New convergence theorems for maximal monotone operators in Banach spaces

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***Abstract*—The purpose of this paper is to introduce a new hybrid iterative scheme for resolvents of maximal monotone operators in Banach spaces by using the notion of generalized f-projection. Next, we apply this result to the convex minimization and variational inequality problems in Banach spaces. The results presented in this paper improve and extend important recent results in the literature.**

I. INTRODUCTION

Maximal monotone operators have frequently proven to be a key class of objects in Optimization and Analysis. Constructing iterative algorithms to approximate zeros of maximal monotone operators is a very active topic in applied mathematics. The problem for finding a zero point of a maximal monotone operator is defined as follows : Given a Banach space E and a maximal monotone operator T, we consider the problem for finding a point $u \in E$ such that:

$$0 \in T(u). \tag{1}$$

The set of all points $u \in E$ such that $0 \in T(u)$ denote by $T^{-1}0$. This problem is very important in optimization theory and related fields. For example, if $F : E \to (-\infty, \infty]$ is a proper lower semicontinuous convex function. In this case, the equation $0 \in \partial F(u)$ is equivalent to the problem of minimizing F over E.

The proximal point algorithm (PPA) is one of the popular methods for solving (1), which was first proposed by Martinet [1] and further developed by Rockafellar [2] in a framework of maximal monotone operators in a Hilbert space. A variety of problems, for exmaple, convex programming and variational inequalities can be formulated as finding a zero point of maximal monotone operators. Many research study and extend the proximal methods for various models (see, for example, [3], [4] and [5]). Both numerical experiments and theoretical analysis have demonstrated that the PPA is robust and has nice convergence properties. The proximal point algorithm (PPA) according to Rockafellar [2] generates a sequence $\{x_n\}$ via the rule

$$x_0 \in H, \quad x_{n+1} = J_{r_n} x_n, \quad n = 1, 2, 3, \dots, \tag{2}$$

where $J_{r_n} = (I + r_n T)^{-1}$ and $\{r_n\} \subset (0, \infty)$, then the sequence $\{x_n\}$ converges weakly to an element of $T^{-1}(0)$.

If $T = \partial F$ where $F : E \to (-\infty, \infty]$ is a proper lower semicontinuous convex function, then (2), is reduced to

$$x_{n+1} = argmin_{y \in H}\{F(y) + \frac{1}{2r_n}\|x_n - y\|^2, \ \ n = 1, 2, 3, \dots.$$

To get the results of strong convergence, Solodov and Sviater [4] modified the proximal point algorithm and projection in a Hilbert space. In 2003, Kohsaka and Takahashi [13] obtained a strong convergence theorem for maximal monotone operators in a Banach space, which extended the result of Solodov and Svaiter in a Hilbert space.

On the other hand, Alber [6], [7] introduced and studied the generalized projections $\pi_C : E^* \to C$ and $\Pi_E : E \to C$ in uniformly convex and uniformly smooth Banach spaces. In 2005, Li [9] extended the generalized projection operator from uniformly convex and uniformly smooth Banach spaces to reflexive Banach spaces. Later, Wu and Huang [10] introduced a new generalized f-projection operator in Banach spaces. They extended the definition of the generalized projection operators introduced by Abler [7] and proved some properties of the generalized f-projection operator. In 2009, Fan et al. [11] presented some basic results for the generalized f-projection operator, and discussed the existence of solutions and approximation of the solutions for generalized variational inequalities in noncompact subsets of Banach spaces. Recently, Li et al. [12] proved some property of the generalized f-projection operator and proved strong convergence theorems for relatively nonexpansive mappings in Banach spaces.

In 2005, Matsushita and Takahashi [14] proposed the following hybrid iteration method (it is also called the CQ method) with generalized projection for relatively nonexpansive mapping in a Banach space E. They obtained a strong convergence theorem for relatively nonexpansive mapping in a Banach space. In 2010, Li et al. [12] introduced and proved the strong convergence theorem for approximation of fixed point of relatively nonexpansive mapping using the properties of generalized f-projection operator in a uniformly smooth real Banach space which is also uniformly convex. We remark here that the results of Li et al. [12] extended and improved on the results of Matsushita and Takahashi [14]. Recently, Saewan and Kumam [15] extended the ideal of the generalized f-projection operator to hybrid Ishikawa iteration process for finding a common element of the fixed point set for two countable families of weak relatively nonexpansive mappings and the set of solutions of finding a common element of the fixed point set for two countable families of weak relatively nonexpansive mappings and the set of solutions of the system of generalized Ky Fan inequalities in a uniformly convex and uniformly smooth Banach space.

Motivated by the previously known results, we introduce a new hybrid iterative scheme of the generalized f-projection operator for finding a zero point of a maximal monotone operator in a Banach space. We prove that if $T^{-1}0$ is nonempty, then our new iterative sequence converges strongly to an element of $T^{-1}0$. As applications, we apply our results to obtain some new results for the convex minimization and the variational inequality problems in a Banach space. The results present in this paper are extension, improvement and generalization of the previously results.

II. Preliminaries

Let E be a real Banach space with dual E^* E is said to be *strictly convex* if $\|\frac{x+y}{2}\| < 1$ for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. The *modulus of convexity* of E is the function $\delta : [0, 2] \to [0, 1]$ defined by

$$\delta(\varepsilon) = inf\{1-\|\frac{x+y}{2}\| : x, y \in E, \|x\| = \|y\| = 1, \|x-y\| \geq \varepsilon\}.$$

A Banach space E is said to be *uniformly convex* if and only if $\delta(\varepsilon) > 0$ for all $\varepsilon \in (0, 2]$.

