Mixed H_2/H_∞ Strategy in Control Law Parameter Design for Linear Strictly Metzlerian Systems

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***Abstract*—The paper provides linear matrix inequality conditions in mixed H_2/H_∞ control design for strictly Metzlerian linear systems. The goal of this formulation is to design the state controller guaranteing H_∞ norm disturbance attenuation and optimized H_2 norm performance. The problem is formulated multi-objective, respecting the constraints implying from H_2 and H_∞ fulfillment, as well as from the parameter constraints defined by the system matrix structures in the strictly Metzlerian system description. The design character guaranties asymptotic stability realized in a strictly Metzlerian closed-loop system form. It is shown that enhanced design conditions span such a synthesis framework for strictly Metzlerian linear system, where matrix variables take diagonal form.**

***Index Terms*—Metzlerian systems, state feedback stabilization, linear matrix inequalities, asymptotic stability, H_∞ norm, H_2 norm attenuation.**

I. Introduction

Unknown disturbance suppression, as well as input gain attenuation, are very important topics in control theory and so tasks formulating H_2 and H_∞ control are presented by many authors (see, e.g., [7], [18], and references therein) In real cases it is interpreted in that way that while H_∞ control design prespecifies frequency-domain performance, H_2 control attenuation precribes performance on transient behavior of a system [19]. To state the H_∞ norm disturbance suppression problem together with optimization of the closed-loop system H_2 norm a mixed H_2/H_∞ control was formulated in [12]. Prioritizing these properties, the development of appropriate control architectures, and associated controller design algorithms, was reflected by a mixed H_2/H_∞ closed-loop performance criterion [8], [9] or, more complex, by formulating the LMI-based computational technique under the obvious additional assumption that the plant is controllable. Some of its results have been recast in more general frameworks [2], [21]. However, these formalisms do not generally provide the same computational power.

Positive system models are exploited in description of industrial and engineering variables pointing out to strictly positive quantities [6]. Restricting to Metzler structure of system matrices when dealing with positive systems, and to nonnegative input and output matrices, then, in abbreviated terms, these systems are often denoted as Metzlerian systems. Consequently, well-tried linear techniques cannot be straightly nominated to positive systems and new approaches have to been derived. Stability and stabilization of Metzlerian systems cover new methods reported [11], [25] and various methods based on linear programming, or a combination of linear programming with linear matrix inequalities, are proposed for positive system stabilization (see, e.g. [1], [5], [20], [24]). Most recently, a principle based purely on the LMI formulation, and used to derive design conditions for controllers and estimators with positive parameter constraints, was presented in [13], [14], respectively, where more detailed formulations and associated results can be found. Besides specific meaningful control application [17], exploitation of the latter principle for designing residual filters in the diagnosis of positive linear systems can be found in [16].

Application of the LMI-based idea to control law parameter setting for linear Metzlerian systems, and reflection by the system structure given constraints, a mixed H_2/H_∞ design strategy is outlined in the article. Main idea lies in adapting the principle used in [15] to set out LMI design criterion for solving problems in stabilization of linear Metzlerian continuous-time systems. Hence, besides the computational aspects induced by the matrix representation of parameter constraints, the design conditions are formulated in the paper using strictly sharp matrix inequalities, respecting diagonal stabilization principle in control design of linear Metzlerian systems [23].

The outline of this paper is as follows. Section II briefly introduces basic fundamental properties of linear Metzlerian systems and the principle of their stabilization. In Sec. III there are H_2 and H_∞ principles reformulated to be suitable for linear strictly Metzlerian systems and the design condition for mixed H_2/H_∞ control of this group of systems is derived. A numerical example demonstrates the results of this paper in Sec. IV and Sec. V presents conclusions with a discussion and summary.

Throughout the paper, the notations is narrowly standard in such way that $\boldsymbol{x}^T$, $\boldsymbol{X}^T$ denotes the transpose of the vector $\boldsymbol{x}$ and matrix $\boldsymbol{X}$, respectively, $diag[\,\cdot\,]$ denotes a block diagonal matrix, for a square matrix $\boldsymbol{X} \prec 0$ means that $\boldsymbol{X}$ is a symmetric negative definite matrix, the symbol $\boldsymbol{I}_n$ indicates the n-th order unit matrix, $\boldsymbol{T}$ refers the permutation matrix, $\|\cdot\|_F$ provides the Frobenius matrix norm, and $\mathbb{R}^n_n$, $\mathbb{R}^{n\times r}_+$ point to the set of all n-dimensional real non-negative vectors and $n \times r$ real non-negative matrices.

II. Fundamentals of Metzlerian Systems

To explain some properties, through this section there are considered strictly linear Metzlerian systems in disturbance-free regime in the state-space form

$$\dot{\boldsymbol{q}}(t) = \boldsymbol{A}\boldsymbol{q}(t) + \boldsymbol{B}\boldsymbol{u}(t) \tag{1}$$

$$\boldsymbol{y}(t) = \boldsymbol{C}\boldsymbol{q}(t) \tag{2}$$

where $\boldsymbol{q}(t) \in I\!R^n_+$, $\boldsymbol{u}(t) \in I\!R^r_+$, $\boldsymbol{y}(t) \in I\!R^m_+$ and $\boldsymbol{A} \in I\!R^{n\times n}$, $\boldsymbol{B}_+ \in I\!R^{n\times r}$, $\boldsymbol{C}_+ \in I\!R^{m\times n}$. The transfer function matrix to (1), (2) is given in the standard way as

$$\boldsymbol{G}(s) = \boldsymbol{C}(s\boldsymbol{I}_n - \boldsymbol{A})^{-1}\boldsymbol{B} \tag{3}$$

Although there are different definitions of Metzler's square matrix structure, specification by Definition 1 is preferred in this paper. This structure results in the smallest number of structural boundaries for the matrix elements, because any further deviation from such defined structure leads to further boundary which has to be added to analysis or synthesis [16]. Whilst non-negativity of matrices $\boldsymbol{B} \in I\!R^{n\times r}_+$, $\boldsymbol{C} \in I\!R^{m\times n}_+$ means that all its entries are nonnegative and at least one is positive, a Metzler square matrix is signum indefinite. Using the following definition, the assignment $\boldsymbol{A} \in I\!R^{n\times n}_{-+}$ can be used highlighting negative diagonal elements and positive non-diagonal elements of a Metzler matrix $\boldsymbol{A}$.

