

On the correctness of one model of the Stefan-Type filtration theory

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Abstract: In this paper a mathematical model of filtration theory with phase transitions is investigated. When using surface-active substances (surfactants) for the development of oil and gas fields in the reservoir occur sorption processes at the interfaces of individual phases (surfactants and oil, or surfactants and soil). In real processes, a finite time is required for achievement equilibrium. Therefore considering the mathematical model was called the mathematical model with phase relaxation. The solvability of the mathematical model, the limiting transition in relaxation time are investigated. It is proved that in the limiting case, the original problem is a problem of Stefan type.

Keywords. Sorption, adsorption, surfactant, relaxation time, mass transfer processes, numerical experiments.

I INTRODUCTION

This paper is a logical continuation of studying the mathematical model presented in [1]. However, between phases, many authors believe that either Henry's law or Langmuir's law is fulfilled. Based on the results of [2], below we assume that there is some characteristic relaxation time for achieving equilibrium between the phases:

$$\frac{\partial s}{\partial t} = \frac{1}{\tau}(H(c) - s), \tag{1}$$

where τ positive constant and called relaxation time. Then the concentration of

surfactant $c(x,t)$ is a solution to the following equation:

$$m \cdot \frac{\partial c}{\partial t} = D \cdot \Delta c - v \cdot \nabla c - \frac{\partial s}{\partial t}, \tag{2}$$

where m, D – positive constants, v – rate of fluid filtration in a porous medium, function $H(c) = 1$ if $c(x, t) > c_*$, $H(c) = 0$ if $c(x, t) < c_*$ and in the case of equilibrium processes $s = H(c)$. Then the system (1), (2) reduces to the well-known Stefan problem. Similar mathematical models were investigated in [1,2,7].

II. FORMULATION OF THE PROBLEM

Let Ω – is a bounded domain in R^m with a sufficiently smooth border $\Gamma, Q_T = \Omega \times (0, T), \Gamma_T = \Gamma \times (0, T)$. Required to find functions $c(x, t), s(x, t)$ (surfactant concentrations in liquid and solid phases), defined in the area Q_T , satisfying equations (1), (2) and initial conditions

$$c(x, 0) = c_0(x), \quad s(x, 0) = s_0(x), \quad x \in \Omega \tag{3}$$

and one of the boundary conditions

$$c(x, t) = c_\Gamma(x, t), \quad (x, t) \in \Gamma_T \tag{4}$$

$$\frac{\partial c}{\partial n} - v \cdot c(x, t) = c_\Gamma(x, t), \quad (x, t) \in \Gamma_T \tag{5}$$

Here n – internal normal vector to S .

Further, under task I is understood as task (1)-(4), and task II is understood as task II (1)-(3), (5).

Definition 1. The solution of problem I (problem II) is a pair of functions $\{c, s\}$ such that:

1. $c \in W_q^{2,1}(Q_T)$, $1 < q < \infty$, $s_t, s \in L_\infty(Q_T)$;
2. Equations (1), (2) are performed almost everywhere (a. e.) in Q_T ;
3. Initial and boundary conditions for $c(x, t)$ are accepted in the sense of traces of functions from the indicated classes, for s the initial condition is taken as follows $\|s(x, t) - s_0\|_{\infty, \Omega} \rightarrow 0$ at $t \rightarrow 0$.

Designations norms and spaces of functions coincide with the notation in [2].

III. AUXILIARY SENTENCES

Lemma 1. Let $u \in W_p^1(Q)$, Q - is a bounded domain in R^k , $p > 1$, $A_\varepsilon = \{x \in Q \mid |u(x)| \leq \varepsilon\}$. Then $\nabla u = 0$ a. e. in A_0 .

Lemma 2. Let Q - is a bounded domain in R^k , $v_n, v, g \in L_p(Q)$, $p > 1$, $\forall x \in Q \setminus A$, $\lim_{n \rightarrow \infty} v_n(x) = f(x)$ and $\forall n \in N, |v_n(x)| \leq g(x)$, where $mes A = 0$; $v_n \rightarrow v$ weakly in $L_p(Q)$. Then $v \geq f$ a. e. in Q .

