On A New Symbolic Method for Solving Two-point Boundary Value Problems with Variable Coefficients

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Abstract—In this paper, we discuss a simple and efficient symbolic method to find the Green's function of a two-point boundary value problem for linear ordinary differential equations with inhomogeneous Stieltjes boundary conditions. The proposed method is also applicable to find an approximate solution of a two-point boundary value problem for non-linear differential equations. Certain examples are presented to illustrate the proposed method. The method is easy to implement the manual calculations in commercial mathematical softwares, such as Maple, Mathematica, Singular, SCIlab etc. Implementation of the proposed algorithm in Maple is also discussed and sample computations are shown using the Maple implementation.

Keywords—Boundary value problem, Initial value problem, Green's function, Interpolation, Symbolic method.

I. INTRODUCTION

Symbolic computation is playing important role in the scientific field to solve the mathematical equations, especially the problems involving differential equations. The science and technology had a very swift progress in various fields, for example, in computing. One of the biggest success in the research of symbolic computation is the development of significant software systems. Many researchers and engineers have vigorously studied the boundary value problems and its applications, for example, the models of electrical circuits, multi-body systems, diffusion processes, robotic modelling and mechanical systems, nuclear reactors, dissipative operators, vibrating wires in magnetic fields etc. and developed several methods, see for example [16], [14], [20], [13], [10], [21], [17], [7], [9], [15], [22] for various symbolic algorithms and implementations. Also there exist a variety of numerical methods for approximating solutions of two-point boundary value problems [18], [19], [23], [11], [12], [8]. A Green's function, in general, is an integral kernel which represents the analytic solutions of the initial/boundary value problems. Green's function allows us to view the visual interpretation as a result of the actions associated to a source of force or to a charge concentrated at a point [6]. Many scientists and engineers have been studied various types of boundary value problems. Most of them have considered ordinary boundary conditions, i.e., boundary conditions at point evaluation. In this paper, we consider the boundary value problems with Stieltjes boundary conditions, the combination of ordinary conditions, differential conditions and integral conditions. Our focus on this paper is to find the Green's function of a given *boundary value problem* (BVP) for linear ordinary differential equations with inhomogeneous Stieltjes boundary conditions over algebras. The key steps to compute the Green's function are the computation of solution of a given initial value problem and the interpolation technique. In [7], [17], [21], [20], authors discussed the initial value problems for higher-order linear differential systems over integro-differential algebra, and in [10], [13], [16], [15] authors discussed a symbolic method of interpolation with different conditions. Using these techniques, we develop a simple and efficient symbolic method for BVP. The proposed method is also applicable to the BVP for non-linear differential equations. In case of non-linear differential equation, one can use a well-known method for initial value problems to obtain an approximate solution. For example, one can use Picard's iterative method or Eluer's method or Runge-kutta method to obtain an approximate solution of the given initial value problem.

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Following is the plan of the paper: In Section II, we propose a new symbolic algorithm to solve BVP; and Section III presents the certain numerical examples to illustrate the proposed method. We present Maple implementation of the proposed algorithm in Section IV and sample computations are presented using the Maple implementation.

II. A NEW SYMBOLIC ALGORITHM

To work with BVPs in symbolic computation, we need an algebraic structure having differentiation along with integration, so-called *integro-differential algebra*, see for example, [7], [9], [17] for more details. An example of integro-differential algebra is $(\mathcal{F}, \mathtt{D}, \mathtt{A})$, where $\mathcal{F} = C^\infty[a,b]$, $[a,b] \subset \mathbb{R}$, $\mathtt{D}f = \frac{df}{dx}$ and $\mathtt{A}f = \int_a^x f dx$, for a fixed $a \in \mathbb{R}$.