Definition 1: [3] A square matrix $\boldsymbol{A} \in I\!R^{n\times n}_{-+}$ is called strictly Metzler matrix if all its diagonal elements are negative and all its off-diagonal elements are positive. A Metzler matrix is stable if it is Hurwitz.

Definition 2: (positive linear systems) System (1), (2) is said to be positive if and only if for every nonnegative initial state and nonnegative input its state and output are nonnegative.

Proposition 1: [6] A solution $\boldsymbol{q}(t)$ of (1) for $t \geq 0$ is positive and asymptotically stable if for non-negative $\boldsymbol{B} \in I\!R^{n\times r}_+$ and stable strictly Metzler $\boldsymbol{A} \in I\!R^{n\times n}_{-+}$ then variable $\boldsymbol{q}(t) \in I\!R^n_+$ with applying $\boldsymbol{u}(t) \in I\!R^r_+$ and $\boldsymbol{q}(0) \in I\!R_+$. The linear system (1), (2) is asymptotically stable and positive if $\boldsymbol{A}$ is strictly Metzler and Hurwitz, $\boldsymbol{B} \in I\!R^{n\times r}_+$, $\boldsymbol{C} \in I\!R^{m\times n}_+$ are non-negative matrices and $\boldsymbol{y}(t) \in I\!R^m_+$ for all $\boldsymbol{u}(t) \in I\!R^r_+$ and $\boldsymbol{q}(0) \in I\!R_+$.

An interesting case that has been widely studied corresponds to the case in which the full state control input for Metzlerian system (1), (2) is defined as

$$\boldsymbol{u}(t) = -\boldsymbol{K}\boldsymbol{q}(t) \tag{4}$$

where $\boldsymbol{K} \in I\!R^{r\times n}_+$ is positive (non-negative). Thus, it yields

$$\dot{\boldsymbol{q}}(t) = (\boldsymbol{A} - \boldsymbol{B}\boldsymbol{K})\boldsymbol{q}(t) = \boldsymbol{A}_c\boldsymbol{q}(t) \tag{5}$$

$$y(t) = \boldsymbol{C}\boldsymbol{q}(t) \tag{6}$$

where $\boldsymbol{A}_c \in I\!R^{n\times n}_{-+}$ is adjusted as

$$\boldsymbol{A}_c = \boldsymbol{A} - \boldsymbol{B}\boldsymbol{K} \tag{7}$$

Note that controller has to stabilize the system with rendering that matrix $\boldsymbol{A}_c \in I\!R^{n\times n}_{-+}$ is Metzler and Hurwitz.

The following theorem defines the LMI conditions to obtain a stable, strictly Metzlerian closed-loop system. The results reflect the diagonal stabilization principle in control design of Metzlerian systems [23].

Theorem 1: [13] The closed-loop system (5), (6) is strictly Metzlerian and stable if the system (1), (2) is strictly Metzlerian and there exist positive definite diagonal matrices $\boldsymbol{Q}, \boldsymbol{R}_k \in I\!R^{n\times n}$ such that for $h = 1, 2, \dots n-1$ and $k = 1, 2 \dots, r$,

$$\boldsymbol{Q} = \operatorname{diag}\begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix} \succ 0 \tag{8}$$

$$\boldsymbol{R}_k = \operatorname{diag}\begin{bmatrix} r_{k1} & r_{k2} & \cdots & r_{kn} \end{bmatrix} \succ 0 \tag{9}$$

$$\boldsymbol{A}(i,i)_{(1\leftrightarrow n)/n}\boldsymbol{Q} - \sum_{k=1}^{r} \boldsymbol{B}_{dk}\boldsymbol{R}_k \prec 0 \tag{10}$$

$$\boldsymbol{T}^h\boldsymbol{A}(i,i+h)_{(1\leftrightarrow n)/n}\boldsymbol{T}^{hT}\boldsymbol{Q} - \sum_{k=1}^{r} \boldsymbol{T}^h\boldsymbol{B}_{dk}\boldsymbol{T}^{hT}\boldsymbol{R}_k \succ 0 \tag{11}$$

$$\boldsymbol{A}\boldsymbol{Q} + \boldsymbol{Q}\boldsymbol{A}^T - \sum_{k=1}^{r}(\boldsymbol{B}_{dk}\boldsymbol{l}\boldsymbol{l}^T\boldsymbol{R}_k + \boldsymbol{R}_k\boldsymbol{l}\boldsymbol{l}^T\boldsymbol{B}_{dk}) \prec 0 \tag{12}$$

where

$$\begin{aligned}
&\boldsymbol{l}^T = \begin{bmatrix} 1 & \cdots & 1 \end{bmatrix}\\
&\boldsymbol{T} = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1\\ 1 & 0 & \cdots & 0 & 0\\ & & \ddots & & \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}\\
&\boldsymbol{B} = \begin{bmatrix} \boldsymbol{b}_1 & \boldsymbol{b}_2 & \cdots & \boldsymbol{b}_r \end{bmatrix}\\
&\boldsymbol{B}_{dk} = \operatorname{diag}\begin{bmatrix} b_{k1} & b_{k2} & \cdots & b_{kn} \end{bmatrix}\\
&\boldsymbol{A} = \begin{bmatrix} a_{11} & \cdots & a_{1n}\\ & \vdots & \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}\\
&\boldsymbol{A}(j+h,j)_{(1\leftrightarrow n)/n} =\\
&= \operatorname{diag}\begin{bmatrix} a_{1+h,1} & \cdots & a_{n,n-h} & a_{1,n-h+1} & \cdots & a_{h,n} \end{bmatrix}
\end{aligned} \tag{13}$$

while $\boldsymbol{T}^{-1} = \boldsymbol{T}^T$.