Let $U = \{x \in E : \|x\| = 1\}$ be the unit sphere of E. Then a Banach space E is said to be *smooth* if the limit $\lim\limits_{t\to 0} \frac{\|x+ty\|-\|x\|}{t}$ exists for each $x, y \in U$. E is also said to be *uniformly smooth* if the limit exists uniformly in $x, y \in U$. Let $\langle \cdot, \cdot \rangle$ denote the duality pairing of E^* and E, the *normalized duality mapping* $J : E \to 2^{E^*}$ is defined by

$$J(x) = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2, \|x^*\| = \|x\|\}.$$

If E is a Hilbert space, then $J = I$, where I is the identity mapping and $\langle \cdot, \cdot \rangle$ denotes an inner product on E. Consider the functional defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2, \quad \text{for } x, y \in E, \tag{3}$$

where $J : E \to 2^{E^*}$ is the *normalized duality mapping*.

As well know that if C is a nonempty closed convex subset of a Hilbert space H and $P_C : H \to C$ is the metric projection of H onto C. This fact actually characterizes Hilbert spaces and consequently, it is not available in more general Banach spaces. In this connection, Alber [6], [7] recently introduced the *generalized projection* $\Pi_C : E \to C$ is a map that assigns to an arbitrary point $x \in E$ the minimum point of the functional $\phi(x, y)$, that is, $\Pi_C x = \bar{x}$, where $\bar{x}$ is the solution to the minimization problem

$$\phi(\bar{x}, x) = \inf_{y \in C} \phi(y, x). \tag{4}$$

The existence and uniqueness of the operator Π_C follows from the properties of the functional $\phi(y, x)$ and strict monotonicity of the mapping J (see, for example, [6], [8], [17], [16]). It is obvious from the definition of function ϕ that

$$(\|y\| - \|x\|)^2 \leq \phi(y, x) \leq (\|y\| + \|x\|)^2, \quad \forall x, y \in E. \tag{5}$$

We also known that

$$\phi(x, y) = \phi(x, z) + \phi(z, y) + 2\langle x - z, Jz - Jy \rangle. \tag{6}$$

If E is a Hilbert space, then $\phi(y, x) = ||y - x\|^2$ and Π_C becomes the metric projection of E onto C.

Remark 1: Let E be a Banach space. Then we know that

1. if E is an arbitrary Banach space, then J is monotone and bounded;
2. if E is a strictly convex, then J is strictly monotone;
3. if E is a smooth, then J is single valued and semi-continuous;
4. if E is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subset of E;
5. if E is reflexive, smooth and strictly convex, then the normalized duality mapping J is single valued, one-to-one and onto;
6. if E is uniformly smooth, then E is smooth and reflexive;
7. E is uniformly smooth if and only if E^* is uniformly convex;

see [8] for more details.

Remark 2: If E is a reflexive, strictly convex and smooth Banach space, then for $x, y \in E$, $\phi(x, y) = 0$ if and only if $x = y$. It is sufficient to show that if $\phi(x, y) = 0$ then $x = y$. From 3, we have $\|x\| = \|y\|$. This implies that $\langle x, Jy \rangle = \|x\|^2 = \|Jy\|^2$. From the definition of J, one has $Jx = Jy$. Therefore, we have $x = y$; see [8], [16] for more details.

*Lemma 1: (*Kamimura and Takahashi *[17]). Let E be a uniformly convex and smooth Banach space and let $\{x_n\}$ and $\{y_n\}$ be two sequences of E. If $\phi(x_n, y_n) \to 0$ and either $\{x_n\}$ or $\{y_n\}$ is bounded, then $x_n - y_n \to 0$.*

Let $G : C \times E^* \longrightarrow \mathbf{R} \cup \{+\infty\}$ be a functional defined as follows:

$$G(y, \varpi) = \|y\|^2 - 2\langle y, \varpi \rangle + \|\varpi\|^2 + 2\rho f(y), \tag{7}$$

where $y \in C$, $\varpi \in E^*$, ρ is positive number and $f : C \to \mathbf{R} \cup \{+\infty\}$ is proper, convex and lower semicontinuous. From definitions of G and f, it is easy to see the following properties:

1) $G(y, \varpi)$ is convex and continuous with respect to ϖ when y is fixed;
2) $G(y, \varpi)$ is convex and lower semicontinuous with respect to y when ϖ is fixed.

Let E be a real Banach space with its dual E^*. Let C be a nonempty closed convex subset of E. We say that $\pi_C^f : E^* \to 2^C$ is *generalized f-projection operator* if

$$\pi_C^f \varpi = \{u \in C : G(u, \varpi) = \inf_{y \in C} G(y, \varpi), \ \forall \varpi \in E^*\}.$$

*Lemma 2: (*Wu and Hung *[10]).* Let E be a reflexive Banach space with its dual E^* and C be a nonempty closed convex subset of E. The following statements hold:

1) $\pi_C^f \varpi$ is nonempty closed convex subset of C for all $\varpi \in E^*$;
2) if E is smooth, then for all $\varpi \in E^*$, $x \in \pi_C^f \varpi$ if and only if
$$\langle x - y, \varpi - Jx \rangle + \rho f(y) - \rho f(x) \geq 0, \ \forall y \in C;$$
3) if E is strictly convex and $f : C \to \mathbf{R} \cup \{+\infty\}$ is positive homogeneous (i.e., $f(tx) = tf(x)$ for all $t > 0$ such that $tx \in C$ where $x \in C$), then π_C^f is single valued mapping.