From Lemma 1 it follows that in the domain $E_c = \{(x, t) \in Q_T \mid c(x, t) = 0\}$ equalities are performed: $c_t = \Delta c = 0$. Then from equations (1), (2) are displayed: $H(c(x, t)) = s(x, t)$ for a. e. $(x, t) \in E_c$. From the definition of the function $H(c)$, in particular, it follows that $0 \leq s(x, t) \leq 1$ for a. e. $(x, t) \in E_c$.

3. The correctness of the mathematical model. Replacement of independent variables and sought functions is done:

$$t' = \frac{t}{\tau}, \quad x' = x\sqrt{m/\tau D}, \quad s' = s, \quad c' = cm, \quad H(c) = H'(mc).$$

Also, the rate of fluid filtration everywhere below is considered positive constant and the strokes are omitted in the equations.

$$s_t = H(c) - s, \quad (6)$$

$$c_t - \Delta c + v \cdot \nabla c + s_t = 0. \quad (7)$$

Beyond the change area of independent variables, the initial and boundary data old designations are preserved. Further, the problem I' (I'') is understood as task I (II) in which equations (1), (2) are replaced by (6), (7).

Theorem 1. Let the border $\Gamma \in O^2$ function $u \in W_q^{2,1}(Q_T)$ conditions (3), (4) (respectively (3), (5)), $s_0, c_0, c_\Gamma(x, t)$ measurable and $0 \leq \{s_0(x), c_0, c_\Gamma(x, t)\} \leq 1, x \in \Omega$. Then the problem I' (accordingly the task II') has a unique solution. In this case the following estimates are fair:

$$\|c\|_{q, Q_T}^{(2)} \leq K_1(1 + \|u\|_{q, Q_T}^{(2)}), \quad 0 \leq c(x, t) \leq 1 \quad (8)$$

$$0 \leq s(x, t) \leq 1, \quad |s_t| \leq 1. \quad (9)$$

Positive constant K_1 depends only q, Ω and T . It should be noted, that when $q > (m + 2)/2$ the solution $c \in H^\alpha(Q_T)$ for some $\alpha > 0$. When $q > m + 2$ it becomes Holders and ∇c .

4. Existence. The function $H(c)$ is approximated by continuous monotonic functions $H_n(c)$ at $c > \frac{1}{n} + c_*$, $c < c_*$, $n = 1, 2, 3, \dots$. Through $(6)_n, (7)_n$ denoted the equation (6), (7), where instead of the function H function H_n is considered. For each n an approximate problem $(6)_n, (7)_n, (3), (4)$ is considered. The operator is determined $P: W_q^{2,1}(Q_T) \rightarrow W_q^{2,1}(Q_T)$ fixed point which gives a solution to this problem.

Let $c, g \in W_q^{2,1}(Q_T)$, then by definition $c = P(g)$, if c satisfies the equation (7)

$$s(x, t) = s_0(x) \cdot e^{-t} + \int_0^t H_n(g(x, \mu)) \cdot e^{\mu-t} d\mu$$

and conditions (3), (4).

For $s(x, t)$, from the written submission, the assessment should be performed (9). Then for $c(x, t)$ assessment is performed (8). Conditions (3), (4), (8) determine in $W_q^{2,1}(Q_T)$ some convex, closed, bounded subset, which the operator P translates into itself. Since P is completely continuous, by the Schauder theorem there is a fixed point of the operator P , which gives the solution to the problem $(6)_n$, $(7)_n$, (3), (4). We denote it by $\{c_n, s_n\}$. The second estimate in (8) follows from the maximum principle. In this to obtain the upper estimate, the cutting function is introduced: $\hat{c} = \max\{0, c - 1\}$. To obtain a lower estimate, the cutting function has the form: $\hat{c} = \min\{0, c\}$. Estimates (8), (9) for c_n, s_n and limitation $H_n(c_n)$ allow us to select subsequences n_k , such that:

$$c_{n_k} \rightarrow c \text{ a.e. in } Q_T, \quad \frac{\partial c_{n_k}}{\partial t} \rightarrow \frac{\partial c}{\partial t}, \quad \nabla c_{n_k} \rightarrow \nabla c, \quad \Delta c_{n_k} \rightarrow \Delta c \text{ weakly in } L_q(Q_T),$$

$$s_{n_k} \rightarrow s, \quad \frac{\partial s_{n_k}}{\partial t} \rightarrow \frac{\partial s}{\partial t}, \quad H_{n_k}(c_{n_k}) \rightarrow h$$

* weakly in $L_\infty(Q_T)$.