With a feasible solution, the controller gain $\boldsymbol{K} \in I\!R^{r\times n}_+$ that solves the design task is

$$\boldsymbol{K}_k = \boldsymbol{R}_k\boldsymbol{Q}^{-1}, \quad \boldsymbol{k}_k^T = \boldsymbol{l}^T\boldsymbol{K}_k, \quad \boldsymbol{K} = \begin{bmatrix} \boldsymbol{k}_1^T\\ \vdots\\ \boldsymbol{k}_r^T \end{bmatrix} \tag{14}$$

□

Such the design task formulation guarantees that matrix $\boldsymbol{A}_c$ is Metzler and Hurwitz if the above given set of inequalities is feasible that is, if

$$\boldsymbol{A}_c = \left[\{a_{ctl}\}_{t,\,l=1}^{n}\right] \tag{15}$$

it yields

$$a_{cll} = a_{ll} - \sum_{k=1}^{m} b_{lk}k_{kl} < 0, \quad l, \in \langle 1, \dots, n\rangle \tag{16}$$

$$a_{ctl} = a_{tl} - \sum_{k=1}^{m} b_{tk}k_{kl} > 0, \quad t \neq l, \;\; t, l, \in \langle 1, \dots, n\rangle \tag{17}$$

Quantification of effects of the input onto the output of a linear system, related to the system transfer function $\boldsymbol{G}(s)$, may be characterized through its H_2 or H_∞ norm.

Definition 3: [10], If $\boldsymbol{A}$ has no imaginary eigenvalues the H_∞ norm of (3) is

$$\|\boldsymbol{G}(s)\|_\infty = \sup_{\omega \in I\!R} \sigma_1(\boldsymbol{G}(j\omega)) \tag{18}$$

where the i-th singular value of the complex matrix $\boldsymbol{G}(j\omega)$ is the nonnegative square-root of the i-th largest eigenvalue of $\boldsymbol{G}^*(j\omega)\boldsymbol{G}(j\omega)$, while the singular values $\sigma_i(\boldsymbol{G}(j\omega))$ of the transfer function matrix are evaluated on the imaginary axis and it is assumed that the singular values are ordered such that $\sigma_i \geq \sigma_{i+1},\ i = 1, 2, \cdots, n-1$.

Definition 4: [7] If $\boldsymbol{A}$ has no imaginary eigenvalues then $\boldsymbol{G}(j\omega)$ is defined for all $\omega \in R$, where ω is the frequency variable, $j = \sqrt{-1}$, and the H_2 norm of (3) is

$$\begin{aligned}\|\boldsymbol{G}(s)\|_2 &= \sqrt{\tfrac{1}{2\pi}\int_{-\infty}^{\infty} \|\boldsymbol{G}(j\omega)\|_F^2\,\mathrm{d}\omega} = \\ &= \sqrt{\tfrac{1}{2\pi}\int_{-\infty}^{\infty} tr(\boldsymbol{G}^*(j\omega)\boldsymbol{G}(j\omega))\mathrm{d}\omega}\end{aligned} \tag{19}$$

where $\boldsymbol{G}^*(j\omega) = \boldsymbol{G}^T(-j\omega)$ and $\|\cdot\|_F$ denotes the Frobenius matrix norm.

To study the state trajectories of linear systems on infinite time interval, the stability conditions related to Definition 3 and Definition 4 are associated.

Lemma 1: [22] $\boldsymbol{A}$ is Hurwitz and $\|\boldsymbol{G}(s)\|_\infty < \gamma_\infty$ if there exists a symmetric positive definite matrix $\boldsymbol{P} \in I\!R^{n\times n}$ and a positive scalar $\gamma_\infty \in I\!R$ such that

$$\boldsymbol{P} = \boldsymbol{P}^T \succ 0 \tag{20}$$

$$\begin{bmatrix} \boldsymbol{A}^T\boldsymbol{P} + \boldsymbol{P}\boldsymbol{A} & * & * \\ \boldsymbol{B}^T\boldsymbol{P} & -\gamma_\infty \boldsymbol{I}_r & * \\ \boldsymbol{C} & \boldsymbol{0} & -\gamma_\infty \boldsymbol{I}_m \end{bmatrix} \prec 0 \tag{21}$$

Hereafter, $*$ denotes the symmetric item in a symmetric matrix.

Lemma 2: [4] If $\boldsymbol{A}$ is Hurwitz and if exists a positive definite square matrix $\boldsymbol{W}_c$ such that

$$\boldsymbol{A}\boldsymbol{W}_c + \boldsymbol{W}_c\boldsymbol{A}^T + \boldsymbol{B}\boldsymbol{B}^T = \boldsymbol{0} \tag{22}$$

then

$$\gamma_2^2 = \mathrm{tr}(\boldsymbol{C}\boldsymbol{W}_c\boldsymbol{C}^T) \tag{23}$$

where $\gamma_2 \in I\!R_+$ is H_2 norm of $\boldsymbol{G}(s)$ and $\boldsymbol{W}_c$ is the controllability Gramian steady value.

