# Solving Inverse and Ill-Posed Problems by Regularization Methods based on Explicit Preconditioned Conjugate Gradient and Approximate Inverse Preconditioners

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***Abstract***

*A modified Tikhonov-Phillips regularization method based on explicit preconditioned Conjugate Gradient and approximate inverse preconditioners for solving inverse problems is presented. Several algorithmic procedures using termination criteria for explicit preconditioned CG (truncated EPCG) and the shifted structure of linear systems (shifted EPCG) are presented. A synoptic theoretical analysis on the convergence of modified TP method is presented. The numerical solution of a class of selected inverse problems indicates the performance of the proposed algorithms.*

***Key words-phrases:*** *Explicit Preconditioned Conjugate Gradient, ill-posed problems, Tikhonov-Phillips regularization, inverse problems, approximate inverse preconditioner*

## I. INTRODUCTION

The concept of well-posed and ill-posed problems has been introduced by Hadamard (1923). The well-posed problems have unique solutions depending on various system parameters and arbitrary small perturbation data parameters cannot cause arbitrary large solution perturbations. The ill-posed problems often arise in the form of inverse problems in various areas of science and engineering in particular when the determination of internal structure of physical systems from system's measured behaviour or the unknown input gives rise to measured output signals (computerized tomography, image restoration, signal processing, electromagnetic scattering, geophysics, optics, acoustics, astrometry) (Groetsh, 1993).

A well-posed mathematical model can have a unique solution stable with respect to noise in the input data, otherwise the problem is characterized as ill-posed problem and is unstable (Hadamard, 1923-1932). In computer mathematics and its applications there are several ill-problems and their corresponding numerical algorithms are often divergent. In order to overcome such difficulties special regularization techniques can be used taking advantage of a priori information (Whitney, 2009). A class of regularization techniques can be designed for solving ill-posed inverse problems leading to regularized learning algorithms. These algorithms are easily implemented kernel methods having a common derivation, with different computational and theoretical properties.

The direct problems of natural sciences include the well-posed problems having unique solutions intensive to small changes in these problems. The inverse problems have the characteristic that aim to find the cause of given effects or finding laws of evolution given the cause and effect. There are indirect measurements such as determination of internal characteristics from measurements on their boundaries, the determination of system parameters from input/output measurements, the reconstruction of past events from measurements of present state. These inverse problems are often ill-posed and frequently are modelled by integral equations of the first kind. Consequently, research work has been focused in the study of integral equations, inverse problems and ill-posed problems (Groetsch, 1993).

## III. REGULARIZATION TECHNIQUE: CONCEPTS AND APPLICATIONS

Computing methodologies studying several topics, such as inverse problems, deterministic inverse problems (regularization, worst case convergence, no assumptions on noise), statistics (estimators, average case analysis, noise is random variable, specific structures), Bayesian inverse problems (posteriori distribution, specific assumptions on noise and prior), control theory (F(control)=state, convergence of state not control, no assumptions) can be interesting research topics using regularization techniques (Kindermann, 2006).

In order to solve an ill-posed problem in a stable manner a priori information can be used, i.e. to construct a regularization solution to the given problem. This a priori information as smoothness of the solution generates the so-called Tikhonov regularization variational technique, allowing to obtain stable approximate solutions for ill-posed problems by stabilizing functional (Whitney, 2009; Kindermann, 2006; Hofman, 1996).

Regularization techniques are processes of introducing additional information for solving ill-posed problems or prevent over-fitting in the fields of machine learning and inverse problems ith application in general in many areas of mathematics, statistics and computer science (Neumaier, 1998; Bühlmann and Van De Geer, 2011). Regularization techniques can be used in the field of classification, where empirical learning of classifiers and learning from finite data sets, are underdetermined problems.

## III. Tikhonov-Phillips regularization methods for inverse problems

In several scientific fields, such as least squares methods, integral equations etc., simple forms of regularization can be used to learn simpler models, induce models to be sparse, introduce group structures in learning problems. These processes are known as Tikhonov regularizations (Tikhonov, 1963) and are also used in non-linear regularization (total variation regularization) for fitting data sets and reducing solution norms. In general regularization methods can be motivated as computational techniques improving the generalizability of learned models.

