Delay SIR Model with Nonlinear Incident Rate and Varying Total Population

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***Abstract*—**Recently, models describing the behavior of SIR epidemic with nonlinear incident rate have been revisited. We study the behavior of the model with time delay in which the total population size varies. Lyapunov functions are constructed to establish the global asymptotic stability of all steady states of the model. Numerical simulations are shown to confirm the main results.

***Keywords*—**Epidemic model, Nonlinear incidence rate, Delay, Stability, SIR.

I. Introduction

RECENTLY, many research works on mathematical models in biology have focused on existence and asymptotic stability of nonnegative equilibrium points through bifurcation analysis of the model [1]-[14]. To make the model more realistic, many assumptions on the model have been imposed. For example, in an SIR model, new population entering the system is not only due to the new born but communicable diseases may be introduced into a population by the arrival of infectives from outside the population, as seen in [2]-[5]. Incidence rate also plays an important role in the dynamical modeling of infections diseases. It has been suggested by several authors ([6]-[11]) that the disease transmission process may have a nonlinear incidence rate. In many epidemic models [6]-[8], the bilinear incidence rate βSI, where S is the number of susceptible individuals, and I is the number, of infective members and the standard incidence rate $\beta SI/N$, where N is the total population size, are frequently used. The bilinear incidence rate is based on the law of mass action. This contact law is suitable for communicable diseases such as influenza and so on, but not for sexually transmitted diseases. It has been pointed out that the standard incidence rate may give a good approximation if the number of available partners is large enough but not if everybody could not make more contacts than it's practically feasible.

After studying the cholera epidemic spread in 1973, Capasso and Serio [9] introduced a saturated incidence rate $g(I)S$ into epidemic models, where $g(I)$ tends to a saturation level when I gets large, that is $g(I) = \beta I/(1+\omega I)$, where βI measures the infection force of the disease and $1/(1+\omega I)$ measures the inhibition effect from the behavioral change of the susceptible individuals when their number increases or from the crowding effect of the infective individuals. This incidence rate seems more reasonable than the bilinear incidence rate βSI, because it includes the behavioral change and crowding effect of the infective individuals and prevents the unboundedness of the contact rate by choosing suitable parameters.

We formulate the model based on the assumptions that the total population size does not change, no new population entered the system, none of the population dies and the recovered population has permanent immunity and cannot be infected again. To make it more realistic we add a time delay in the infectious rate, which is an important key factor in the model that also involves immigration. Then, our model has the following form:

$$\begin{aligned}
\frac{dS(t)}{dt} &= (1-p)C - \frac{\beta S(t)I(t-\tau)}{1+\omega I(t-\tau)} - \sigma S(t),\\
\frac{dI(t)}{dt} &= pC + \frac{\beta S(t)I(t-\tau)}{1+\omega I(t-\tau)} - (\gamma+\sigma+\alpha)I(t), \qquad (1)\\
\frac{dR(t)}{dt} &= \gamma I(t) - \sigma R(t),
\end{aligned}$$

where $S(t)$ is the number of individuals susceptible to the disease at time t, $I(t)$ is the number of infective members at time t, $R(t)$ is the number of members who have been removed from the possibility of infection through full immunity at time t, $I(t-\tau)$ is the number of members who have been removed from the possibility of infection through full immunity at time $t-\tau$, $N(t)$ is the total population size at time t, where $N(t) = S(t) + I(t) + R(t)$, p is the fraction of infectives, $0 \le p \le 1$, C is a constant flow of new members into the whole population per unit time which can be susceptible or infective, β is the average number of contacts

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per infective member per day, τ is the time delay during which the infectious agents, or germs, use as incubation time before a human who has been infected can spread the infection to a susceptible human, $1/(1+\omega I(t-\tau))$ measures the inhibition effect from the behavioral change of the susceptible individuals when their number increases or from the crowding effect of the infective individuals, σ is the natural death rate of the population, γ is the recovery rate of infective individuals, α is the death rate due to the disease, and all parameters are positive.

The paper is arranged as follows. In Section II, the basic results are given. In Section III, the stability is analyzed by making use of parts of Kuang's work [12]. In Section IV, numerical results are illustrated by computer simulations. The conclusion is provided in the final section.