The existence conditions such that $\|\boldsymbol{G}(s)\|_2$, $\|\boldsymbol{G}(s)\|_\infty$ be as close to γ_2, γ_∞, are stated above in terms of the given continuous time-invariant system (1), (2), where control system properties are derived for linear controllable systems. Since the principle of controllability can not be applied to Metzlerian linear systems, structures defined in Lemma 1 and Lemma 2 need to be adapted to the principle of diagonal stability of Metzlerian systems.

III. H_2 AND H_∞ CONTROL OF METZLERIAN SYSTEMS

Given system (1), (2), its extended form with the bounded unknown disturbance $\boldsymbol{d}(t) \in I\!R_+^{n\times p}$ is

$$\dot{\boldsymbol{q}}(t) = \boldsymbol{A}\boldsymbol{q}(t) + \boldsymbol{B}\boldsymbol{u}(t) + \boldsymbol{E}\boldsymbol{d}(t) \tag{24}$$

$$\boldsymbol{y}(t) = \boldsymbol{C}\boldsymbol{q}(t) \tag{25}$$

where $\boldsymbol{E} \in I\!R_+^{n\times p}$.

Considering state feedback control (4), where $\boldsymbol{K} \in I\!R_+^{r\times n}$, then under the control law action the linear formulation is obtained

$$\dot{\boldsymbol{q}}(t) = \boldsymbol{A}_c\boldsymbol{q}(t) + \boldsymbol{E}\boldsymbol{d}(t) \tag{26}$$

$$\boldsymbol{y}(t) = \boldsymbol{C}\boldsymbol{q}(t) \tag{27}$$

where $\boldsymbol{A}_c$ is introduced in (7), while

$$\boldsymbol{G}_c(s) = \boldsymbol{C}(s\boldsymbol{I}_n - \boldsymbol{A}_c)^{-1}\boldsymbol{B} \tag{28}$$

$$\boldsymbol{G}_{cd}(s) = \boldsymbol{C}(s\boldsymbol{I}_n - \boldsymbol{A}_c)^{-1}\boldsymbol{E} \tag{29}$$

The objectives under following consideration include H_2 and H_∞ performances. Principally, other additional constraints on the closed-loop properties can be imposed [15].

Theorem 2: (H_2 control of strictly Metzlerian systems) The control (4) to system (24), (25) exists if for given positive scalar $\eta \in I\!R_+$ there exist positive definite diagonal matrices $\boldsymbol{V}, \boldsymbol{R}_k \in I\!R^{n\times n}$, $\boldsymbol{H} \in I\!R^{m\times m}$ such that for $h = 1, 2, \ldots n-1$, $k = 1, 2 \ldots, r$,

$$\boldsymbol{H} = \mathrm{diag}\begin{bmatrix} h_1 & h_2 & \cdots & h_m \end{bmatrix} \succ 0 \tag{30}$$

$$\boldsymbol{V} = \mathrm{diag}\begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix} \succ 0 \tag{31}$$

$$\boldsymbol{R}_k = \mathrm{diag}\begin{bmatrix} r_{k1} & r_{k2} & \cdots & r_{kn} \end{bmatrix} \succ 0 \tag{32}$$

$$\boldsymbol{A}(i,i)_{(1\leftrightarrow n)/n}\boldsymbol{V} - \sum_{k=1}^{r}\boldsymbol{B}_{dk}\boldsymbol{R}_k \prec 0 \tag{33}$$

$$\boldsymbol{T}^h\boldsymbol{A}(i,i+h)_{(1\leftrightarrow n)/n}\boldsymbol{T}^{hT}\boldsymbol{V} - \sum_{k=1}^{r}\boldsymbol{T}^h\boldsymbol{B}_{dk}\boldsymbol{T}^{hT}\boldsymbol{R}_k \succ 0 \tag{34}$$

$$\begin{bmatrix} \boldsymbol{A}\boldsymbol{V} + \boldsymbol{V}\boldsymbol{A}^T - \sum_{k=1}^{r}(\boldsymbol{B}_{dk}\boldsymbol{l}\boldsymbol{l}^T\boldsymbol{R}_k + \boldsymbol{R}_k\boldsymbol{l}\boldsymbol{l}^T\boldsymbol{B}_{dk}) & * \\ \boldsymbol{B}^T & -\boldsymbol{I}_r \end{bmatrix} \prec 0 \tag{35}$$

$$\begin{bmatrix} \boldsymbol{V} & \boldsymbol{V}\boldsymbol{C}^T \\ \boldsymbol{C}\boldsymbol{V} & \boldsymbol{H} \end{bmatrix} \succ 0 \tag{36}$$

$$\boldsymbol{l}^T\boldsymbol{H}^\circ\boldsymbol{l} - \eta < 0 \tag{37}$$

with the structural matric variable

$$\boldsymbol{H}^\circ = \mathrm{diag}\begin{bmatrix} \boldsymbol{H} & \boldsymbol{0} \end{bmatrix} \succeq 0 \tag{38}$$

$\boldsymbol{H}^\circ \in I\!R_+^{n\times n}$ and the design parameters are from (13).