Several basic principles lead to regularization, such as iterative optimization, projection, penalized minimization, enforcing solution stability. The main idea of using regularization techniques is used in machine learning (filter functions; function approximation in signal processing and approximation theory; neural networks, radial basis function) and statistics problems, while the solution of inverse problems is related to the use of Tikhonov regularization

One of the well-known regularization techniques for solving ill-posed inverse problems is the Tikhonov-Phillips method (Tikhonov et al., 1995; Phillips, 1962). Tikhonov regularization can be used for solving linear discrete ill-posed problems. Note that a well-posed problem has a unique solution that changes as the initial conditions changed. Regulation process can stabilize ill-posed problems giving accurate approximate solutions including prior related information. Tikhonov regularization can also produce solutions in the case that the given large data sets contain statistical noises (ridge regression).

*Disadvantages of Regularization*

The disadvantages of Tikhonov regularization include the following:

(i) when this technique is used with Morozov's discrepancy principle (Anzengruber and Ramlau, 2009; Scherzer, 1993) there is repeated matrix manipulation for computing the solutions. The approximate solution uses an inverse (should exist) and start Newton's iterations with proper selection of parameters (overestimated parameter).

(ii) in the process there is reconstruction non-smooth or discontinuous solutions, where different penalty terms should be used.

## IV. The Explicit Preconditioned Conjugate Gradient method

During the last decades, considerable research effort has been directed to the solution of complex linear and nonlinear systems of algebraic equations by using a class of iterative methods. This class includes the conjugate gradient method and its hybrid multi-variants (Reid, 1971; Evans and Lipitakis, 1980; Axelsson 1985; Saad, 1985; Lipitakis and Gravvanis, 1992; Benzi., 1988-2002; Gravvanis, 1995; Saad and Van der Vorst, 2001). The explicit preconditioned conjugate gradient (EPCG) and approximate inverse have been efficiently used for solving large sparse linear systems with unsymmetric matrices of irregular structures (Saad, 2001; Lipitakis, 2016-2017).

An Adaptive Explicit Preconditioned Conjugate Gradient (EPCG) method using the explicit approximate preconditioner can solve the problem $min\left\|b-AR^{-1}x\right\|$, where R is the sparse, non-singular QR factor, while the preconditioned CGLS method can solve the equations: $M=R^TR$, $R^{-T}A^TA\,R^{-1}\,u=R^{-T}A^Tb$ and $u=Rx$. Note that the factor Q cannot be stored, while the only additional computational work is solving the two equations $R^T\,w=v$ and $Rz=w$. All the factorization processes are numerically stable.

In order to compute efficiently the solution of the linear system $Ax = b$, a modified Explicit Preconditioned Conjugate Gradient (mEPCG) method is applied in the following algorithmic form:

**Algorithm mEPCG (A, b, tol, x0, M*, x)**
*Purpose:* a modified EPCG method is used for solving a given system of linear equations
*Input:* A is a symmetric and positive definite coefficient matrix, b is the right hand side vector, tol is the predetermined tolerance, $\mathbf{x_0}$ is the initial guess, M* is the required preconditioner
*Output:* x the solution vector
*Computational Procedure:*
Step 1: Given $x_0$, preconditioner M*
Step 2: set $r_0 = A * x_0 - b$
Step 3: solve $M^* y_0 = r_0$ for $y_0$
Step 4: set $p_0 = -y_0, k = 0$
Step 5: while $r_k \neq 0$
Step 5.1: compute a step length
$$a_k = (r_k^T * y_k) / (\mathrm{p}_k^T A * p_k)$$
Step 5.2: update the approximate solution
$$x_{k+1} = x_k + a_k * p_k$$
Step 5.3: update the residual
$r_{k+1} = r_k + a_k * \mathrm{A} * p_k$
Step 5.4: solve $M^* y_{k+1} = r_{k+1}$
Step 5.5: compute a gradient correction factor $\beta_{k+1} = (r_{k+1}^T * y_{k+1}) / (\mathrm{r}_k^T * \mathrm{y}_k)$
Step 5.6: set the new search direction
$$p_{k+1} = -y_{k+1} + \beta_{k+1} * p_k$$
Step 5.7: set k=k+1
Step 6: end (while)

This algorithm requires the additional work that is needed to solve the linear system

$$\mathrm{M}^* \tilde{\mathrm{r}}_{\mathrm{n}} = \mathrm{r}_{\mathrm{n}} \tag{4.1}$$

once per iteration. Therefore, the preconditioner M* should be chosen such that can be done easily and efficiently.