In this paper, we consider the regular BVPs of the following type.

$$\begin{aligned} &Tu = f, \\ &B_a u = c_a, B_b u = c_b, \end{aligned} \tag{1}$$

where $u \in \mathcal{F}$ is unknown function to be determine, $T = f_2\mathtt{D}^2 + f_1\mathtt{D} + f_0 \in \mathcal{F}[\mathtt{D}]$ is linear differential operator with coefficient functions $f_i \in \mathcal{F}$; B_a, B_b are boundary operators; and $c_a, c_b \in \mathbb{R}$ are constants. Given a forcing function $f \in \mathcal{F}$ and the constants $c_a, c_b \in \mathbb{R}$, we want to solve (1) for u. Since the BVP (1) is in the form of operators, proposed method works on the level of operator, i.e. we get the desired Green's function of (1) by performing calculations on various operators related to it. We can also translate the operator problem into a functional setting using the standard methods (see, for example, [3, pp. 188–190]).

In this paper, we focus on the regular BVPs. One can check the regularity of a BVP algorithmically: suppose $\{u_1, u_2\}$ is a fundamental system for T and $\{B_a, B_b\}$ is linearly independent set of boundary operators, then the BVP (1) is regular if and only if the matrix

$$\mathcal{R} = \begin{pmatrix} B_a(u_1) & B_a(u_2) \\ B_b(u_1) & B_b(u_2) \end{pmatrix}$$

is regular, i.e. non-singular. This matrix $\mathcal{R}$ is called the *evaluation matrix*.

Definition 1. *The monic differential operator* $T = \mathrm{D}^2 + p\mathrm{D} + q \in \mathcal{F}[\mathrm{D}]$*, where* $p, q \in \mathcal{F}$*, is called* regular differential operator *if* $\{u_1, u_2\}$ *is linearly independent.*

The key step for obtaining the desired Green's function is to solve the initial value problem underlaying the given BVP.

A. Initial Value Problems

If f is continuous and locally Lipschitz then by the Picard-Lindelöf theorem, for any $\alpha \in \mathbb{R}$ the initial value problem

$$\begin{aligned} &\frac{d^2}{dx^2}u(x) + p(x)\frac{d}{dx}u(x) + q(x)u(x) = f(x), \\ &u(a) = c_a, \left(\frac{d}{dx}u(x)\right)_{x=a} = \alpha, \end{aligned} \tag{2}$$

will have a unique solution on some interval about $x = a$, where $a \in \mathbb{R}$ is a fixed initial value and $p(x), q(x) \in \mathcal{F}$. If we denote E_a as evaluation operator at a, i.e., $\mathrm{E}_a f = f(a)$ for fixed initial value $a \in \mathbb{R}$, then the operator notations of IVP (2) is

$$\begin{aligned} &Tu = f, \\ &\mathrm{E}_a u = c_a, \mathrm{E}_a \mathrm{D} u = \alpha, \end{aligned} \tag{3}$$

where $T = \mathrm{D}^2 + p\mathrm{D} + q \in \mathcal{F}[\mathrm{D}]$ is a monic differential operator, $p, q \in \mathcal{F}$. Now the evaluation matrix and Wronskian matrix are

$$\mathcal{R} = \begin{pmatrix} \mathrm{E}_a u_1 & \mathrm{E}_a u_2 \\ \mathrm{E}_a \mathrm{D} u_1 & \mathrm{E}_a \mathrm{D} u_2 \end{pmatrix}, \qquad W = \begin{pmatrix} u_1 & u_2 \\ \mathrm{D}u_1 & \mathrm{D}u_2 \end{pmatrix},$$

where $\{u_1, u_2\}$ is a fundamental system for T. Using *variation of parameters*, we can solve the IVP (3) uniquely as given in the following theorem.