With a feasible solution, the controller gain $\boldsymbol{K} \in I\!R_+^{r\times n}$ that solves the design task is

$$\boldsymbol{K}_k = \boldsymbol{R}_k\boldsymbol{V}^{-1}, \quad \boldsymbol{k}_k^T = \boldsymbol{l}^T\boldsymbol{K}_k, \quad \boldsymbol{K} = \begin{bmatrix} \boldsymbol{k}_1^T \\ \vdots \\ \boldsymbol{k}_r^T \end{bmatrix} \tag{39}$$

where $\boldsymbol{K} \in I\!R_+^{r\times n}$. □

Proof: Constructing Lyapunov function and evaluating its derivative function to parse system stability in the following form

$$v(\boldsymbol{q}(t)) = \boldsymbol{q}^T(t)\boldsymbol{P}\boldsymbol{q}(t) > 0 \tag{40}$$

$$\dot{v}(\boldsymbol{q}(t)) = \dot{\boldsymbol{q}}^T(t)\boldsymbol{P}\boldsymbol{q}(t) + \boldsymbol{q}^T(t)\boldsymbol{P}\dot{\boldsymbol{q}}(t) < 0 \tag{41}$$

where $\boldsymbol{P} \in I\!R^{n\times n}$ is a positive definite diagonal matrix, then, with respect to autonomous part of (24),

$$\dot{v}(\boldsymbol{q}(t)) = \boldsymbol{q}^T(t)(\boldsymbol{A}^T\boldsymbol{P} + \boldsymbol{P}\boldsymbol{A})\boldsymbol{q}(t) < 0 \tag{42}$$

Inequality (42) admits a stable solution with a Hurwitz matrix $\boldsymbol{A}$ if

$$\boldsymbol{A}^T\boldsymbol{P} + \boldsymbol{P}\boldsymbol{A} \prec 0 \tag{43}$$

$$\boldsymbol{A}\boldsymbol{V} + \boldsymbol{V}\boldsymbol{A}^T \prec 0 \tag{44}$$

where $\boldsymbol{V} = \boldsymbol{P}^{-1}$. Defining the matrix

$$\boldsymbol{V}_o = \boldsymbol{W}_c\boldsymbol{A} + \boldsymbol{A}^T\boldsymbol{W}_c \tag{45}$$

and subtracting (45) from (43) then

$$\boldsymbol{V}_{oo} = (\boldsymbol{V} - \boldsymbol{W}_c)\boldsymbol{A} + \boldsymbol{A}^T(\boldsymbol{V} - \boldsymbol{W}_c) \tag{46}$$

and $\boldsymbol{V}_{oo}$ be negative definite if $\boldsymbol{A}$ is Hurwitz and

$$\boldsymbol{V} \succ \boldsymbol{W}_c \tag{47}$$

Since $\boldsymbol{V} \succ \boldsymbol{W}_c$ for any $\boldsymbol{W}_c$ satisfying (22) implies

$$\boldsymbol{A}\boldsymbol{V} + \boldsymbol{V}\boldsymbol{A}^T + \boldsymbol{B}\boldsymbol{B}^T \prec 0 \tag{48}$$

and it can easily see that guaranteing (48) it is also satisfied

$$\begin{bmatrix} \boldsymbol{A}\boldsymbol{V} + \boldsymbol{V}\boldsymbol{A}^T & \boldsymbol{B} \\ \boldsymbol{B}^T & -\boldsymbol{I}_r \end{bmatrix} \prec 0 \tag{49}$$

Thus, replacing $\boldsymbol{A}$ in (49) by (7) modifies this LMI as

$$\begin{bmatrix} (\boldsymbol{A} - \boldsymbol{B}\boldsymbol{K})\boldsymbol{V} + \boldsymbol{V}(\boldsymbol{A} - \boldsymbol{B}\boldsymbol{K})^T & \boldsymbol{B} \\ \boldsymbol{B}^T & -\boldsymbol{I}_r \end{bmatrix} \prec 0 \tag{50}$$

and, using (37), it can be set

$$(\boldsymbol{A} - \boldsymbol{B}\boldsymbol{K})\boldsymbol{V} = \boldsymbol{A}\boldsymbol{V} - \boldsymbol{B}\boldsymbol{R} = \boldsymbol{A}\boldsymbol{V} - \sum_{k=1}^{r}\boldsymbol{b}_k\boldsymbol{r}_k^T \tag{51}$$

while

$$\boldsymbol{R} = \boldsymbol{K}\boldsymbol{V},\ \boldsymbol{B}\boldsymbol{R} = \sum_{k=1}^{r}\boldsymbol{b}_k\boldsymbol{r}_k^T = \sum_{k=1}^{r}\boldsymbol{B}_{dk}\boldsymbol{l}\boldsymbol{l}^T\boldsymbol{R}_{dk} \tag{52}$$

Since instead of the inequality (12), a generalized Lyapunov matrix inequality also ensures stability of a controlled Metzlerian system [13], with (51), (52) then (50) immediately implies the inequality (36).

Equality (23) and inequality (47) result in

$$\text{tr}(\boldsymbol{C}\boldsymbol{V}\boldsymbol{C}^T) > \text{tr}(\boldsymbol{C}\boldsymbol{W}_c\boldsymbol{C}^T) = \gamma_2^2 \tag{53}$$

and using the change of variable

$$\boldsymbol{H} \succ \boldsymbol{C}\boldsymbol{V}\boldsymbol{C}^T = \boldsymbol{C}\boldsymbol{V}\boldsymbol{V}^{-1}\boldsymbol{V}\boldsymbol{C}^T \tag{54}$$

$$\text{tr}(\boldsymbol{H}) = \text{tr}(\boldsymbol{H}^\circ) = \boldsymbol{l}^T\boldsymbol{H}^\circ\boldsymbol{l} < \eta \tag{55}$$

with $\boldsymbol{H} \in I\!R^{m\times m}$ being diagonal positive definite, then (54), (55) imply (36), (37). This concludes the proof. ■