Fan et al. [11] show that the condition f is positive homogeneous which appeared in [11, Lemma 2.1 (iii)] can be removed.

Lemma 3: (Fan et al. [11]). Let E be a reflexive Banach space with its dual E^* and C be a nonempty closed convex subset of E. If E is strictly convex, then $\pi_C^f\varpi$ is single valued.

Recall that J is single value mapping when E is a smooth Banach space. There exists a unique element $\varpi \in E^*$ such that $\varpi = Jx$ where $x \in E$. This substitution for (7) give

$$G(y, Jx) = \|y\|^2 - 2\langle y, Jx\rangle + \|x\|^2 + 2\rho f(y). \quad (8)$$

It is obvious from the definition of G that

$$G(y, Jx) = G(y, Jz) + \phi(z, x) + 2\langle y - z, Jz - Jx\rangle, \quad (9)$$

for all $x, y, z \in E$.

Next, we consider the second generalized f-projection operator in Banach spaces.

Definition 1: (Li et al. [12]). Let E be a real smooth Banach space and C be a nonempty closed convex subset of E. We say that $\Pi_C^f : E \to 2^C$ is generalized f-projection operator if

$$\Pi_C^f x = \{u \in C : G(u, Jx) = \inf_{y\in C} G(y, Jx)\}, \ \forall x \in E.$$

*Lemma 4: (*Deimling *[18])*. Let E be a Banach space and $f : E \to \mathbf{R} \cup \{+\infty\}$ be a lower semicontinuous convex functional. Then there exist $x^* \in E^*$ and $\alpha \in \mathbf{R}$ such that

$$f(x) \geq \langle x, x^*\rangle + \alpha, \ \forall x \in E.$$

*Lemma 5: (*Li et al. *[12])*. Let E be a reflexive smooth Banach space and C be a nonempty closed convex subset of E. The following statements hold:

1) $\Pi_C^f x$ is nonempty closed convex subset of C for all $x \in E$;
2) for all $x \in E$, $\hat{x} \in \Pi_C^f x$ if and only if

$$\langle \hat{x}-y, Jx-J\hat{x}\rangle+\rho f(y)-\rho f(\hat{x}) \geq 0, \ \forall y \in C; \quad (10)$$

3) if E is strictly convex, then Π_C^f is single valued mapping.

*Lemma 6: (*Li et al. *[12])*. Let E be a reflexive smooth Banach space and C be a nonempty closed convex subset of E and let $x \in E$, $\hat{x} \in \Pi_C^f x$. Then

$$\phi(y, \hat{x}) + G(\hat{x}, Jx) \leq G(y, Jx), \ \forall y \in C.$$

Remark 3: Let E be a uniformly convex and uniformly smooth Banach space and $f(x) = 0$ for all $x \in E$. Then Lemma 6 reduces to the property of the generalized projection operator considered by Alber [6].

*Lemma 7: (*Li et al. *[12])*. Let E be a Banach space and $f : E \to \mathbf{R}\cup\{+\infty\}$ be a proper, convex and lower semicontinuous mapping with convex domain $D(f)$. If $\{x_n\}$ is a sequence in $D(f)$ such that $x_n \rightharpoonup \hat{x} \in D(f)$ and $\lim_{n\to\infty} G(x_n, Jy) = G(\hat{x}, Jy)$, then $\lim_{n\to\infty} \|x_n\| = \|\hat{x}\|$.

An operator T is said to be *closed* if for any sequence $\{x_n\} \subset C$ with $x_n \to x$ and $Tx_n \to y$, then $Tx = y$.

An operator $T \subset E \times E^*$ is said to be *monotone* if $\langle x - y, x^* - y^*\rangle \geq 0$ whenever $(x, x^*), (y, y^*) \in T$. We denote the set $\{x \in E : 0 \in Tx\}$ by $T^{-1}0$. A monotone T is said to be *maximal* if its graph $G(T) = \{(x, y) : y \in Tx\}$ is not properly contained in the graph of any other monotone operator. If T is maximal monotone, then the solution set $T^{-1}0$ is closed and convex. Let E be a reflexive, strictly convex and smooth Banach space, it is knows that T is a maximal monotone if and only if $R(J + rT) = E^*$ for all $r > 0$. Define the *resolvent* of T by $J_r = (J + rT)^{-1}J$ for all $r > 0$. J_r is a single-valued mapping from E to $D(T)$. Also, $T^{-1}(0) = F(J_r)$ for all $r > 0$, where $F(J_r)$ is the set of all fixed points of J_r. Define, for $r > 0$, the *Yosida approximation* of T by $T_r = (J-JJ_r)/r$. We know that $T_r x \in T(J_r x)$ for all $r > 0$ and $x \in E$.