From the function definition $H_n(c_n)$ and $c_{n_k} \rightarrow c$ a.e. in Q_T , follows that $h(x, t) = H(c(x, t))$ a.e. in $Q_T \setminus E_c$. On the set E_c function $h(x, t) = s(x, t)$ and by definition of function $H(c)$ and lemma 2: $0 \leq h(x, t) \leq 1$. So on set E_c function $h(x, t)$ coincides with $(c(x, t))$. Passing to the limit in $(6)_n$, $(7)_n$, (3), (4) at $n_k \rightarrow \infty$ we obtain, that the limit functions $c(x, t)$, $s(x, t)$ are the desired solution of the problem I' .

5. Uniqueness. To prove the uniqueness the solvability of the adjoint problem is shown.

Let $c_i, s_i, i = 1, 2$ – two problem solutions I' . We put

$$c = c_1 - c_2, \quad s = s_1 - s_2, \quad H = H(c_1) - H(c_2).$$

Functions c, s, H satisfy equations (6), (7) and conditions:

$$c, s|_{t=0} = 0, \quad c|_{\Gamma_T} = 0, \quad H|_E = s \tag{10}$$

where $E = E_{c_1} \cap E_{c_2}$. Auxiliary functions are introduced F_δ и $F_{\varepsilon, \delta}$: $F_\delta = H/c$ on set $E = \{(x, t) \in Q_T \mid |c(x, t)| \geq \delta\}$ and $F_\delta = 0$ on $Q_T \setminus E$, and the function $F_{\varepsilon, \delta}$ is selected from the conditions:

$$F_{\varepsilon, \delta} \in \dot{C}^\infty(Q_T), \quad 0 \leq F_{\varepsilon, \delta} \leq F_\delta, \quad \lim_{\varepsilon \rightarrow 0} \|F_{\varepsilon, \delta} - F_\delta\|_{1, Q_T} = 0.$$

In Q_T functions φ, ψ are considered, fairly smooth, satisfying the conditions:

$$\varphi, \psi|_{t \in [T_0, T]} = 0, \quad \varphi|_{\Gamma_T} = 0, \quad 0 < T_0 \leq T \tag{11}$$

From (6), (7), (10), (11), the equality is derived:

$$\int_{Q_T} \{c \cdot M_1(\varphi, \psi) + s \cdot M_2(\varphi, \psi) + (H - F_{\varepsilon, \delta} c) \cdot \psi\} dx dt = 0 \tag{12}$$

where $M_1(\varphi, \psi) = \varphi_t + \Delta \varphi + v \cdot \nabla \varphi + F_{\varepsilon, \delta} \psi$, $M_2(\varphi, \psi) = \varphi_t + \psi_t - \psi$.

Let be $G_1, G_2 \in \dot{C}^\infty(Q_{T_0})$ and in Q_{T_0} equations are considered

$$M_i(\varphi, \psi) = G_i, \quad i = 1, 2 \tag{13}$$

Solvability of problem (11), (13) is obtained in the standard way, based on the local existence theorem and a priori estimates. From (13) view is displayed for ψ :

$$\psi(x, t) = \int_t^{T_0} \{\psi(x, \mu) + G_2(x, \mu)\} d\mu - \varphi(x, t) \tag{14}$$