Theorem 2. *[7], [9], [2] Let* $(\mathcal{F}, \mathrm{D}, \mathrm{A})$ *be an ordinary integro-differential algebra. Given a regular differential operator* $T = \mathrm{D}^2 + p\mathrm{D} + q$ *and a independent set of fundamental system* $\{u_1, u_2\}$*, the initial value problem*

$$\begin{aligned} &Tu = f, \\ &E_a u = c_a, E_a D u = \alpha, \end{aligned}$$

has the unique solution

$$u = c_a + (x - a)\alpha + T^\star f, \tag{4}$$

where $T^\star f = \frac{1}{\det(W)}(u_2 \mathrm{A} u_1\ f - u_1 \mathrm{A} u_2\ f)$*,* A *is integral operator and* W *is the Wronskian matrix of* $\{u_1, u_2\}$*.*

Proof: See, for example, [2, p. 87]. ■

The following section presents an algorithm to compute the desired Green's function of the given BVP (1).

B. Boundary Value Problems

Recall the BVP given by (1),

$$\begin{aligned} &Tu = f, \\ &B_a u = c_a, B_b u = c_b, \end{aligned} \tag{5}$$

The BVP (5) will have a solution if and only if there exists $\alpha \in \mathbb{R}$ such that (i) the maximal interval of existence of the unique solution of IVP (3) contains the interval $[a, b]$, and (ii) the unique solution u of IVP (3) satisfies $B_b u = c_b$. Since we are focused on regular BVP, the condition $B_b u = c_b$ must be satisfied by the solution (4). We have

$$\begin{aligned} B_b u &= c_a + B_b(x - a)\alpha + B_b T^\star f \quad \text{or} \\ \alpha &= \frac{c_b - c_a - B_b T^\star f}{B_b(x - a)}, \end{aligned} \tag{6}$$

where $B_b T^\star f = \frac{1}{\det(W)}(u_2 B_b \mathrm{A} u_1\ f - u_1 B_b \mathrm{A} u_2\ f)$. Now the required Green's function of a given BVP (5) is obtained from the equations (4) and (6) as follows

$$u = c_a + \frac{c_b - c_a - B_b T^\star f}{B_b(x - a)}(x - a) + T^\star f.$$

In particular, if $B_a = \mathrm{E}_a, B_b = \mathrm{E}_b$ then we have

$$\alpha = \frac{c_b - c_a - \mathrm{E}_b T^\star f}{b - a},$$

where $\mathrm{E}_b T^\star f = \frac{1}{\det(W)}(u_2 \mathrm{E}_b \mathrm{A} u_1\ f - u_1 \mathrm{E}_b \mathrm{A} u_2\ f)$ and $\mathrm{E}_b \mathrm{A} u_1 = \int_a^b u_1\ dx$, $\mathrm{E}_b \mathrm{A} u_2 = \int_a^b u_2\ dx$. Now the Green's function is

$$u = c_a + \frac{c_b - c_a - \mathrm{E}_b T^\star f}{b - a}(x - a) + T^\star f.$$

We generalize the above formulation in the following theorem.

Theorem 3. *Let* $(\mathcal{F}, \mathrm{D}, \mathrm{A})$ *be an ordinary integro-differential algebra. Given a regular differential operator* $T = \mathrm{D}^2 + p\mathrm{D} + q$ *and independent sets of fundamental system* $\{u_1, u_2\}$ *and boundary operators* $\{B_a, B_b\}$*, the boundary value problem*

$$\begin{aligned} &Tu = f, \\ &B_a u = c_a, B_b u = c_b, \end{aligned}$$

has the unique solution

$$u = c_a + \frac{c_b - c_a - B_b T^\star f}{B_b(x - a)}(x - a) + T^\star f, \tag{7}$$

where $T^\star f = \frac{1}{\det(W)}(u_2 \mathrm{A} u_1\ f - u_1 \mathrm{A} u_2\ f)$*,* A *is integral operator and* W *is the Wronskian matrix of* $\{u_1, u_2\}$*.*

Algorithm 1. *To find the solution of a given boundary value problem of the type* (5)*, compute the function on* $[a, b]$ *defined by* (7)*.*

If the BVP (1) has no solution then for any solution u of the ordinary differential equation in BVP (1) that satisfies $B_a u = c_a$ and exists on $[a, b]$ the number on the right hand side of (7), hence the value of α specified by (6), exists, but will not be equal to the originally determined value of $\mathrm{E}_a \mathrm{D} u$. Symbolic iterative algorithm for singular BVPs of the type (1) with evaluation boundary operators have been discussed by H. Semiyari and D. S. Shafer in [4] similar to the algorithm presented in Theorem 3 using a Picard iteration scheme for system of first order equations. We recall the algorithm presented in [4], for sake of simplicity.