Theorem 3: (H_∞ control of strictly Metzlerian systems) The feedback control (4) to the system (24), (25) exists and $\|\boldsymbol{G}_{cd}(s)\|_\infty < \gamma_\infty$ if there exist positive definite diagonal matrices $\boldsymbol{Q}, \boldsymbol{R}_k \in I\!R_+^{n\times n}$ and a positive scalar $\gamma_\infty \in I\!R_+$ such that for $h = 1, 2, \ldots n-1$, $k = 1, 2\ldots, r$,

$$\gamma_\infty > 0 \tag{56}$$

$$\boldsymbol{Q} = \text{diag}\begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix} \succ 0 \tag{57}$$

$$\boldsymbol{R}_k = \text{diag}\begin{bmatrix} r_{k1} & r_{k2} & \cdots & r_{kn} \end{bmatrix} \succ 0 \tag{58}$$

$$\boldsymbol{A}(i,i)_{(1\leftrightarrow n)/n}\boldsymbol{Q} - \sum_{k=1}^{r}\boldsymbol{B}_{dk}\boldsymbol{R}_k \prec 0 \tag{59}$$

$$\boldsymbol{T}^h\boldsymbol{A}(i,i+h)_{(1\leftrightarrow n)/n}\boldsymbol{T}^{hT}\boldsymbol{Q} - \sum_{k=1}^{r}\boldsymbol{T}^h\boldsymbol{B}_{dk}\boldsymbol{T}^{hT}\boldsymbol{R}_k \succ 0 \tag{60}$$

$$\begin{bmatrix} \boldsymbol{A}\boldsymbol{Q}+\boldsymbol{Q}\boldsymbol{A}^T-\sum_{k=1}^{r}(\boldsymbol{B}_{dk}\boldsymbol{l}\boldsymbol{l}^T\boldsymbol{R}_k+\boldsymbol{R}_k\boldsymbol{l}\boldsymbol{l}^T\boldsymbol{B}_{dk}) & * & * \\ \boldsymbol{E}^T & -\gamma_\infty\boldsymbol{I}_p & * \\ \boldsymbol{C}\boldsymbol{Q} & \boldsymbol{0} & -\gamma_\infty\boldsymbol{I}_m \end{bmatrix} \prec 0 \tag{61}$$

where (13) stipulates design parameter structure.

When the above conditions hold, the control gain matrix $\boldsymbol{K} \in I\!R_+^{r\times n}$ is given by (14).

Proof: Reformulating (21) with $\boldsymbol{E}$ and $\boldsymbol{I}_p$ and defining matrix $\boldsymbol{T}$ with only block diagonal nonzero entries such that

$$\boldsymbol{T} = \text{blockdiag}\begin{bmatrix} \boldsymbol{Q} & \boldsymbol{I}_p & \boldsymbol{I}_m \end{bmatrix}, \quad \boldsymbol{Q} = \boldsymbol{P}^{-1} \tag{62}$$

then, pre-multiplying the left side and post-multiplying the ride side by $\boldsymbol{T}$, the inequality (21) reformulated in this way originates the result

$$\begin{bmatrix} \boldsymbol{A}\boldsymbol{Q} + \boldsymbol{Q}\boldsymbol{A}^T & * & * \\ \boldsymbol{E}^T & -\gamma_\infty\boldsymbol{I}_p & * \\ \boldsymbol{C}\boldsymbol{Q} & \boldsymbol{0} & -\gamma_\infty\boldsymbol{I}_m \end{bmatrix} < 0 \tag{63}$$

This allows to obtain a modified condition expressed in terms of LMI by reflecting (7)

$$\begin{bmatrix} (\boldsymbol{A} - \boldsymbol{B}\boldsymbol{K})\boldsymbol{Q} + \boldsymbol{Q}(\boldsymbol{A} - \boldsymbol{B}\boldsymbol{K})^T & * & * \\ \boldsymbol{E}^T & -\gamma_\infty\boldsymbol{I}_p & * \\ \boldsymbol{C}\boldsymbol{Q} & \boldsymbol{0} & -\gamma_\infty\boldsymbol{I}_m \end{bmatrix} < 0 \tag{64}$$

Referring to closed-loop system matrix by notations

$$(\boldsymbol{A} - \boldsymbol{B}\boldsymbol{K})\boldsymbol{Q} = \boldsymbol{A}\boldsymbol{Q} - \boldsymbol{B}\boldsymbol{R} = \boldsymbol{A}\boldsymbol{Q} - \sum_{k=1}^{r}\boldsymbol{b}_k\boldsymbol{r}_k^T \tag{65}$$

$$\boldsymbol{R} = \boldsymbol{K}\boldsymbol{Q},\ \boldsymbol{B}\boldsymbol{R} = \sum_{k=1}^{r}\boldsymbol{b}_k\boldsymbol{r}_k^T = \sum_{k=1}^{r}\boldsymbol{B}_{dk}\boldsymbol{l}\boldsymbol{l}^T\boldsymbol{R}_{dk} \tag{66}$$

the linear matrix inequality (64) implies (61), where instead of the inequality (12), bounded real lemma form (61) is used in design. This concludes the proof. ■

Putting the above presented standard algorithms in the common design context, an enhanced H_2/H_∞ design principle for linear strictly Metzlerian systems, reflecting H_2 and H_∞ norms, makes use of the mixed approach.

Theorem 4: (mixed H_2/H_∞ control of strictly Metzlerian systems) The state feedback control (4) to the system (24), (25) exists and $\|\boldsymbol{G}_{dd}(s)\|_\infty < \gamma_\infty$ if for given positive scalar $\eta \in I\!\!R_+$ there exist positive definite diagonal matrices $\boldsymbol{Q}, \boldsymbol{R}_k \in I\!\!R_+^{n\times n}$, a positive definite diagonal matrix $\boldsymbol{H} \in I\!\!R^{m\times m}$ and a positive scalar $\gamma_\infty \in I\!\!R$ such that for $h = 1, 2, \dots n-1$, $k = 1, 2 \dots, r$,

$$\gamma_\infty > 0 \tag{67}$$

$$\boldsymbol{H} = \text{diag}\begin{bmatrix} h_1 & h_2 & \cdots & h_m \end{bmatrix} \succ 0 \tag{68}$$