Lemma 8: (*Kohsaka and Takahashi* [13]). Let E be a smooth, strictly convex and reflexive Banach space, let C be a nonempty closed convex subset of E and let $T \subset E\times E^*$ be a monotone operator satisfying $D(T) \subset C \subset J^{-1}(\cap_{r>0}R(J + rT))$. Let $r > 0$, let J_r and T_r be the resolvent and the Yosida approximation of T, respectively. Then the following hold:

(i) $\phi(u, J_r x) + \phi(J_r x, x) \leq \phi(u, x), \forall x \in C, u \in T^{-1}0;$
(ii) $(J_r x, T_r x) \in T, \forall x \in C.$

Lemma 9: Let E be a smooth, strictly convex and reflexive Banach space, let $T \subset E \times E^*$ be a monotone operator with $T^{-1}0 \neq \emptyset$, and let $J_r = (J + rT)^{-1}J$ for each $r > 0$. Then

$$G(p, JJ_r x) + \phi(J_r x, x) \leq G(p, Jx), \ \forall x \in E, p \in T^{-1}0.$$

Proof. Let $r > 0$, $p \in T^{-1}0$, and $x \in E$. By the monotonicity of T and (9), we have

$$\begin{aligned} G(p, Jx) &= G(p, JJ_r x) + \phi(J_r x, x) + 2\langle p - J_r x, JJ_r x - Jx\rangle \\ &= G(p, JJ_r x) + \phi(J_r x, x) + 2r\langle p - J_r x, -T_r x\rangle \\ &\geq G(p, JJ_r x) + \phi(J_r x, x). \end{aligned}$$

The proof is complete.

III. MAIN RESULTS

Theorem 1: Let C be a nonempty closed convex subset of a uniformly convex and uniformly smooth Banach space E. Let $T \subset E \times E^*$ be a maximal monotone operator satisfying $D(T) \subset C$ and let $J_{r_n} = (J + r_n T)^{-1}J$ for all $r_n > 0$. Let $f : E \to \mathbf{R}$ be a convex and lower semicontinuous function with $C \subset int(D(f))$. Assume that $T^{-1}0 \neq \emptyset$, for arbitrary point $x_1 \in C_1$ with $C_1 = C$, generate a sequences $\{x_n\}$ by

$$\begin{cases} y_n = J^{-1}(\alpha_n Jx_n + (1 - \alpha_n)JJ_{r_n} x_n), \\ C_{n+1} = \{z \in C_n : G(z, Jy_n) \leq G(z, Jx_n)\}, \\ x_{n+1} = \Pi^f_{C_{n+1}} x_1 \end{cases} \quad (11)$$

where $\{\alpha_n\}$ is a sequence in $[0, 1]$ and $\{r_n\}$ is a sequence in $(0, \infty)$. If $\liminf_{n\to\infty}(1 - \alpha_n) > 0$ and $\lim_{n\to\infty} r_n = \infty$, then $\{x_n\}$ converges strongly to $\Pi^f_{T^{-1}0} x_1$.

Proof. We first show that C_{n+1} is closed and convex for each $n \geq 1$. Clearly $C_1 = C$ is closed and convex. Suppose that C_n is closed and convex for each $n \in \mathbf{N}$. Since for any $z \in C_n$, we know that $G(z, Jy_n) \leq G(z, Jx_n)$ is equivalent to

$$2\langle z, Jx_n - Jy_n\rangle \leq \|x_n\|^2 - \|y_n\|^2.$$

Therefore, C_{n+1} is closed and convex. This implies that $\Pi^f_{C_{n+1}}x_1$ is well defined.

Next, we will show by induction that $T^{-1}0 \subset C_n$ for all $n \in \mathbf{N}$. It is obvious that $T^{-1}0 \subset C_1 = C$. Suppose that $T^{-1}0 \subset C_n$ for some $n \in \mathbf{N}$. Let $q \in T^{-1}0$, put $v_n = J_{r_n}x_n$ for all $n \geq 1$ and by lemma 9, we have

$$\begin{aligned} G(q, Jy_n) &= G(q, \alpha_n Jx_n + (1-\alpha_n)Jv_n) \\ &= \|q\|^2 - 2\langle q, \alpha_n Jx_n + (1-\alpha_n)Jv_n\rangle \\ &\quad + \|\alpha_n Jx_n + (1-\alpha_n)Jv_n\|^2 + 2\rho f(q) \\ &\leq \|q\|^2 - 2\alpha_n\langle q, Jx_n\rangle - 2(1-\alpha_n)\langle q, Jv_n\rangle \\ &\quad + \alpha_n\|Jx_n\|^2 + (1-\alpha_n)\|Jv_n\|^2 + 2\rho f(q) \\ &= \alpha_n G(q, Jx_n) + (1-\alpha_n)G(q, Jv_n) \\ &= \alpha_n G(q, Jx_n) + (1-\alpha_n)G(q, JJ_{r_n}x_n) \\ &\leq \alpha_n G(q, Jx_n) + (1-\alpha_n)G(q, Jx_n) \\ &= G(q, Jx_n). \end{aligned} \tag{12}$$