Consider the following system of first order differential equations

$$\begin{aligned} u' &= v, \quad \mathtt{E}_a u = c_a, \\ v' &= f, \quad \mathtt{E}_a v = \alpha, \end{aligned}$$

which is equivalent to the IVP

$$u'' = f(x,u,u'), \quad \mathtt{E}_a u = c_a, \quad \mathtt{E}_a \mathtt{D} = \alpha.$$

Now the algorithm for approximating solutions BVP is

$$\begin{aligned} u^{[0]}(x) &\equiv c_a, \\ v^{[0]}(x) &\equiv \frac{c_b - c_a}{b-a} \end{aligned} \tag{8}$$

and

$$\begin{aligned} \alpha^{[k+1]} &= \frac{1}{b-a}\left(c_b - c_a - \mathtt{E}_b\mathtt{A}(b-x)f(x,u^{[k]},v^{[k]})\right), \\ u^{[k+1]}(x) &= c_a + \mathtt{A}v^{[k]}, \\ v^{[k+1]}(x) &= \alpha^{[k+1]} + \mathtt{A}f(x,u^{[k]},v^{[k]}). \end{aligned} \tag{9}$$

Algorithm 2. *[4] To approximate the solution of the boundary value problem* $u'' = f(x,u,u'), \quad \mathtt{E}_a u = c_a, \quad \mathtt{E}_a D = \alpha$ *iteratively compute the sequence of functions on* $[a,b]$ *defined by* (8) *and* (9).

III. Examples

Example 1. *(A simple classical example) Consider one of the classical examples that are most often used for introducing the concepts of ordinary linear BVPs [5, p. 42].*

$$\begin{aligned} u'' &= f, \\ u(0) &= \alpha, u(1) = \beta. \end{aligned} \tag{10}$$

It can be interpreted as describing one-dimensional steady heat conduction *in a homogeneous rod. In its functional formulation, we have to solve the BVP* (10) *for the temperature* $u \in C^\infty[0,1]$ *with a given heat source* $f \in C^\infty[0,1]$.

The operator representation of (10) *is*

$$\begin{aligned} &Tu = f, \\ &B_a u = c_a, B_b u = c_b, \end{aligned}$$

where $T = D^2$, $B_a = E_0, B_b = E_1$ *and* $c_a = \alpha, c_b = \beta$. *Following the proposed algorithm, we have*

$$T^\star = xAf - Axf.$$

Using Green's function (7)*, we have*

$$\begin{aligned} u &= \alpha + \frac{\beta - \alpha - E_1 T^\star}{1-0}(x-0) + T^\star \\ &= \alpha + x\beta - x\alpha - xE_1T^\star + T^\star \\ &= (1-x)\alpha + x\beta - xE_1T^\star + T^\star \\ &= (1-x)\alpha + x\beta - xE_1xAf + xE_1Axf + xAf - Axf \\ &= (1-x)\alpha + x\beta - xE_1Af + xE_1Axf + xAf - Axf \\ &\qquad\qquad (\because E_1xAf = E_1xE_1Af = E_1Af) \end{aligned}$$

Translation of the Green's function in functional setting is given by

$$\begin{aligned} u = (1-x)\alpha + x\beta - x\int_0^1 f(x)\ dx + x\int_0^1 xf(x)\ dx \\ + x\int_0^x f(x)\ dx - \int_0^x xf(x)\ dx. \end{aligned}$$