Function $\Phi(x, t)$ determined by the equality:

$$\Phi(x, t) = \varphi(x, t) \cdot e^t \tag{15}$$

The following equality is a consequence of (13)-(15):

$$\begin{aligned} &\Phi \cdot \Phi_t + \Phi \cdot \Delta \Phi + v \cdot \Phi \cdot \nabla \Phi - (1 + F_{\varepsilon, \delta}) \cdot \\ &\Phi^2 = \{G_1 - F_{\varepsilon, \delta} \cdot \int_t^{T_0} (\psi + G_2) d\mu\} \cdot e^t \cdot \Phi \end{aligned} \tag{16}$$

Consider (16) at the point of the internal maximum of the function Φ . The first two terms at this point are nonpositive and the third term turn to zero, so taking into account (15), we obtain the estimate:

$$|\varphi| \leq |\Phi| \leq \left\{ \int_0^{T_0} (|\psi| + |G_2|) dt + |G_1| \right\} \cdot e^{T_0} \tag{17}$$

From (14) and (17) using the Gromwell inequality, the following estimate is obtained:

$$\|\psi\|_{\infty, Q_T} \leq e^{T_0} \|G_1\|_{\infty, Q_{T_0}} + K_2 \cdot T_0 \tag{18}$$

Constant K_2 depends on T and norms $\|G_i\|_{\infty, Q_{T_0}}, i=1,2$. After the passage to the limit $\rightarrow 0$, then by $\delta \rightarrow 0$ by virtue of estimate (18) and conditions (10), equality (12) takes the form:

$$\int_Q \{c \cdot G_1 + s \cdot G_2\} dxdt + \int_E \psi \cdot s dxdt = 0. \tag{19}$$

We can assume that $G_i, i = 1,2$ – arbitrary functions from $L_\infty(Q_{T_0})$. Suppose $T_0 = \min(T,$

$\frac{1}{2 \cdot K})$. Choosing $G_1 = 0, G_2 = \text{sign } s$, from (18), (19) we get, that $s = 0$ in Q_{T_0} . At $G_2 = 0, G_1 = \text{sign } c$ from (19) $c = 0$ in Q_{T_0} . Similarly, uniqueness is shown in $\Omega x[T_0, 2 \cdot T_0], \Omega x[2 \cdot T_0, 3 \cdot T_0]$ etc. For a finite number of steps, we obtain uniqueness in Q_T .

IV. THE CONTINUITY OF THE SOLUTION ON THE INITIAL AND BOUNDARY DATA.

Let be $c_i, s_i, i = 1,2$ – two solutions of the problem II' , satisfying the initial and boundary conditions:

$$\begin{aligned} c_i(x, 0) &= u_i(x, 0), \quad s_i(x, 0) \\ &= s_{0i}(x), \quad x \in \Omega. \end{aligned}$$

$$\left. \frac{\partial c_i}{\partial n} \right|_{\Gamma_T} = \left. \frac{\partial u_i}{\partial n} \right|_{\Gamma_T}, \quad i = 1,2.$$

We introduce the notation:

$$\begin{aligned} c &= c_1 - c_2, \quad s = s_1 - s_2, \quad H \\ &= H(c_1) - H(c_2), \quad s_0 \\ &= s_{01} - s_{02}, \end{aligned}$$

$$\begin{aligned} u &= u_1 - u_2, \quad \delta = \|u\|_{q, Q_T}^{(2)} + \\ \|s_0\|_{\infty, \Omega}, \quad c_0 &= c_{01} - c_{02}, \quad c_\Gamma = c_{\Gamma 1} - \\ &c_{\Gamma 2}. \end{aligned}$$

Then the functions c, s, H satisfy equations (6), (7) and conditions (3), (5).

Theorem 2. If the assumptions of Theorem1 are fulfilled for the functions $u_i, s_{0i}, i = 1,2$ then the following estimates are true:

$$\|c\|_{q, Q_T}^{(2)} \leq K_3 \cdot \delta^{1/q}, \tag{20}$$

$$\|s_t\|_{p, Q_T} + \|s\|_{p, Q_T} \leq K_4 \cdot \delta^{1/p} \tag{21}$$

where $1 \leq p < \infty$, constants K_3, K_4 depend on $q, T, \Omega, \|u_i\|_{q, Q_T}^{(2)}, \|s_{0i}\|_{\infty, \Omega}, i = 1,2$.