II. Basic Results

We rewrite the system (1) into the form

$$\begin{aligned}\frac{dS(t)}{dt} &= (1-p)C - \frac{\beta S I_\tau}{1+\omega I_\tau} - \sigma S,\\ \frac{dI(t)}{dt} &= pC + \frac{\beta S I_\tau}{1+\omega I_\tau} - \eta I,\\ \frac{dR(t)}{dt} &= \gamma I - \sigma R,\end{aligned} \tag{2}$$

where $S = S(t), I = I(t), I_\tau = I(t-\tau), R = R(t)$ and $\eta = \gamma + \sigma + \alpha$. The initial conditions of (2) are given as

$$\begin{aligned}&S(\theta) = \Phi_1(\theta), I(\theta) = \Phi_2(\theta), R(\theta) = \Phi_3(\theta),\\ &\Phi_i(\theta) \ge 0, \theta \in [-\tau, 0], \Phi_i(0) > 0, i = 1,2,3,\end{aligned} \tag{3}$$

where $\Phi_1(\theta), \Phi_2(\theta)$ and $\Phi_3(\theta)$ are continuous on $[-\tau, 0]$. First, we consider the steady states of the system (2). These steady states (S^*, I^*, R^*) are determined analytically by setting $\dot{S} = \dot{I} = \dot{R} = 0$. Then, we obtain

$$\begin{aligned}S^* &= \frac{(1-p)C(\alpha + \omega(C - \sigma N^*))}{(\beta + \sigma\omega)(C - \sigma N^*) + \sigma\alpha},\\ I^* &= (C - \sigma N^*)/\alpha,\\ R^* &= \gamma(C - \sigma N^*)/\sigma\alpha\end{aligned}$$

where $N^* = S^* + I^* + R^*$. Letting

$$F(N^*) = \frac{\alpha\sigma\beta N^* - \beta(C - \sigma N^*)(\sigma + \gamma)}{\sigma(\alpha + \omega(C - \sigma N^*))} - \eta,$$

we then have

$$0 = pC + \frac{1}{\alpha} F(N^*)(C - \sigma N^*). \tag{4}$$

If the system (2) has a solution where $N^* \in (0, C/\sigma]$, then the system (2) has a physically feasible equilibrium solution.

By substitution, the following can be easily shown.

Proposition 1. *If $p = 0$, then the system (2) has a unique disease-free equilibrium point, namely $E_0 = (C/\sigma, 0, 0)$.*

Proposition 2. *If $R_0 = \dfrac{\beta C}{\sigma\eta}$, and*

(i) $p = 0$ and $R_0 > 1$, or

(ii) $0 < p < 1$ and $N^* \in (0, C/\sigma)$, or

(iii) $p = 1$

Then the system (2) has an endemic equilibrium point.

Proof:

(i) From system (2), if $p = 0$, the steady state $E_1 = (S_1, I_1, R_1)$ is determined analytically by setting $\dot{S} = \dot{I} = \dot{R} = 0$ and $R_0 > 1$. Then we obtain

$$\begin{aligned}S_1 &= (\eta + \omega C)/(\beta + \sigma\omega),\\ I_1 &= (\beta C - \sigma\eta)/[\eta(\beta + \sigma\omega)] > 0,\\ R_1 &= [\gamma(\beta C - \sigma\eta)]/[\sigma\eta(\beta + \sigma\omega)] > 0.\end{aligned}$$

(ii) Consider the system (2) when $0 < p < 1$ and $N^* \in (0, C/\sigma)$. Multiplying both sides of (4) by $\alpha/(C - \sigma N^*)$, we obtain $F(N^*) + \dfrac{\alpha pC}{(C - \sigma N^*)} = 0$. Let

$$H(N^*) = F(N^*) + \frac{\alpha pC}{(C - \sigma N^*)},$$

Then,

$$\frac{dH(N^*)}{dN^*} = \frac{dF(N^*)}{dN^*} + \frac{\alpha\sigma pC}{(C - \sigma N^*)^2}.$$

Since all parameters are positive, $N^* \in (0, C/\sigma)$ and $\dfrac{dF(N^*)}{dN^*} > 0$, then $\dfrac{dH(N^*)}{dN^*} > 0$. This means $H(N^*)$ is an increasing function of N^*. Now, consider the limit of $H(N^*)$ as $N^* \to 0$. We obtain $\lim\limits_{N^* \to 0^+} H(N^*) = \lim\limits_{N^* \to 0^+} F(N^*) + \alpha p$.