So, $p \in C_{n+1}$. That is $T^{-1}0 \in C_{n+1}$. Consequently, $T^{-1}0 \subset C_n$, for all $n \in \mathbf{N}$. This implies that $\{x_n\}$ is well defined. Since $f : E \to \mathbf{R}$ is convex and lower semicontinuous mapping, from Lemma 4, we known that there exist $x^* \in E^*$ and $\alpha \in \mathbf{R}$ such that

$$f(x) \geq \langle x, x^*\rangle + \alpha, \forall x \in E.$$

For $x_n \in E$, it follows that

$$\begin{aligned} G(x_n, Jx_1) &= \|x_n\|^2 - 2\langle x_n, Jx_1\rangle + \|x_1\|^2 + 2\rho f(x_n) \\ &\geq \|x_n\|^2 - 2\langle x_n, Jx_1\rangle + \|x_1\|^2 + 2\rho\langle x_n, x^*\rangle \\ &\quad + 2\rho\alpha \\ &= \|x_n\|^2 - 2\langle x_n, Jx_1 - \rho x^*\rangle + \|x_1\|^2 \\ &\quad + 2\rho\alpha \\ &\geq \|x_n\|^2 - 2\|x_n\|\|Jx_1 - \rho x^*\| + \|x_1\|^2 \\ &\quad + 2\rho\alpha \\ &= (\|x_n\| - \|Jx_1 - \rho x^*\|)^2 + \|x_1\|^2 \\ &\quad - \|Jx_1 - \rho x^*\|^2 + 2\rho\alpha. \end{aligned} \tag{13}$$

For each $q \in T^{-1}0 \subset C_n$ and $x_n = \Pi^f_{C_n}x_1$, by the definition of C_n it follows from (13) that

$$\begin{aligned} &G(q, Jx_1) \\ &\quad \geq G(x_n, Jx_1) \\ &\quad \geq (\|x_n\| - \|Jx_1 - \rho x^*\|)^2 + \|x_1\|^2 \\ &\quad\quad - \|Jx_1 - \rho x^*\|^2 + 2\rho\alpha. \end{aligned}$$

This implies that $\{x_n\}$ is bounded and so are $\{y_n\}$ and $\{G(x_n, Jx_1)\}$.

By the fact that $x_{n+1} = \Pi^f_{C_{n+1}}x_1 \in C_{n+1} \subset C_n$ and $x_n = \Pi^f_{C_n}x_1$, it follows by Lemma 6, we get

$$\begin{aligned} 0 &\leq (\|x_{n+1} - \|x_n\|)^2 \\ &\leq \phi(x_{n+1}, x_n) \\ &\leq G(x_{n+1}, Jx_1) - G(x_n, Jx_1). \end{aligned} \tag{14}$$

This implies that $\{G(x_n, Jx_1)\}$ is nondecreasing. So, $\lim_{n\to\infty} G(x_n, Jx_1)$ exist.

For any $m > n$, $x_n = \Pi^f_{C_n}x_1$, $x_m = \Pi^f_{C_m}x_1 \in C_m \subset C_n$ and from (14), we have

$$\phi(x_m, x_n) \leq G(x_m, Jx_1) - G(x_n, Jx_1).$$

Taking $m, n \to \infty$, we have $\phi(x_m, x_n) \to 0$. From Lemma 1 we get that $\|x_n - x_m\| \to 0$. Hence, $\{x_n\}$ is a Cauchy sequence. Since E is a Banach space and C_n is closed and convex, we can assume that there exists $p \in C$ such that $x_n \to p \in C$ as $x_n \to \infty$. In particular, since $\lim_{n\to\infty} G(x_n, Jx_1)$ exist from (14), we also have

$$\lim_{n\to\infty} \phi(x_{n+1}, x_n) = 0. \tag{15}$$

It follow from Lemma 1 that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{16}$$

Since J is uniformly norm-to-norm continuous on bounded subsets of E, we obtain that

$$\lim_{n\to\infty} \|Jx_{n+1} - Jx_n\| = 0. \tag{17}$$

From definition of C_{n+1} and $x_{n+1} = \Pi^f_{C_{n+1}}x_1$, we have

$$G(x_{n+1}, Jy_n) \leq G(x_{n+1}, Jx_n).$$

Therefore, we obtain that

$$\begin{aligned} &\|x_{n+1}\|^2 - 2\langle x_{n+1}, Jy_n\rangle + \|y_n\|^2 + 2\rho f(x_{n+1}) \\ &\leq \|x_{n+1}\|^2 - 2\langle x_{n+1}, Jx_n\rangle + \|x_n\|^2 + 2\rho f(x_{n+1}) \end{aligned}$$

then

$$\begin{aligned} &\|x_{n+1}\|^2 - 2\langle x_{n+1}, Jy_n\rangle + \|y_n\|^2 \\ &\leq \|x_{n+1}\|^2 - 2\langle x_{n+1}, Jx_n\rangle + \|x_n\|^2 \end{aligned}$$