Proof. Multiply equation (7) by $c \cdot (c^2 + \varepsilon)^{-\frac{1}{2}}, \varepsilon > 0$, and integrate over the region Q_θ :

$$\begin{aligned} &\int_{Q_\theta} \{[(c^2 + \varepsilon)^{\frac{1}{2}}]_t + \varepsilon \cdot (\nabla c)^2 \cdot (c^2 + \varepsilon)^{-\frac{3}{2}} + \\ &H \cdot c \cdot (c^2 + \varepsilon)^{-\frac{1}{2}} - s \cdot c \cdot (c^2 + \varepsilon)^{-\frac{1}{2}}\} dxdt = \\ &= \int_\Gamma c_\Gamma \cdot c \cdot (c^2 + \varepsilon)^{-\frac{1}{2}} d\Gamma + \varepsilon \cdot v \cdot \\ &\int_{Q_\theta} \nabla \left[(c^2 + \varepsilon)^{\frac{1}{2}} \right] dxdt. \end{aligned}$$

Drop the nonnegative term in this equality $\varepsilon \cdot (\nabla c)^2 \cdot (c^2 + \varepsilon)^{-\frac{3}{2}}$ and in the resulting inequality we go to the limit by $\varepsilon \rightarrow 0$:

$$\int_{Q_\theta} \{ |c|_t + |H| - |s| \} dxdt \leq \int_\Gamma |c_\Gamma| d\Gamma. \tag{22}$$

$$\int_{Q_\theta} \{ |s|_t + |s| - |H| \} dxdt \leq 0. \tag{23}$$

From inequalities (22), (23) follows that

$$\int_\Omega \{ |c(x, \theta)| + |s(x, \theta)| \} dx \leq \|c_0\|_{1,\Omega} + \|s_0\|_{1,\Omega} + \|c_\Gamma\|_{1,\Gamma_\theta} \leq K \cdot \delta. \tag{24}$$

From the estimates (22), (24) the inequality is derived: $\|H\|_{1,Q_T} \leq K \cdot \delta$. From here and from equation (6) is estimated: $\|s_t\|_{1,Q_T} \leq K \cdot \delta$. Evaluation (21) is a consequence of the resulting assessment and the limitations of the norm $\|s\|_{\infty,Q_T}$. The considered function $c(x, t)$, as a solution to problem (7), (3), (5) and, taking into account (21), we obtain the estimate (20).

It should be noted, to get ordinary continuity you need to show that $\lim_{\delta \rightarrow 0} \|s_t\|_{\infty,Q_T} = 0$. In general, this equality does not performed.

Suppose we have two solutions to the problem under consideration: $c \equiv 0, s \equiv 0$ и $c(x, t) = 1 - s(x, t)$,

$$\begin{cases} 0 < \varepsilon < 1, & s(x, t) = \\ 1 - e^{-t}, & 0 \leq t \leq -\ln(1 - \varepsilon), \\ \varepsilon, & -\ln(1 - \varepsilon) \leq t \leq T. \end{cases}$$

The second solution satisfies the following initial and boundary data:

$$\begin{aligned} c|_{t=0} = \varepsilon, \quad s|_{t=0} = 0, \quad \left(\frac{\partial c}{\partial n} - v \cdot c \right) |_{\Gamma_T} \\ = 0. \end{aligned}$$

Obviously, in this case $\delta = \varepsilon$, а $\|s_t\|_{\infty,Q_T} = 1$.

1. Limit transation in relaxation time. Below, we will study the behavior of the solution as $\tau \rightarrow 0$ using the example of problem 1. For problem II a similar result is true. We introduce $K(Q_T)$ – function space with the norm:

$$\|u\|_{K,T} = \|u\|_{\infty,Q_T} + \|u_{tt}\|_{1,Q_T} + \|\nabla u\|_{2,Q_T} + \|u_t\|_{2,Q_T} + \|\nabla u_t\|_{2,1,Q_T}.$$

Functions $c^\tau(x, t), s^\tau(x, t)$ satisfying equations (1), (2) and conditions are considered:

$$(c^\tau(x, t) - c_0^\tau)|_{\Gamma_T \cup \{t=0\}} = 0, \quad s^\tau|_{t=0} = s_0^\tau \tag{25}$$

$$\text{where } c_0^\tau \in W_q^{2,1}(Q_T) \cap K(Q_T), \quad s_0^\tau \in L_\infty(\Omega) \text{ and } s_0^\tau(x) = H(c_0^\tau(x)), \quad x \in \Omega. \tag{26}$$

Without loss of generality, constants everywhere below are considered, equal to one.