$$\boldsymbol{Q} = \text{diag}\begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix} \succ 0 \tag{69}$$

$$\boldsymbol{R}_k = \text{diag}\begin{bmatrix} r_{k1} & r_{k2} & \cdots & r_{kn} \end{bmatrix} \succ 0 \tag{70}$$

$$\boldsymbol{A}(i,i)_{(1\leftrightarrow n)/n}\boldsymbol{Q} - \sum_{k=1}^{r} \boldsymbol{B}_{dk}\boldsymbol{R}_k \prec 0 \tag{71}$$

$$\boldsymbol{T}^h\boldsymbol{A}(i,i+h)_{(1\leftrightarrow n)/n}\boldsymbol{T}^{hT}\boldsymbol{Q} - \sum_{k=1}^{r} \boldsymbol{T}^h\boldsymbol{B}_{dk}\boldsymbol{T}^{hT}\boldsymbol{R}_k \succ 0 \tag{72}$$

$$\begin{bmatrix} \boldsymbol{AQ}+\boldsymbol{QA}^T - \sum_{k=1}^{r}(\boldsymbol{B}_{dk}\boldsymbol{l}\boldsymbol{l}^T\boldsymbol{R}_k + \boldsymbol{R}_k\boldsymbol{l}\boldsymbol{l}^T\boldsymbol{B}_{dk}) & * & * \\ \boldsymbol{E}^T & -\gamma_\infty \boldsymbol{I}_p & * \\ \boldsymbol{CQ} & \boldsymbol{0} & -\gamma_\infty \boldsymbol{I}_m \end{bmatrix} \prec 0 \tag{73}$$

$$\begin{bmatrix} \boldsymbol{AQ}+\boldsymbol{QA}^T - \sum_{k=1}^{r}(\boldsymbol{B}_{dk}\boldsymbol{l}\boldsymbol{l}^T\boldsymbol{R}_k + \boldsymbol{R}_k\boldsymbol{l}\boldsymbol{l}^T\boldsymbol{B}_{dk}) & * \\ \boldsymbol{B}^T & -\boldsymbol{I}_r \end{bmatrix} \prec 0 \tag{74}$$

$$\begin{bmatrix} \boldsymbol{V} & \boldsymbol{VC}^T \\ \boldsymbol{CV} & \boldsymbol{H} \end{bmatrix} \succ 0 \tag{75}$$

$$\eta - \boldsymbol{l}^T\boldsymbol{H}^\circ\boldsymbol{l} > 0, \quad \boldsymbol{H}^\circ = \text{diag}\begin{bmatrix} \boldsymbol{H} & \boldsymbol{0} \end{bmatrix} \tag{76}$$

where (13) stipulates design parameter structure.

When the above conditions hold, a controller gain $\boldsymbol{K} \in I\!\!R_+^{r\times n}$ that solves the design task is given by (14).

Proof: The proof easily follows considering that $\boldsymbol{V} = \boldsymbol{Q}$. Then (30)-(37) and (56)-(61) implies (67)-(76). This concludes the proof. ■

Note, the mixed H_2/H_∞ allows to minimize H_∞ norm constraint, while H_2 norm does'nt exceed η when is optimized over a state-feedback gain $\boldsymbol{K}$.

IV. Illustrative example

To illustrate design strategy, system (24), (25) is considered with the matrix parameters [13]

$$\boldsymbol{A} = \begin{bmatrix} -3.3800 & 0.2080 & 6.7150 & 5.6760 \\ 0.5810 & -4.2900 & 2.0500 & 0.6750 \\ 1.0670 & 4.2730 & -6.6540 & 5.8930 \\ 0.0480 & 2.2730 & 1.3430 & -2.1040 \end{bmatrix}$$

$$\boldsymbol{B} = \begin{bmatrix} 0.0400 & 0.0189 \\ 0.0568 & 0.0203 \\ 0.0114 & 0.0315 \\ 0.0114 & 0.0170 \end{bmatrix}, \ \boldsymbol{E} = \begin{bmatrix} 0.0140 \\ 0.0150 \\ 0.0223 \\ 0.0061 \end{bmatrix}, \ \boldsymbol{C}^T = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

It is easy to verify that the matrix $\boldsymbol{A}$ is not Hurwitz since

$$\rho(\boldsymbol{A}) = \{\ 1.9761, \quad -9.4392, \quad -4.4824 \pm 1.2499\,\mathrm{i}\ \}.$$

Through the design task, the auxiliary parameters are

$$\boldsymbol{T} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad \boldsymbol{l} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\boldsymbol{A}(i,i)_{(1\leftrightarrow 4)/4} = -\text{diag}\begin{bmatrix} 3.3800 & 4.2900 & 6.6540 & 2.1040 \end{bmatrix}$$
$$\boldsymbol{A}(i,i+1)_{(1\leftrightarrow 4)/4} = \text{diag}\begin{bmatrix} 0.2080 & 2.0500 & 5.8930 & 0.0480 \end{bmatrix}$$
$$\boldsymbol{A}(i,i+2)_{(1\leftrightarrow 4)/4} = \text{diag}\begin{bmatrix} 6.7150 & 0.6750 & 1.0670 & 2.2730 \end{bmatrix}$$
$$\boldsymbol{A}(i,i+3)_{(1\leftrightarrow 4)/4} = \text{diag}\begin{bmatrix} 5.6760 & 0.5810 & 4.2730 & 1.3430 \end{bmatrix}$$
$$\boldsymbol{B}_{d1} = \text{diag}\begin{bmatrix} 0.0400 & 0.0568 & 0.0114 & 0.0114 \end{bmatrix}$$
$$\boldsymbol{B}_{d2} = \text{diag}\begin{bmatrix} 0.0189 & 0.0203 & 0.0315 & 0.0170 \end{bmatrix}$$