The preconditioner M*=G that results in a minimal memory use. The storage requirement was the vectors r, x, y, p and the upper triangular matrix G, in the data implementation. The convergence rate of preconditioned CG is independent of the order of equations and the matrix vector products are orthogonal and independent. The preconditioned CG method in not self-correcting and the numerical errors accumulate every round. Therefore, to minimize the numerical errors in the EPCG, it was used double precision variables at the cost of memory use. Note that in the case of undertermined linear systems the left preconditioned method can be applied, while if there is an overdetermined linear system then the right preconditioned can be applied (Saad, 1988).

## V. Tikhonov-Phillips Regularization Method for ill-posed problems

Several ill-posed problems arise in large variety of applications when the considered problem is modelled by integral equations of the first kind with smooth kernels arising from inverse problems [medical images (Louis, 1992; Natterer, 1986), scattering problems (Colton and Kress, 1992), geophysics problems (Garmany, 1979), applications in differential equations (Kunisch and Sachs, 1992)].

The Tikhonov-Phillips regularization requires the solution of

$$(\mathrm{A}^*\mathrm{A} + \alpha \mathrm{I})\mathrm{x} = \mathrm{A}^* \mathrm{y}^+, \tag{5.1}$$

for several values of α. This system can be solved by CG-type methods, by selecting proper termination criteria and solving various shifted structures of system (5.1) for different values of parameter α.
The inverse problems can be modelled by the system

$$Ax = y, \tag{5.2}$$

where A is a compact operator between Hilbert spaces, x is the searched quantity and y describes the given data occurring from measurement of limited precision (perturbed data y+ with known error bound

$$\left\| y - y^+ \right\| \leq \varepsilon^+ \tag{5.3}$$

The inverse problem is ill-posed when A is not continuously invertible or equivalently the set

$$\{x \in X \mid \left\| Ax - y \right\| \leq \varepsilon^+\} \tag{5.4}$$

is inbounded quantity.
The Tikhonov-Phillips (TP) regularization technique (Tikhonov and Yagola, 1998) results when the system (5.2) is replaced by

$$(\mathrm{A}^*\mathrm{A} + \alpha \mathrm{I})x = A^* y^+, \tag{5.5}$$

where α is the regularization parameter and the TP solution is

$$x_a^+ = (A^* A + aI)^{-1} A^* y^+ \tag{5.6}$$

Note that a large value of α suppresses the data errors but increases the approximation errors, while the computational complexity of the corresponding algorithm is determined by the efficiency of the method used for solving the operator equations (5.5).

Several fast CG-type methods for determining the optimal value of parameter α and computing the TP solution. These methods include (i) a truncated CG method solving approximately the equation (5.5) and (ii) a shifted CG method based on the three term recurrence Lanczos process for computing the solution $x_\alpha^+$ (Frommer and Maas, 1999).

Let us consider the TP regularization for solving equation (5.2), i.e.

$$(A^* A + aI)x = A^* y^+ \tag{5.7}$$

where $\|y - y^+\| \le \varepsilon$, and A is a compact operator between Hilbert spaces X, Y (Engl et al., 1996; Hanke and Hansen, 1993; Louis, 1992).

Consider x* the exact solution of equation (5.2) with unperturbed data. Then the error estimate states

$$\|x^* - x_a^+\| \le (\varepsilon / [2a^{1/2}]) + ca^v \tag{5.8}$$

showing that small values of parameter α leads to strong amplification of data error (Neubauer, 1988; Maas and Rieder, 1997).

The theoretically optimal value for α of the order

$$\alpha_{opt} = O(\varepsilon^{[2/(2v+1)]}). \tag{5.9}$$

Since the exact value of ν is unknown a priori the type of parameter has to investigated (Neubauer, 1988). The selection of parameter α can be chosen according to Morozov discrepancy principle, such as

$$\|Ax_a^+ - y^+\| = r\varepsilon, \text{ where } r>1, \tag{5.10}$$

leading to optimal convergence rate as ε→0 (Maas and Rieder, 1997).

The TP regularization methods can be implemented by using a class of algorithmic methods consisting of two nested iterations: the outer iteration running over different values of parameter α and the inner iteration (CG iteration) solving the regularized linear system (5.1).