Lemma 3. To solve problem (1), (2), (25), (26), the following estimates are valid.:

$$\|c^\tau\|_{\infty,Q_T} \leq K_5, \tag{27}$$

$$\|c_t^\tau\|_{2,Q_T} + \max_{[0,T]} \|\nabla c^\tau\|_{2,\Omega} \leq K_6, \tag{28}$$

$$\|H(c^\tau) - s^\tau\|_{1,Q_T^\delta} \leq K_7 \cdot \delta^{-1/2} \cdot \tau, \quad \delta > 0 \tag{29}$$

where $Q_T^\delta = \Omega^\delta \times (0, T), \quad \Omega^\delta = \{x \in \Omega \mid \text{dist}(x, \mu) > \delta\}, \delta > 0\}$, and the constants $K_i, i=5, 6, 7$ depend only on T, Ω and $\|c_0^\tau\|_{K,T}$.

Proof. From equation (1), (2) and condition (26) we get follow estimates:

$$\begin{aligned} 0 \leq s^\tau(x, t) \leq 1, \quad s_t^\tau \cdot (c^\tau - c^*) \geq \\ 0, \quad \int_0^\theta s_t^\tau \cdot (c^\tau - c^*)_t dt \geq 0, \theta \leq T. \end{aligned}$$

From the maximum principle should also be estimated: $0 \leq c^\tau(x, t) \leq 1$.

Here and further, where there is no misunderstanding, the index τ of the functions c_0^τ, c^τ, s^τ – is omitted.

Let us show how the last one turns out:

$$\begin{aligned} \tau \int_0^\theta s_t \cdot (c - c^*)_t dt &= \int_0^\theta (H(c) - s) \cdot \\ (c - c^*)_t dt &= (H(c) - s) \cdot (c - c^*)|_0^\theta + \\ + \int_0^\theta s_t \cdot (c - c^*) dt &\geq 0. \end{aligned}$$

From the resulting estimates and on the basis of the results from [2] the estimate (28) follows. To obtain the estimate (29), the function is introduced:

$$f \in W_2^1(\Omega), \quad f|_{\Omega^\delta} = 1, \quad \|\nabla f\|_{2,\Omega} \leq K(\Omega) \cdot \delta^{-1/2}$$

Further, equation (2) is multiplied by $f \cdot c \cdot (c^2 + \varepsilon)^{-\frac{1}{2}}$, $\varepsilon > 0$ and integrated by Q_T :

$$\begin{aligned} \int_{Q_\theta} \{ [f \cdot (c^2 + \varepsilon)^{\frac{1}{2}}]_t + \varepsilon \cdot f \cdot (\nabla c)^2 \cdot \\ (c^2 + \varepsilon)^{-\frac{3}{2}} + s_t \cdot f \cdot c \cdot (c^2 + \varepsilon)^{-\frac{1}{2}} \} dx dt = \\ = \varepsilon \cdot v \cdot \int_{Q_\theta} \nabla f \cdot \nabla [(c^2 + \varepsilon)^{\frac{1}{2}}] dx dt + \\ \int_{Q_\theta} \nabla f \cdot \nabla c \cdot c \cdot (c^2 + \varepsilon)^{-\frac{1}{2}} dx dt. \end{aligned}$$

Passing to the limit at $\varepsilon \rightarrow 0$ and discarding the nonnegative terms in the left-hand side, we obtain the estimate (29). The lemma is proved.