Prescribing $\eta = 0.75$ then the conditions (67)–(76) are satisfied if LMI matrix variables are

$$\boldsymbol{H} = \text{diag}\begin{bmatrix} 0.2863 & 0.2849 \end{bmatrix}, \quad \gamma_\infty = 0.6665$$
$$\boldsymbol{Q} = \text{diag}\begin{bmatrix} 0.3985 & 0.0406 & 0.0904 & 0.0370 \end{bmatrix}$$
$$\boldsymbol{R}_1 = \text{diag}\begin{bmatrix} 0.8202 & 0.0594 & 1.7990 & 0.0348 \end{bmatrix}$$
$$\boldsymbol{R}_2 = \text{diag}\begin{bmatrix} 0.3348 & 0.2106 & 3.4578 & 0.9646 \end{bmatrix}$$

These results enforce the following strictly positive control gain matrix

$$\boldsymbol{K} = \begin{bmatrix} 2.0581 & 1.4613 & 19.9059 & 0.9418 \\ 0.8400 & 5.1850 & 38.2616 & 26.0717 \end{bmatrix}$$

and

$$\boldsymbol{A}_c = \begin{bmatrix} -3.4782 & 0.0517 & 5.1964 & 5.1461 \\ 0.4471 & -4.4782 & 0.1428 & 0.0923 \\ 1.0172 & 4.0933 & -8.0838 & 5.0621 \\ 0.0103 & 2.1682 & 0.4660 & -2.5582 \end{bmatrix}$$

$$\rho(\boldsymbol{A}_c) = \{ -0.4990 \quad -9.1615 \quad -4.4689 \pm 1.2388\,\mathrm{i} \}$$

that is, the matrix $\boldsymbol{A}_c$ is strictly Metzler and Hurwitz.

Moreover, it can verify that obtained $\gamma_\infty > 0.0120$, whilst $\gamma = 0.0120$ is the real H_∞ norm of $\boldsymbol{G}_{cd}(s)$ and the constraint tr$(\boldsymbol{H}) = 0.5712 >$ tr$(\boldsymbol{CQC}^T) = 0.0776$, when tr$(\boldsymbol{H}) < \eta = 0.75$, which reflects improving of the control performances.

Figure 1 and Fig. 2 show the evolution of the system state and output over time. It is evident that the vector $\boldsymbol{q}(t)$ as well as the output vector $\boldsymbol{y}(t)$ are positive when the input is

$$\boldsymbol{u}(t) = -\boldsymbol{Kq}(t) + \boldsymbol{Ww}(t)$$

Using the static decoupling principle, signal gain matrix $\boldsymbol{W}$, system non-negative initial state $\boldsymbol{q}(0)$ and positive vector $\boldsymbol{w}$ are given as

$$\boldsymbol{W} = \text{diag}\begin{bmatrix} 158.1888 & -94.6102 \\ -230.2012 & 167.2121 \end{bmatrix}, \quad \boldsymbol{w} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$
$$\boldsymbol{q}(0) = \begin{bmatrix} 0 & 0 & 1.8 & 0.5 \end{bmatrix}^T$$

Since the control is based on the diagonal stabilization of linear Metzler systems, the above mentioned principle can only be used if the output matrix $(\boldsymbol{CQC}^T)$ is a diagonal

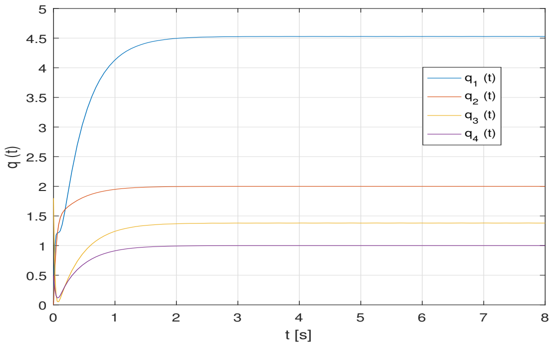

Fig. 1: Corresponding state of the controlled system

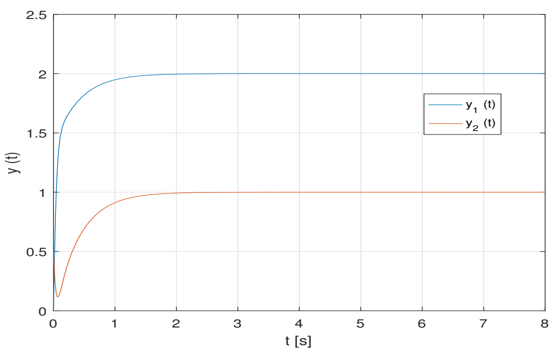

Fig. 2: Corresponding output of the controlled system

matrix (each row of a matrix $\boldsymbol{C}$ can contain no more than one non-zero element). Since even in this case $\boldsymbol{W}$ is generally signum-indefinite, to make the system output non-negative it is necessary to set a suitable non-negative vector of the scheme initial state.

V. Concluding Remarks

The presented approach shows that an optimized disturbance attenuation for systems with linear strictly Metzlerian model is solvable, although control synthesis for this group of system is based on the diagonal stabilization. To accomplish that the closed-loop system Metzler matrix be Hurwitz and the control gain be a positive matrix, the design conditions are linked to positive definite diagonal matrix variables. Novelty is made in the design conditions characterizing continuous-time stability, disturbance attenuations and H_2 objective to be directly applicable to strictly Metzlerian linear continuous-time systems. Fully formulated by means of LMIs, it is interesting to note that the proposed algorithm differs from those exploiting linear programming. Simulations are carried out to evaluate the performance of the proposed scheme.

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