so,

$$\phi(x_{n+1}, y_n) \quad \leq \quad \phi(x_{n+1}, x_n).$$

From Lemma 1 and (15), it follows that

$$\lim_{n\to\infty} \|x_{n+1} - y_n\| = 0, \tag{18}$$

and

$$\lim_{n\to\infty} \|Jx_{n+1} - Jy_n\| = 0. \tag{19}$$

From (12), we have

$$G(q, Jv_n) \geq \frac{1}{1-\alpha_n}(G(q, Jy_n) - \alpha_n G(q, Jx_n))$$

and from Lemma 9, we observe that

$$\begin{aligned} \phi(v_n, x_n) &= \phi(J_{r_n}x_n, x_n) \\ &\leq G(q, Jx_n) - G(q, JJ_{r_n}x_n) \\ &= G(q, Jx_n) - G(q, Jv_n) \\ &\leq G(q, Jx_n) - \tfrac{1}{1-\alpha_n}(G(q, Jy_n) - \alpha_n G(q, Jx_n) \\ &= \tfrac{1}{1-\alpha_n}(G(q, Jx_n) - G(q, Jy_n)) \\ &= \tfrac{1}{1-\alpha_n}(\|x_n\|^2 - \|y_n\|^2 - 2\langle q, Jx_n - Jy_n\rangle) \\ &\leq \tfrac{1}{1-\alpha_n}((\|x_n\|^2 - \|y_n\|^2) + 2|\langle q, Jx_n - Jy_n\rangle|) \\ &\leq \tfrac{1}{1-\alpha_n}((\|x_n\| - \|y_n\|)(\|x_n\| + \|y_n\|) \\ &\quad + 2\|q\|\|Jx_n - Jy_n\|) \\ &\leq \tfrac{1}{1-\alpha_n}((\|x_n\| - \|y_n\|)(\|x_n\| + \|y_n\|) \\ &\quad + 2\|q\|\|Jx_n - Jy_n\|). \end{aligned} \tag{20}$$

Applied from (18), (19) and $\liminf_{n\to\infty}(1-\alpha_n) > 0$, we get

$$\lim_{n\to\infty} \phi(v_n, x_n) = 0. \tag{21}$$

From Lemma 1, we get

$$\lim_{n\to\infty} \|v_n - x_n\| = 0. \tag{22}$$

Since J is uniformly norm-to-norm continuous, we have

$$\lim_{n\to\infty} \|Jx_n - Jv_n\| = 0. \tag{23}$$

Noticing the condition $r_n > 0$, it follows that

$$\lim_{n\to\infty} \tfrac{1}{r_n}\|Jx_n - Jv_n\| = 0. \tag{24}$$

Therefore,

$$\begin{aligned}\lim_{n\to\infty} \|T_{r_n} x_n\| &= \lim_{n\to\infty} \tfrac{1}{r_n}\|Jx_n - JJ_{r_n} x_n\| \\ &= \lim_{n\to\infty} \tfrac{1}{r_n}\|Jx_n - Jv_n\| = 0.\end{aligned} \tag{25}$$

For $(w, w^*) \in T$, from the monotonicity of T, we have $\langle w - v_n, w^* - T_{r_n} x_n\rangle \geq 0$ for all $n \geq 0$. Letting $n \to \infty$, we get $\langle w - p, w^*\rangle \geq 0$. From the maximality of T, we have $p \in T^{-1}0$. Finally, we show that $p = \Pi^f_{T^{-1}0} x_1$. Since F is closed and convex set from Lemma 5, we have $\Pi^f_{T^{-1}0} x_1$ is single value, denote by v. From $x_n = \Pi^f_{C_n} x_1$ and $v \in F \subset C_n$, we also have

$$G(x_n, Jx_1) \leq G(v, Jx_1), \forall n \geq 1.$$

By definition of G and f, we know that, for each given x, $G(y, Jx)$ is convex and lower semicontinuous with respect to y. So

$$\begin{aligned} G(p, Jx_1) & \\ &\leq \liminf_{n\to\infty} G(x_n, Jx_1) \\ &\leq \limsup_{n\to\infty} G(x_n, Jx_1) \\ &\leq G(v, Jx_1). \end{aligned}$$

From definition of $\Pi^f_{T^{-1}0} x_1$ and $p \in T^{-1}0$, we can conclude that $v = p = \Pi^f_{T^{-1}0} x_1$ and $x_n \to p$ as $n \to \infty$. This completes the proof.

Taking $f(y) = 0$ for all $y \in E$, we have $G(y, Jx) = \phi(y, x)$ and $\Pi^f_C x = \Pi_C x$. From Theorem 1 we obtain the following corollary.

Corollary 1: Let C be a nonempty closed convex subset of a uniformly convex and uniformly smooth Banach space E. Let $T \subset E \times E^*$ be a maximal monotone operator satisfying $D(T) \subset C$ and let $J_{r_n} = (J + r_n T)^{-1} J$ for all $r_n > 0$. Assume that $T^{-1}0 \neq \emptyset$. For arbitrary point $x_1 \in C_1$ with $C_1 = C$, generate a sequences $\{x_n\}$ by

$$\begin{cases} y_n = J^{-1}(\alpha_n Jx_n + (1 - \alpha_n) JJ_{r_n} x_n), \\ C_{n+1} = \{z \in C_n : \phi(z, Jy_n) \leq \phi(z, Jx_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_1 \end{cases} \tag{26}$$

where $\{\alpha_n\}$ is a sequence in $[0, 1]$ and $\{r_n\}$ is a sequence in $(0, \infty)$. If $\liminf_{n\to\infty}(1 - \alpha_n) > 0$ and $\lim_{n\to\infty} r_n = \infty$, then $\{x_n\}$ converges strongly to $\Pi_{T^{-1}0} x_1$.

IV. Applications

A. Convex minimization problem

In this section, we study the problem for finding a minimizer of a proper lower semicontinuous convex function in Banach spaces.

A proper function $F : E \to (-\infty, \infty]$ is said to be *convex* if

$$F(\alpha x + (1 - \alpha)y) \leq \alpha F(x) + (1 - \alpha)F(y), \tag{27}$$

for all $x, y \in E$ and $\alpha \in (0, 1)$.