## VI. Modified TP Regularization Algorithms

Several algorithmic implementations of the TP regularization method have been presented (Frommer and Maas, 1999). In the following a class of modified TP regularization algorithms based on the explicit preconditioned conjugate gradient and approximate inverse preconditioners is presented.

The modified standard TP regularization algorithm with EPCG can be described as follows:

**Algorithm mTPR-1 ($\alpha_0$, q, r, x)**

*Purpose:* This algorithm applies the TP regularization with truncated EPCG

*Input:* parameter $\alpha_0$, q, residual r

*Output:* solution x*

*Computational Procedure*:

Step 1: Choose $\alpha_0$, q, r

Step 2: For k=0,1,… until convergence

Step 2.1: Compute $\alpha_k$,

Step 2.2: Solve $(A^* A + a_k I)x = A^* y^+$ using EPCG method.

Step 3: End

Note that the termination criterion can be

$$\|Ax_a^+ - y^+\| = r\varepsilon$$

The CG-type methods with appropriate termination criteria yield regularization methods by themselves (Hanke, 1995).

Other related algorithmic variants requiring considerably fewer computational operations (matrix-vector multiplications) will be presented.

**Algorithm mTPR-2 ($\alpha_0$, q, r, x*)**

*Purpose*: This algorithm applies the TP regularization with truncated EPCG

*Input:* parameter $\alpha_0$, q, residual r

*Output*: solution x*

*Computational Procedure*:

Step 1: choose $\alpha_0$, q, r

Step 2: set $x_{a_{-1}}^* = 0$

Step 3: for k=0,1,… until convergence $\|Ax_a^+ - y^+\| \le r\varepsilon$

Step 3.1: compute $\alpha_\kappa$

Step 3.2: set $x_{ak}^0 = x_{a(k-1)}^*$

Step 3.3: perform EPCG iteration on

$$(A^* A + a_k I)x = A^* y^+$$

Step 3.3.1: check for each iterate $x_{ak}^j$ the condition $\|x^* - x_a^+\| \le (\varepsilon / [2a^{1/2}]) + ca^v$ holds or if the iteration has converged

Step 3.4: call the last EPCG iterate $x_{ak}^*$

Step 4: end

In the case that $(A^* A)$ is (nearly) singular, then $(A^* A + aI)$ is ill-conditioned when the value of parameter α is small.

Another more stable algorithm than the standard EPCG implementation for the regularized system can be described as follows:

**Algorithm mEPCG-TPR-3 ($x_0$, $\alpha_0$, q, r, x*)**
*Purpose*: This algorithm applies the TP regularization with truncated EPCG
*Input:* initial vector $x_0$, parameter $\alpha_0$, q, residual r
*Output:* solution vector x*
*Computational Procedure:*
Step 1: choose $x_0$,
Step 2: compute $z_0 = y - Ax_0$; $r_0 = A^* z_0 - \alpha x_0$
Step 3: set $p_0 = r_0$
Step 4: for j =0,1,…,Nmax /until convergence/
Step 4.1: $q_j = A p_j$
Step 4.2: $\beta_j = (r_j, r_j) / [(q_j, q_j) + \alpha (p_j, p_j)]$
Step 4.3: $x_{j+1} = x_j + \beta_j p_j$
Step 4.4: $z_{j+1} = z_j - \beta_j q_j$
Step 4.5: $r_{j+1} = A^* z_{j+1} - \alpha x_{j+1}$
Step 4.6: $w_j = (r_{j+1}, r_{j+1}) / (r_j, r_j)$
Step 4.7: $p_{j+1} = r_{j+1} + w_j p_j$
Step 4: end

The algorithm mEPCG- TPR- 3 is more stable than the modified standard EPCG algorithm for the regularization system since updates recursively the quantity $z_j = y^+ - Ax_j$ rather than the EPCG residuals, requiring only the computation of the norm $\|z_j\|$.
The shifted structure of systems (5.1) can be used by considering the general linear system

$$Mx = b, \qquad (6.1)$$

where M is symmetric, positive definite matrix. Then, the following TP regularization with modified EPCG method for the shifted system can be used:

**Algorithm TPR-SHIFTED MEPCD-4 ($u_0$, $\alpha_0$, q, r, u*)**
*Purpose:* This algorithm applies the TP regularization with shifted EPCG
*Input:* initial vector $u_0$, parameter $\alpha_0$, q, residual r
*Output*: solution vector u*
*Computational Procedure*:
Step 1: for $k = 0,1,\ldots,k_{\max}$
Step 1.1: set $a_k = a_0 q^k$, $x_{ak}^0 = 0$
Step 1.2: set $u_0^* = y^+$; compute $v_0^* = A^* u_0^*$
Step 1.3: set $\beta_0 = \|v_0^*\|, v_0 = v_0^*/\beta_0$, $u_0 = u_0^*/\beta_0$,
Step 1.4: set $p_{ak}^0 = r_{ak}^0 = v_0^*, u^{-1} = 0$,
Step 1.5: for j=0,1,… until convergence then the j-th Lanczos step/
Step 1.5.1: set $q_j = Av_j, d_j = (q_j, q_j)$
Step 1.5.2: compute $u_{j+1}^* = q_j - d_j u_j - \beta_j u_{j-1}$
Step 1.5.3: set $v_{j+1}^* = A^* u_{j+1}^*$
Step 1.5.4: compute $\beta_{i+1} = \|v_{i+1}^*\|$,
Step 1.5.5: compute $v_{j+1} = v_{j+1}^* / \beta_{j+1}$; $u_{j+1} = u_{j+1}^* / \beta_{j+1}$
Step 1.6: for k=0,1,…, $k_{max}$, if the system has not converged then compute (j+1) EPCG iterate and check for convergence
Step 2: end

## VII. ON THE CONVERGENCE OF MODIFIED TP REGULARIZATION METHOD

In this section a synoptic theoretical analysis on the convergence rate of modified TP regularization method is presented. A similar analysis has been given for the case of TP regularization methods (Frommer and Maass, 1999).

Let us consider a class of inverse problems mathematically modelled by the relationship

$$A x = y, \qquad (7.1)$$

where A denotes a compact operator between Hilbert spaces X, Y, i.e. A: X→Y, x is the solution vector and y is the set of given data arising from measurements with limited precision, i.e. perturbed data with known available error bounds, i.e.

$$\|y - y^\varepsilon\| \le \varepsilon \qquad (7.2)$$

$$\{x \in X \mid \|Ax - y\| \le \varepsilon\} \qquad (7.3)$$

is unbounded, and the solution of the instability of the inverse problem requires regularization methods. In this case the equation (1.1) can be replaced by

$$(A^*A + \alpha I)x = A^* y^\varepsilon, \qquad (7.4)$$

where α is the regularization parameter, while the TP solution (1.4) can be denoted by

$$x^{\varepsilon,a} = [A^*A + aI)^{-1} A^* y^\varepsilon \qquad (7.5)$$

The convergence rate of the modified TP regularization method is governed by the following theorem:

**Theorem 1:** Let us assume that A is a compact injective* operator and let $x^{\varepsilon,a}$ denotes the solution of the equation of (7.5). Then for an arbitrary x∈X it holds

$$\|Ax^{\varepsilon,\alpha} - y^\varepsilon\| \ge \|Ax - y^\varepsilon\| - E^{\varepsilon,a}, \qquad (7.6)$$

where

$$E^{\varepsilon,\alpha} = \|(A^*A + \alpha I)x - A^* y^\varepsilon\| \qquad (7.6a)$$

(*) Note that the term injective refers to functions preserving distinctness, i.e. never maps distinct elements of its domain to the same element of its codomains.

**Proof**

Let us consider $\{u_n, v_n, \mu_n\}$ be a generalized eigenvalue decomposition (svd) of A: X→Y, i.e. un is an orthogonal basis of X, and vn is an orthonormal system in Y such that $Au_n = \mu_n v_n$ and $A^* v_n = \mu_n u_n$ The singular values $\mu_n$ are nonnegative and $\mu_n \to 0$ for compact operators A. Then,

$$Az = \sum_{\mu_n > 0} \mu_n < z\,,\, u_n > v_n \tag{7.7}$$

Since $\mu^2 = [\mu + \alpha\mu^{-1}]^{-2}[\mu^2 + \alpha]^2$ (7.8)

with α, μ>0 and $1/[\mu + \alpha\mu^{-1}] \le 1/[2\alpha^{1/2}]$ (7.9)

we have that

$$\|Az\|^2 \le \{\sum_{\mu n > 0} [\mu_n^2 + a]^2 |< z, u_n >|^2\} / 4a \tag{7.10}$$

In the case that A is injective, then we have that

$$z = \sum_{\mu n > 0} < z, u_n > u_n \tag{7.11}$$

and

$$\|(A^* A + aI)z\|^2 = \sum_{\mu n > 0} (\mu_n^2 + a)^2 |< z, u_n >|^2 \tag{7.12}$$

from which we obtain

$$\|Az\| \le \{\|(A^* A + aI)z\|\} / [2a^{1/2}] \tag{7.13}$$

From these relationships the conclusion of the theorem easily follows■

Let $\lambda_{min}$ and $\lambda_{max}$ denote the smallest and largest eigenvalues of $(A^* A)$.

