process in Equation (1). The matrix C_{3y} contains the cumulants of the output sequence and the matrix C_{fyy} contains the cross-cumulants of the input and output sequences.

Equation (7) can be written as

$$\boldsymbol{c} = [-C_{3y} \;\; C_{fyy}]\begin{bmatrix}\boldsymbol{a}_p \\ \boldsymbol{b}_q\end{bmatrix}$$

$$\boldsymbol{c} = C\boldsymbol{\theta} \tag{8}$$

The vector $\boldsymbol{c}$ contains third order cumulants at $n = m = 0$, the data composite matrix C contains the cumulants of the output sequence and the cross-cumulants of the input and output sequences, $\boldsymbol{\theta}$ is a vector represents the parameters

$$\boldsymbol{\theta} = [1, a_1, \ldots, a_p, \; 1, b_1, \ldots, b_q]^{\mathrm{T}} \tag{9}$$

Notice that the columns in the linear statistical model in (8) are linearly independent. Hence, we can multiply both sides of (8) by C^T, then

$$C^T C\boldsymbol{\theta} = C^T \boldsymbol{c} \tag{10}$$

Now, every $(m \times n)$ matrix C $(m \geq n)$ can be factored into the product of a matrix Q $(m \times m)$, having orthonormal vectors for its columns, and an upper triangular matrix R $(m \times n)$. The product

$$C = QR \tag{11}$$

is the QR factorization of C.

Substituting Eq. (11) into (10), we obtain

$$(QR)^T(QR)\boldsymbol{\theta} = (QR)^T \boldsymbol{c} \tag{12}$$

$$\boldsymbol{\theta} = R^{-1}Q^T \boldsymbol{c} \tag{13}$$

Now, the matrix C (where $C \in R^{m \times n}$) may be written as the product of an orthogonal matrix, Q, and an upper triangular matrix, R as follows:

$$C = [\boldsymbol{c}_1 \; \boldsymbol{c}_2 \cdots \boldsymbol{c}_n] =$$

$$[\boldsymbol{q}_1 \; \boldsymbol{q}_2 \cdots \boldsymbol{q}_n]\begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1n} \\ 0 & r_{22} & \cdots & r_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & r_{nn} \end{bmatrix} \tag{14}$$

In this representation, the vectors $(\boldsymbol{c}_1 \; \boldsymbol{c}_2 \cdots \boldsymbol{c}_n)$ are linearly independent. Since R has nonzero diagonal elements, then it is nonsingular matrix. This factorization represents the n^{th} column of C by linear combination of orthogonal columns.

III. SIMULATION EXAMPLES

Several examples were simulated to test the QR factorization technique at different levels of signal-to-noise ratio (SNR) on the output. The input sequence was generated as a zero-mean, independent and identically distributed (i.i.d), and exponentially distributed non-Gaussian process. The length of the signal is N=1500. A comparison of the performance of the proposed algorithm with the GM and the QSS methods were made at different SNRs on the output signal.

Example 1. The time series to be considered is given by [10]

$$y(n) + 0.39y(n-1) + 0.30y(n-2) = f(n) - 0.90f(n-1) \quad (16)$$

This model has two poles and one zero; ARMA (2,1). The poles are located at $0.548e^{\pm j110^0}$. The zero is located at 0.9. The modeling data were generated by exciting this system with a non-Gaussian process; namely exponential random process. These measurements were in turn corrupted by additive Gaussian noise at signal-to-noise ratio (SNR) of 10 dB on the output signal. Using the data as generated above; the proposed procedure was used to estimate the ARMA (2,1) model at various excitations and response SNR of 20 dB on the output signal. The ARMA parameters where then estimated using the GM and the proposed methods. These estimates were obtained by averaging 100 independent Monte Carlo simulations. Table 1 shows the True and estimated parameters for both GM and the proposed QR methods.

Table 1. True and estimated ARMA (2,1) parameters (Averaging 100 Simulations)

	True	GM	Proposed QR
a(1)	0.39	0.375	0.906
a(2)	0.30	0.294	0.301
b(1)	-0.90	-0.935	-.896

IV. CONCLUSION

The paper presented an algorithm for computing the ARMA parameters using QR factorization of a data covariance matrix of third order cumulants from only the output sequence. A comparison of the proposed method with the GM method is also presented. Numerical simulation results were presented to demonstrate the performance of the proposed method.

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