Since $\lim\limits_{N^* \to 0^+} F(N^*) = -\infty$, we have $\lim\limits_{N^* \to 0^+} H(N^*) < 0$. As $N^* \to C/\sigma$, we obtain

$$\lim_{N^* \to \frac{C}{\sigma}^-} H(N^*) = F(\frac{C}{\sigma}) + \lim_{N^* \to \frac{C}{\sigma}^-} \frac{\alpha pC}{(C - \sigma N^*)}.$$

Since $\lim\limits_{N^* \to \frac{C}{\sigma}^-} \dfrac{\alpha pC}{(C - \sigma N^*)} = +\infty$ we have $\lim\limits_{N^* \to \frac{C}{\sigma}^-} H(N^*) > 0$.

Thus, $H(N^*)$ has a unique positive root $N^* \in (0, C/\sigma)$ which means that the endemic equilibrium point will exist and is unique.

Let $E_2 = (S_2, I_2, R_2)$ be the endemic equilibrium point of the system (2) where $0 < p < 1$ and $N_2 = S_2 + I_2 + R_2 \in (0, C/\sigma)$, so that

$$\begin{aligned}S_2 &= \frac{(1-p)C(\alpha + \omega(C - \sigma N_2))}{(\beta + \sigma\omega)(C - \sigma N_2) + \sigma\alpha},\\ I_2 &= (C - \sigma N_2)/\alpha,\\ R_2 &= \gamma(C - \sigma N_2)]/\sigma\alpha.\end{aligned}$$

(iii) From the system (2), if $p = 1$ we have the steady state $E_3 = (S_3, I_3, R_3)$ where $S_3 = 0, I_3 = C/\eta$ and $R_3 = \gamma C/\sigma\eta$. □

III. Stability Analysis

A. Local Stability

In what follows, we analyze the model system in terms of its stability relying in parts on the work of Kuang's [12].

Theorem 3. *Let* $R_0 = \frac{\beta C}{\sigma\eta}$. *If* $p = 0$ *and*

(i) $R_0 < 1$, *then the disease-free equilibrium point* E_0 *is locally asymptotically stable for any time delay* $\tau \geq 0$.

(ii) $R_0 > 1$, *then the disease-free equilibrium point* E_0 *is unstable for any time delay* $\tau \geq 0$.

(iii) $R_0 = 1$, *the disease-free equilibrium point* E_0 *is stable.*

Here, R_0 *is the average number of secondary infections generated by one primary infection in a susceptible population.*

Proof: To discuss the local asymptotic stability of the disease-free equilibrium point $E_0 = (C/\sigma, 0, 0)$, let us consider the following coordinate transformation

$$x(t) = S(t) - S^*, y(t) = I(t) - I^*, z(t) = R(t) - R^*,$$

where (S^*, I^*, R^*) denotes any equilibrium of the system (2). Hence, we have the corresponding linearized system of (2) of the form

$$\begin{aligned} \dot{x}(t) &= (\frac{-\beta I^*}{1+\omega I^*} - \sigma)x(t) \frac{-\beta S^*}{(1+\omega I^*)^2} y(t-\tau), \\ \dot{y}(t) &= \frac{\beta I^*}{1+\omega I^*} x(t) + \frac{\beta S^*}{(1+\omega I^*)^2} y(t-\tau) - \eta y(t), \\ \dot{z}(t) &= \gamma y(t) - \sigma z(t), \end{aligned} \qquad (5)$$

The associated transcendental characteristic equation of (5) at the disease-free equilibrium point $E_0 = (S^*, I^*, R^*) = (C/\sigma, 0, 0)$ becomes

$$(\lambda + \sigma)[\lambda^2 + (\eta + \sigma)\lambda + \sigma\eta + (-\frac{\beta C}{\sigma}\lambda - \beta C)e^{-\lambda\tau}] = 0 \quad (6)$$

It is clear that (6) has a characteristic root $\lambda_1 = -\sigma$, which is always negative since $\sigma > 0$. Now, we consider the transcendental polynomial equation

$$\lambda^2 + (\eta + \sigma)\lambda + \sigma\eta + (-\frac{\beta C}{\sigma}\lambda - \beta C)e^{-\lambda\tau} = 0 \qquad (7)$$

(i) We will show that if $R_0 < 1$ then E_0 is locally asymptotically stable. Consider the case $\tau = 0$ in (7). We have

$$\lambda^2 + (\eta + \sigma - \frac{\beta C}{\sigma})\lambda + \sigma\eta - \beta C = 0$$

Since $R_0 < 1$, from the Routh-Hurwitz criterion, all roots of (7) have negative real parts for $\tau = 0$.