After the normalization of A, we have $\lambda_{max} \approx 1$ and $\lambda_{min} \approx 0$ and assuming that

$$\lambda_{\min} << a << 1 \tag{7.14}$$

the condition numbers of matrices $(A^* A + aI)$ can be computed as

$$cond(A^* A + aI) = [(\lambda_{\max} + a)/(\lambda_{\min} + a)] \approx 1/a \tag{7.15}$$

The number of EPCG iterations achieving a predetermined accuracy is proportional to the square root of condition number. If $N(k)^*$ is the number of EPCG iterative steps required on system k until convergence, then $N(k)^*$ is proportional to 1/ ($\alpha^{1/2}$), and $N(k)^* \approx c[1/q^{1/2}]^k$.

Then,

$$\sum_{k=0}^{kopt} N(k)^* \approx c[1/(q^{kopt+1})^{1/2} - 1]/[1/q^{1/2} - 1] \tag{7.16}$$

which is more than $N(k)^*$.

The algorithms represent a stable implementation of the EPCG method for the regularized system $[(A^* A + aI)x - A^* y^\varepsilon]$.

## VIII. MODEL PROBLEMS

The presented TP regularization algorithms based on EPCG methods can be applied to several ill-posed model problems (Frommer et al., 1999), such as

(i) the solution of inverse ill-posed problem defined by A x = y, with the compact operator

$TA: L^2([0,1]) \to L^2([0,1]),\ x \to \int_0^1 k(\mu, t)dt$,

where $k(s,t) = \begin{cases} t(1-\mu), for\, t \le \mu \\ \mu(1-\tau), for\, \mu \le t \end{cases}$ (8.1)

(ii) the hyperthermia treatment planning model (for non-invasive cancer therapy) is defined by:

$$E_0(x) = E(x) + \int_G [F(x,y)]dy, \tag{8.2}$$

where

$$F(x,y) = [w^2 \mu_0 (\varepsilon_0 - \varepsilon(y)) E(y) \Phi(x,y) + [(E(y)\nabla\varepsilon(y))\nabla y \Phi(x,y)]] / \varepsilon(y), \tag{8.3}$$

Force acoustic waves to travel on half, G is the volume containing the patient, ε is the dielectricity and

$$\Phi(x,y) = [e^{i\|x-y\|w/2}]/[4\pi\|x-y\|] \tag{8.4}$$

This equation governs the electric field generated by antennae (Bolomey et al., 1990).

(iii) Seismic travel time inversion model: The geophysical subsurface structure consists of measuring travel times of acoustic waves and acoustic pulses are emitted at points x on the surface while travel times are measured. The linear model force the acoustic waves to travel on half circles. Small deviations n(x,y) of expected linearly increasing velocity field is linked to measured data by the following integral equation:

$$\nabla g_{i,j} = \int_0^{\pi} n[x_i + j_h \cos(\phi) - j_h \sin(\phi)] d\phi \quad (8.5)$$

where Δg denotes the difference between measured values and those predicted by linear model. The inverse problem of recovering n from the measured data is an ill-posed one (Bortfeld, 1983).

## IX. CONCLUSIONS

A modified Tikhonov-Phillips regularization method for solving inverse and ill-posed problems has been presented. This method is based on explicit preconditioned conjugate gradient methods and approximate inverse preconditioners. Several algorithmic procedures for implementing the truncated EPCG and shifted EPCG using termination criteria and shifted structures respectively are used A synoptic theoretical analysis on the convergence of modified TP method is presented. The truncated EPCG algorithm seems to perform better than the rest related algorithms. Future research work will be focused on modified Tikhonov-Phillips regularization methods for solving inverse and ill-posed 3D problems and in parallel computer environments.

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