Example 2. *(Damped oscillations) Consider the following BVP*

$$\begin{aligned} u'' + 2u' + u &= f, \\ u(0) = \alpha, u(\pi) &= \beta. \end{aligned} \tag{11}$$

The operator representation of (10) *is*

$$\begin{aligned} &Tu = f, \\ &B_a u = c_a, B_b u = c_b, \end{aligned}$$

where $T = D^2 + 2D + 1$, $B_a = E_0, B_b = E_\pi$ *and* $c_a = \alpha, c_b = \beta$. *Following the proposed algorithm, we have*

$$T^\star = e^{-x}xAe^x f - e^{-x}Axe^x f.$$

Using Green's function (7)*, we have*

$$\begin{aligned} u =& \frac{1}{\pi}(\pi x e^{-x}\int_0^x e^x f(x)\ dx - \pi x e^{-\pi}\int_0^\pi e^x f(x)\ dx \\ &+ xe^{-\pi}\int_0^\pi xe^x f(x)\ dx - \pi x e^{-x}\int_0^x xe^x f(x)\ dx \\ &+ \alpha\pi - \alpha x + \beta x) \end{aligned}$$

Example 3. *Consider a non-linear boundary value problem similar to the Example 4.2 of [4],*

$$\begin{aligned} u'' - \frac{1}{4}uu' &= 16 + (3-2x)^3, \\ u(0) &= \frac{43}{3}, u(1) = 17. \end{aligned} \tag{12}$$

The exact solution of the BVP (12) *is* $u = (3-2x) + 16(3-2x)^{-1}$. *Now we are going to apply the Algorithm 2 to find an approximate solution of the given BVP. Using the equations* (8)-(9)*, we have*

$$\begin{aligned} u^{[0]}(x) &\equiv \frac{43}{3}, \\ v^{[0]}(x) &\equiv \frac{8}{3} \end{aligned}$$

and

$$\begin{aligned} \alpha^{[k+1]} &= \frac{8}{3} - \left(\int_0^1 (1-s)f(s,u^{[k]}(s),v^{[k]}(s))\ ds\right), \\ u^{[k+1]}(x) &= \frac{43}{3} + \int_0^x v^{[k]}(s)\ ds, \\ v^{[k+1]}(x) &= \alpha^{[k+1]} + \int_0^x f(s,u^{[k]}(s),v^{[k]}(s)\ ds. \end{aligned} \tag{13}$$

Now an approximate solution is obtained iteratively for $k = 0, 1, 2, \ldots$ *in* (13). *The first two iterations are given below*

$$\alpha^{[1]} = \frac{8}{3} - \left(\int_0^1 (1-s) f(s, u^{[0]}(s), v^{[0]}(s))\ ds\right) = -\frac{1549}{90}$$
$$u^{[1]} = \frac{43}{3} + \int_0^x v^{[0]}(s)\ ds = \frac{43}{3} + \frac{8}{3}x$$
$$v^{[1]} = \alpha^{[1]} + \int_0^x f(s, u^{[0]}(s), v^{[0]}(s)\ ds$$
$$= -\frac{1549}{90} + \frac{473}{9}x - 2x^4 + 12x^3 - 27x^2.$$

and

$$\alpha^{[2]} = -\frac{16273}{2268}$$
$$u^{[2]} = \frac{43}{3} - \frac{1549}{90}x + \frac{473}{18}x^2 - 9x^3 + 3x^4 - \frac{2}{5}x^5$$
$$v^{[2]} = -\frac{16273}{2268} - \frac{20167}{1080}x + \frac{2457}{40}x^2 - \frac{2777}{324}x^3 + \frac{17}{4}x^4 + \frac{1}{6}x^5 - \frac{2}{9}x^6.$$

At $x = 1$, *after sixth iteration, we have* $|u - u^{[6]}| = 1.0 \times 10^{-11}$. *One can easily observe that the iterates converge to the exact solution.*