If (7) has pure imaginary roots $\lambda = \pm i\theta$ for some $\theta > 0$ and $\tau > 0$, then by substituting $\lambda = i\theta$ into (7), we then have

$$-\theta^2 + \sigma\eta = \frac{\beta C\theta}{\sigma}\sin\theta\tau + \beta C\cos\theta\tau \qquad (8)$$

and

$$\theta(\sigma + \eta) = \frac{\beta C\theta}{\sigma}\cos\theta\tau - \beta C\sin\theta\tau \qquad (9)$$

Squaring both sides of (8) and (9) and adding, we get

$$\theta^4 + \theta^2(\eta^2 + \sigma^2 - \frac{\beta^2 C^2}{\sigma^2}) + \sigma^2\eta^2 - \beta^2 C^2 = 0$$

Since $R_0 < 1$, then we have $\eta^2 + \sigma^2 - \frac{\beta^2 C^2}{\sigma^2} > 0$ and $\sigma^2\eta^2 - \beta^2 C^2 > 0$. From the Routh-Hurwitz criterion, we obtain $\theta^2 < 0$. This contradiction means that any solution of (7) must have negative real part. Hence, the disease-free equilibrium point E_0 is locally asymptotically stable for any time delay $\tau \geq 0$.

(ii) Next, we will show that if $R_0 > 1$, then E_0 is unstable. Let

$$G(\lambda) = \lambda^2 + (\eta + \sigma)\lambda + \sigma\eta + (-\frac{\beta C}{\sigma}\lambda - \beta C)e^{-\lambda\tau}.$$

Note that, since $R_0 > 1$, we have that $G(0) = \sigma\eta - \beta C < 0$ and $G(\lambda) \to \infty$ as $\lambda \to \infty$. It follows from continuity of the function $G(\lambda)$ on $(-\infty, +\infty)$ that the equation $G(\lambda) = 0$ has at least one positive solution. Hence, the characteristic (7) has at least one positive real solution. Hence, E_0 is unstable. This proves the conclusion of (ii).

(iii) Next, we will show that if $R_0 = 1$, and then E_0 is stable. If $R_0 = 1$, the transcendental polynomial (7) becomes

$$G(\lambda) = \lambda^2 + (\frac{\beta C}{\sigma} + \sigma)\lambda + \sigma\eta + (-\frac{\beta C}{\sigma}\lambda - \beta C)e^{-\lambda\tau} + \beta C = 0 \quad (10)$$

It is clear that $\lambda = 0$ is a simple root of $G(\lambda)$. We further show that other solutions of (10) must have negative real parts. In fact, if (10) has an imaginary solution, $\lambda = u \pm i\theta$ for some $u \geq 0, \theta \geq 0$ and $\tau \geq 0$, then by substituting $\lambda = u + i\theta$ into (10), we have

$$u^2 - \theta^2 + (\frac{\beta C}{\sigma} + \sigma)u + \beta C = \frac{\beta C}{\sigma}e^{-u\tau}[(u + \sigma)\cos\theta\tau + \theta\sin\theta\tau] \quad (11)$$

and

$$2u\theta + (\frac{\beta C}{\sigma} + \sigma)\theta = -\frac{\beta C}{\sigma}e^{-u\tau}[(u + \sigma)\sin\theta\tau - \theta\cos\theta\tau]. \quad (12)$$

Squaring both sides of (11) and (12) then adding we get

$$\begin{aligned} &[u^2 - \theta^2 + (\frac{\beta C}{\sigma} + \sigma)u + \beta C]^2 + [2u\theta + (\frac{\beta C}{\sigma} + \sigma)\theta]^2 \\ &\leq \frac{\beta^2 C^2}{\sigma^2}[(u + \sigma)^2 + \theta^2]. \end{aligned} \qquad (13)$$