Let be $U = c + H(c)$ generalized solution of the Stefan problem (see definition, for example, in [9]), satisfying initial and boundary conditions:

$$\begin{aligned} U(x, 0) &= c_0(x, 0) + \\ s_0(x), & \quad x \in \Omega, \end{aligned} \quad (30)$$

$$c(x, t) = c_0(x, t), \quad (x, t) \in \Gamma_T,$$

where $c_0 \in K(Q_T), s_0 \in L_\infty(\Omega), s_0 = H(c_0(x, 0))$.

Theorem 3. If

$$\begin{aligned} \|c_0^\tau - c_0\|_{K,T} + \|s_0^\tau - s_0\|_{1,\Omega} \rightarrow 0 \text{ at } \tau \rightarrow \\ 0, \end{aligned} \quad (31)$$

then $c^\tau + s^\tau$ converges to U at $\tau \rightarrow 0$ in the following sense:

$$c^{\tau_k} \rightarrow c^\circ \text{ weakly in } W_2^1(Q_T), \quad * - \text{ weakly in } L_\infty(Q_T), \quad (32)$$

$$s^{\tau_k} \rightarrow H(c^\circ) \quad * - \text{ weakly in } L_\infty(Q_T) \text{ at } \tau_k \rightarrow 0$$

Really, the functions c^τ, s^τ satisfy the identity::

$$\begin{aligned} \int_{Q_T} \{ c^\tau \cdot \varphi - (c^\tau + s^\tau) \cdot \varphi_t \} dx dt \\ = \int_\Omega (c_0^\tau + s_0^\tau) \cdot \varphi(x, 0) dx \end{aligned}$$

for any $\varphi \in W_2^1(Q_T), \varphi|_{\Gamma_T \cup \{t=0\}} = 0$. Then, from conditions (31) and estimates (27) - (29) we can choose a subsequence that $\tau_k \rightarrow 0$ (32) are fulfilled due to the uniqueness of the whole family converges to solving the Stefan problem as $\tau \rightarrow 0$.

V. NUMERICAL EXPERIMENTS.

Before proceeding to the construction of algorithms for solving the original problem, we will analyze one model problem from [5], which has important practical significance, since it is the basis of the models, taking into account the given equations of the kinetics of changes in any parame.

$$\frac{\partial u}{\partial t} - \Delta u + \chi \frac{\partial w}{\partial t} = 0 \quad (33)$$

$$\frac{\partial w}{\partial t} = \frac{1}{\tau} [H(u) - w] + \lambda \Delta w \quad (34)$$

$$H(u) = \begin{cases} 1, & u > u^* \\ w, & u = u^* \\ 0, & u < u^* \end{cases} \quad (35)$$

where $H(u)$ characterizes the phase transition, τ - relaxation time, χ, λ - positive weight coefficients, taking into account the nature of the contribution process (transience and slowness).

For (33) - (35) the following initial and boundary conditions are given:

$$\begin{aligned} u(x,0) = u_0(x), \quad w(x,0) = w_0(x), \\ x \in \Omega \end{aligned} \quad (36)$$

$$u(x,t) = u_s(x,t), \quad (x,t) \in S_T \quad (37)$$

The correctness of the problem (33) - (37) and the qualitative properties of the solutions (the asymptotic behavior of the solution with an unlimited increase in time, limiting transitions in relaxation time and λ - migration of moisture It is proved that problem (33) - (37) is a problem of Stefan type. When $\lambda \rightarrow 0$, problem (33) - (37) tends to solve the problem without taking into account moisture migration.) were investigated in detail in [5].

It is known, that systems of equations like (33) - (37) have a number of features, in particular, depending on the values $\tau, \chi,$

λ and u^* the solution of the system is either completely absent, or contains jumps, which leads to the expansion or contraction of the phase transition zone. We present a numerical algorithm only for the one-dimensional case with a given type of boundary and initial data, i.e.:

$$u_t - u_{xx} + \chi w_t = 0 \quad (38)$$

$$w_t = \frac{1}{\tau} [H(u) - w] + \lambda w_{xx} \quad (39)$$

$$H(u) = \begin{cases} 1, & u > u^* \\ w, & u = u^* \\ 0, & u < u^* \end{cases} \quad (40)$$

initial and boundary conditions:

$$\begin{aligned} u(x,0) = e^{-x}, \\ w(x,0) = 1 - e^{-x} = w_0(x), \quad x \in \Omega \end{aligned} \quad (41)$$

$$\begin{aligned} u(0,t) = (1+t)^2, \quad u(1,t) = (1+t)^3, \\ (x,t) \in S_T \end{aligned} \quad (42)$$