IV. Maple Implementation

In this section, we discuss Maple implementation of the proposed algorithm by creating different data types with help of the Maple package `IntDiffOp` implemented by Anja Korporal et al. [1]. The data type `DiffOperator(p,q,r)` is created to generate the differential operator T of a given BVP, where `p,q` and `r` are the coefficients of a given differential equation.

```
DiffOperator := proc (p, q, r)
local diffop;
diffop := DIFFOP(r, q, p);
return diffop;
end proc
```

The function `BVPsolution(diffop, fun, b1, b2, a, b, ini_point)` produces the exact solution of a given regular BVP, where `diffop` is the differential operator T, `fun` is a forcing function $f(x)$, `b1,b2` are boundary operators, i.e. $b1 = B_a, b2 = B_b$, `a,b` are boundary data, i.e. $a = c_a, b = c_b$, `ini_point` is initial point.

```
BVPsolution := proc (diffop, fun, b1, b2,
a, b, ini_point)
local fund_right_inv_diffop, ivp_solution,
ivp_sol_1, ivp_sol_2, eval_matrix,
fund_sys, bvp_solution, funMat, evlMat;
fund_right_inv_diffop :=
FundamentalRightInverse(diffop);
fund_sys := FundamentalSystem(diffop);
ivp_sol_1 := ApplyOperator
(fund_right_inv_diffop, fun);
eval_matrix := EvaluationMatrix(fund_sys,
BC(b1, b2));
funMat := convert(fund_sys, Matrix);
evlMat := convert(eval_matrix, Matrix);
ivp_sol_2 := funMat.(1/evlMat).Matrix([[a],
[b]]);
ivp_solution := simplify
(ivp_sol_1+ivp_sol_2[1, 1]);
bvp_solution := simplify
(a+(b-a-ApplyOperator(MultiplyOperator
(b2, fund_right_inv_diffop), fun))
*(x-ini_point)/ApplyOperator(b2,
x-ini_point)+ivp_sol_1);
return bvp_solution
end proc
```

Using the following procedure `RegularTest(diffop, b1, b2)`, one can check the regularity of a given BVP.

```
RegularTest := proc (diffop, b1, b2)
local fund_sys, eval_matrix, detment;
fund_sys := FundamentalSystem(diffop);
eval_matrix := EvaluationMatrix(fund_sys,
BC(b1, b2));
if abs(eval_matrix) = 0 then
print(Singular BVP)
else print(Regular BVP) end if
end proc
```

Recall the BVP (11) presented in Example 2 for sample computations using the maple implementation of the proposed algorithm.

```
T := DiffOperator(1,2,1);
```

$$1 + 2.D + D^2$$

```
B1 := BOUNDOP(EVOP(0, EVDIFFOP(1),
EVINTOP()));
B2 := BOUNDOP(EVOP(π, EVDIFFOP(1),
EVINTOP()));
```

$$\mathtt{E}[0]$$
$$\mathtt{E}[\pi]$$

```
RegularTest(T,B1,B2);
```

$$Regular\ BVP$$

```
BVPSolution(T,f(x),B1,B2,α,β,0);
```

$$\frac{1}{\pi}(\pi x e^{-x} \int_0^x e^x f(x)\ dx - \pi x e^{-\pi} \int_0^\pi e^x f(x)\ dx + x e^{-\pi} \int_0^\pi x e^x f(x)\ dx - \pi x e^{-x} \int_0^x x e^x f(x)\ dx + \alpha\pi - \alpha x + \beta x)$$

V. Conclusion

In this paper, we presented a simple and efficient symbolic method to solve a two-point boundary value problem with inhomogeneous Stieltjes boundary conditions. we also recalled a similar algorithm to find an approximate solution of a two-point boundary value problem for non-linear differential equations. Certain examples are discussed to illustrate the proposed method and the Maple implementation of the proposed algorithm is also discussed and presented sample computations using the implementation.

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