The function F is said to be *lower semicontinuous* if the set $\{x \in E : F(x) \leq r\}$ is closed in E for all $r \in \mathbf{R}$. For a proper lower semicontinuous convex function $F : E \to (-\infty, \infty]$, the *subdifferential* ∂F of F is defined by

$$\partial F(x) = \{x^* \in E^* : F(x) + \langle y - x, x^*\rangle \leq F(y), \forall y \in E\}, \tag{28}$$

for all $x \in E$. It is easy to see that $0 \in \partial F(u)$ if and only if $F(u) = min_{x\in E} F(x)$. Rockafellar [20] proved that the subdifferential mapping $\partial F \subset E \times E^*$ of F is a maximal monotone operator.

Lemma 10: (Takahashi [21]) Let E be a Banach spaces, let $F \to (-\infty, \infty]$ be a proper lower semicontinuous convex function and let $g : E \to \mathbf{R}$ be a continuous convex function.Then

$$\partial(F + g)(x) = \partial F(x) + \partial g(x), \tag{29}$$

for all $x \in E$.

Theorem 2: Let C be a nonempty closed convex subset of a uniformly convex and uniformly smooth Banach space E. Let $F \to (-\infty, \infty]$ be a proper lower semicontinuous convex function and let $f : E \to \mathbf{R}$ be a convex and lower semicontinuous function with $C \subset int(D(f))$. Assume that $(\partial F)^{-1}0 \neq \emptyset$, for arbitrary point $x_1 \in C_1$ with $C_1 = C$, generate a sequences $\{x_n\}$ by

$$\begin{cases} z_n = argmin_{u\in E} F(u) + \frac{1}{2r_n}\|u\|^2 - \frac{1}{r_n}\langle u, Jx_n\rangle \\ y_n = J^{-1}(\alpha_n Jx_n + (1 - \alpha_n) Jz_n), \\ C_{n+1} = \{z \in C_n : G(z, Jy_n) \leq G(z, Jx_n)\}, \\ x_{n+1} = \Pi^f_{C_{n+1}} x_1 \end{cases} \tag{30}$$

where $\{\alpha_n\}$ is a sequence in $[0, 1]$ and $\{r_n\} \subset (0, \infty)$. If $\liminf_{n\to\infty}(1 - \alpha_n) > 0$ and $\lim_{n\to\infty} r_n = \infty$, then $\{x_n\}$ converges strongly to $\Pi^f_{(\partial F)^{-1}0} x_1$.

Proof. We known that $(\partial F)^{-1}0$ is a maximal monotone operator. For $w \in E$ and $r > 0$, let J_r be the resolvent of (∂F), we have

$$Jw \in J(J_r w) + r(\partial F)(J_r w).$$

Hence

$$0 \in (\partial F)(J_r w) + \frac{1}{r} J(J_r w) - \frac{1}{r} Jw = \partial(F + \frac{1}{2r}\|.\|^2 - \frac{1}{r} Jw)(J_r w)$$

that is

$$J_z w = argmin_{u\in E}\{F(u) + \frac{1}{2r}\|u\|^2 - \frac{1}{r}\langle u, Jw\rangle\}.$$

Since $z_n = J_{r_n} x_n$ for all $n = 1, 2, 3, ...$ By Theorem 1, $\{x_n\}$ converges strongly to $\Pi^f_{(\partial F)^{-1}0} x_1$.

B. Variational inequality problem

Let E be a real Banach space and let C be a nonempty closed and convex subset of E and $A : C \to E^*$ be an operator. *The variational inequality problem* for an operator A is to find $x \in C$ such that

$$\langle y - x, Ax, \rangle \geq 0, \quad \forall y \in C. \tag{31}$$

The set of solution of 31 is denote by $VI(A, C)$.

Let A be a monotone mapping of C into E^* which is said to be *hemicontinuous* if for all $x, y \in C$, the mapping h of $[0, 1]$ into E^*, defined by $h(t) = A(tx+(1-t)y)$, is continuous with respect to the weak* topology of E^*. We define by $N_C(v)$ *the normal cone* for C at a point $v \in C$, that is,

$$N_C(v) = \{x^* \in E^* : \langle y - v, x^* \rangle \leq 0, \quad \forall y \in C\}. \tag{32}$$

*Lemma 11: (*Rockafellar *[19]).* Let C be a nonempty, closed convex subset of a Banach space E and A a monotone, hemicontinuous operator of C into E^*. Let $T \subset E \times E^*$ be an operator defined as follows:

$$Tv = \begin{cases} Av + N_C(v), & v \in C; \\ \emptyset, & otherwise. \end{cases} \tag{33}$$

Then T is maximal monotone and $T^{-1}0 = VI(A, C)$.

Theorem 3: Let C be a nonempty closed and convex subset of a uniformly convex and uniformly smooth Banach space E. Let $A : C \to E^*$ be a single-value, monotone and hemicontinuous operator and let $f : E \to \mathbf{R}$ be a convex and lower semicontinuous function with $C \subset int(D(f))$. Assume that $VI(A, C) \neq \emptyset$. For arbitrary point $x_1 \in C_1$ with $C_1 = C$, generate a sequences $\{x_n\}$ by

$$\begin{cases} z_n = VI(A + \frac{1}{r_n}(J - Jx_n), C), \\ y_n = J^{-1}(\alpha_n Jx_n + (1 - \alpha_n)Jz_n), \\ C_{n+1} = \{z \in C_n : G(z, Jy_n) \leq G(z, Jx_n)\}, \\ x_{n+1} = \Pi^f_{C_{n+1}} x_1, \ n = 1, 2, 3, ..., \end{cases} \tag{34}$$

where $\{\alpha_n\}$ is a sequence in $[0, 1]$ and $\{r_n\} \subset (0, \infty)$. If $\liminf_{n\to\infty}(1 - \alpha_n) > 0$ and $\lim_{n\to\infty} r_n = \infty$, then $\{x_n\}$ converges strongly to $\Pi^f_{VI(A,C)} x_1$.