In system (38) - (42) at the beginning we solve (39) with the corresponding boundary conditions, for a given value u . Further we find u at the found value of w . To do this, we introduce a uniform grid $x_i = x_0 + ih, \quad i = \overline{0, N},$ where $h = \text{const}$ - grid spacing $t^n = n\tilde{\tau}, \quad n = 0,1,2,\dots,$ here $\tilde{\tau}$ - time step. Required functions u, w at the nodes (x_i, t^n) hereinafter referred to as u_i^n, w_i^n respectively. Then we write the difference form of the equation of system (38) and (39) for a constant grid step h :

$$\frac{1}{\tau} (H(u_i^n) - w_i^{n+1}) + \lambda \frac{w_{i+1}^{n+1} - 2w_i^{n+1} + w_{i-1}^{n+1}}{h^2} = \frac{w_i^{n+1} - w_i^n}{\tilde{\tau}} \quad (43)$$

$$\frac{u_i^{n+1} - u_i^n}{\tilde{\tau}} + \chi \frac{w_i^{n+1} - w_i^n}{\tilde{\tau}} = \frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{h^2} \quad (44)$$

This system is solved sequentially by applying the sweep method to each equation. The stability conditions of the sweep method are satisfied.

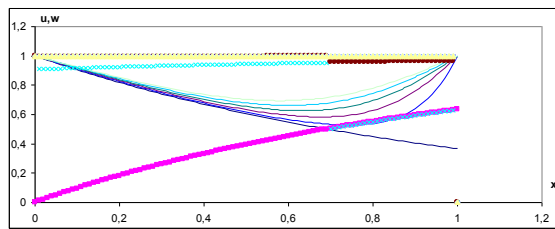
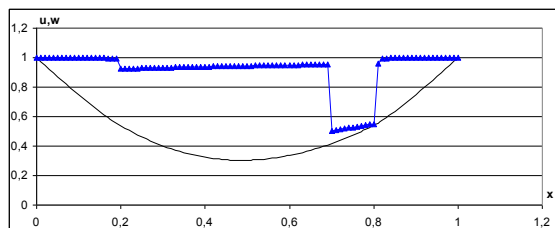
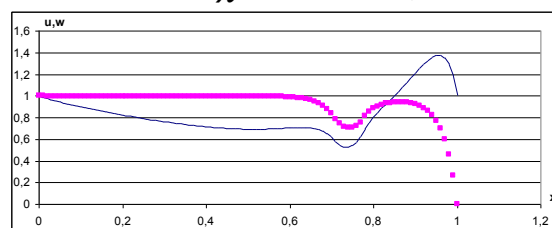


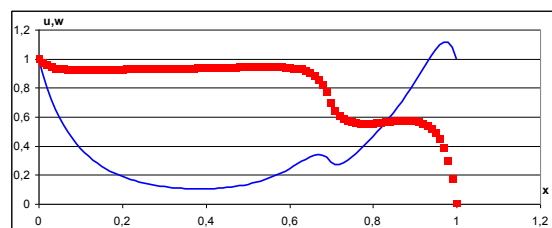
Рисунок 1 – Распределение u (тонкая линия) и w (толстая линия) при $\lambda = 0, \chi = 0, u^* = 0,5$



Picture 2 – Distribution u (thin line) and w (thick line) at $\lambda = 0, \chi = 1, u^* = 0,5$



Picture 3 – Distribution u (thin line) and w (thick line) at $\lambda = 1, \chi = 0, u^* = 0,5$



Picture 4 – Distribution u (thin line) and w (thick line) at $\lambda = 1, \chi = 1, u^* = 0,5$

The obtained numerical results take place for solving the original problem after simple transformations. In addition, the results of [5] are valid for solving the original problem.

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