Proof. For $w \in E$ and $r > 0$, by Theorem 11, we have

$$Jw \in J(J_r w) + rT(J_r w).$$

Hence

$$-A(J_r w) + \frac{1}{r}(Jw - J(J_r w)) \in N_C(J_r w).$$

It follows that $\langle y - J_r w, A(J_r w) + \frac{1}{r}(J(J_r w) - Jw)\rangle \geq 0$, for all $y \in C$. That is $J_r w = VI(A + \frac{1}{r}(J - Jw), C)$. Since $z_n = J_{r_n} x_n$ for all $n = 1, 2, 3, ...$, by Theorem 1, $\{x_n\}$ converges strongly to $\Pi^f_{VI(A,C)} x_1$.

V. Conclusion

In this paper, we extend and improve the iterative scheme for resolvents of maximal monotone operators from using the generalized projection to using the generalized f-projection. We apply this result to the convex minimization and variational inequality problems in Banach spaces.

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References

[1] B. Martinet, Régularisation d inéquations variationelles par approximations succesives. Rev. Fr. Inf. Rech. Opér. 2, 154-159 (1970).

[2] R.T. Rockafellar, *Monotone operators and the proximal point algorithm*, SIAM J. Control Optim. **14** (1976) 877–898.

[3] O. G Güler, *On the convergence of the proximal point algorithm for convex minimization*, SIAMJ. Control Optim. 29 (1991), no. 2, 403-419.

[4] M. V. Solodov and B. F. Svaiter, F*orcing strong convergence of proximal point iterations in a Hilbert space*, Math.Program.87 (2000), 189-202.

[5] S. Kamimura and W. Takahashi, *Approximating solutions of maximal monotone operators in Hilbert spaces*, J.Approx.Theory106 (2000), no. 2, 226-240.

[6] Ya. Alber, *Metric and generalized projection operators in Banach spaces: Properties and applications*, in: A. Kartsatos (Ed.), Theory and Applications of Nonlinear Operators of Accretive and Monotone Type, Dekker, New York, 1996, pp. 15–50.

[7] Ya. Alber, *Generalized projection operators in Banach spaces: Properties and applications*, in: Proceedings of the Israel Seminar, Ariel, Israel, Funct. Differential Equation 1 (1994) 1–21.

[8] I. Cioranescu, *Geometry of Banach spaces, Duality Mappings and Nonlinear Problems*, Kluwer, Dordrecht, 1990.

[9] J. L. Li, *The generalized projection operator on reflexive Banach spaces and its applications*, J. Math. Anal. Appl. 306 (2005) 55–71.

[10] K. Q. Wu, N. J. Huang, *The generalised f-projection operator with an application*, Bull. Aust. Math. Soc. 73(2006) 307-317.

[11] J.H. Fan, X. Liu, J.L. Li, *Iterative schemes for approximating solutions of generalized variational inequalities in Banach spaces*, Nonlinear Anal. TMA 70(2009) 3997-4007.

[12] X. Li, N. Huang, D. O'Regan, *Strong convergence theorems for relatively nonexpansive mappings in Banach spaces with applications*, Comp. Math. Appl. 60(2010), 1322–1331.

[13] F. Kohsaka and W. Takahashi, *Strong convergence of an iterative sequence for maximal monotone operators in a Banach space*, Abstract and Applied Analysis, vol. 2004, no. 3,239–249.

[14] S. Matsushita, W. Takahashi, *A strong convergence theorem for relatively nonexpansive mappings in a Banach space*, J. Approx. Theory 134 (2005) 257-266.

[15] S. Saewan and P. Kumam, *A generalized f-projection method for countable families of weak relatively nonexpansive mappings and the system of generalized Ky Fan inequalities*, J Glob Optim, DOI 10.1007/s10898-012-9922-3.

[16] W. Takahashi, *Nonlinear Functional Analysis*, Yokohama-Publishers, 2000.

[17] S. Kamimura, W. Takahashi, *Strong convergence of a proximal-type algorithm in a Banach space*, SIAM J. Optim. 13 (2002) 938–945.

[18] K. Deimling, *Nonlinear Functional Analysis*, Springer-Verlag, Berlin, Heidelberg, New York, Tokyo, 1985.

[19] R. T. Rockafellar, *On the maximality of sums of nonlinear monotone operators*, Trans. Amer. Math. Soc 149(1970), 75–88.

[20] R. T. Rockafellar, *Characterization of the subdifferentials of convex functions,* Pacific J. Math. 17 (1966), 497-510.

[21] W. Takahashi, Totsu Kaiseki to Fudōten Kinji [Convex Analysis and Approximation of Fixed Points, Sūrikaiseki Shiriizu, vol. 2, Yokohama Publishers, Yokohama, 2000.