Simplifying the above inequality, we then have

$$(u^2 + \theta^2)[(u + \sigma)^2 + 2u\frac{\beta C}{\sigma} + \theta^2] + 2u\beta C(2u + \sigma) \leq 0$$

Since $\beta, u, \theta, \sigma,$ and C are all positive, the above inequality is not true. Hence, the solutions of (10) have negative real parts except for the solution $\lambda = 0$. Hence, E_0 is stable for any time delay $\tau \geq 0$. This proves the conclusion of (iii). □

Theorem 4. *If* $p = 0$ *and* $R_0 > 1$, *then the endemic equilibrium point* E_1 *is locally asymptotically stable for any time delay* $\tau \geq 0$.

Proof: The associated transcendental characteristic equation of system (5) at $(S^*, I^*, R^*) = E_1 = (S_1, I_1, R_1)$ becomes

$$(\lambda+\sigma)[\lambda^2 + (\eta + A_1)\lambda + \eta A_1 - B_1(\lambda+\sigma)e^{-\lambda\tau}] = 0, \quad (14)$$

where $A_1 = \beta I_1/(1+\omega I_1) + \sigma$, $B_1 = \beta S_1/(1+\omega I_1)^2$. It is clear that $\lambda_1 = -\sigma$ is always negative. Now, we consider

$$\lambda^2 + (\eta + A_1)\lambda + \eta A_1 - B_1(\lambda+\sigma)e^{-\lambda\tau} = 0. \quad (15)$$

For $\tau = 0$, we have that

$$\lambda^2 + (\eta + A_1 - B_1)\lambda + \eta A_1 - \sigma B_1 = 0.$$

Substituting

$$S_1 = (\eta + \omega C)/(\beta + \sigma\omega)$$

$$I_1 = (\beta C - \sigma\eta)/[\eta(\beta + \sigma\omega)]$$

we then have

$$\eta + A_1 - B_1 = I_1(\eta\omega + \beta)/(1+\omega I_1) + \sigma > 0$$

and

$$\eta A_1 - \sigma B_1 = \eta I_1(\sigma\omega + \beta)/(1+\omega I_1) > 0.$$

It follows that any solution of (15) has negative real part for $\tau = 0$. If (15) has pure imaginary solution $\lambda = \pm i\theta$ for some $\theta > 0$ and $\tau > 0$, then we have

$$-\theta^2 + \eta A_1 = \sigma B_1 \cos\theta\tau + B_1\theta\sin\theta\tau, \quad (16)$$

and

$$\eta\theta + A_1\theta = B_1\theta\cos\theta\tau - \sigma B_1 \sin\theta\tau. \quad (17)$$

Squaring both sides of equations (16) and (17) and adding, we get

$$\theta^4 + (\eta^2 + A_1^2 - B_1^2)\theta^2 + \eta^2 A_1^2 - \sigma^2 B_1^2 = 0.$$

Letting $z = \theta^2$, we then have

$$z^2 + (\eta^2 + A_1^2 - B_1^2)z + \eta^2 A_1^2 - \sigma^2 B_1^2 = 0. \quad (18)$$

Consider

$$\eta^2 + A_1^2 - B_1^2 = (\frac{\beta I_1}{1+\omega I_1} + \sigma)^2 + [(\frac{\eta(1+\omega I_1)^2 + \beta S_1}{(1+\omega I_1)^4}) * (\eta(1+\omega I_1)^2 - \beta S_1)]$$

and

$$\eta(1+\omega I_1)^2 - \beta S_1 = \frac{1}{\sigma\omega + \beta}\left[\frac{\eta\beta\omega(\beta C - \sigma\eta) + \beta\omega^2 C(\beta C - \sigma\eta)}{\eta(\sigma\omega + \beta)}\right].$$

It is readily seen that if $R_0 > 1$, then $\eta(1+\omega I_1)^2 - \beta S_1 > 0$. So, $\eta^2 + A_1^2 - B_1^2 > 0$ and

$$\eta^2 A_1^2 - \sigma^2 B_1^2 = [\eta(\eta(\frac{\beta I_1}{1+\omega I_1} + \sigma) + \frac{\beta\sigma S_1}{(1+\omega I_1)^2}) \frac{I_1}{1+\omega I_1}(\sigma\omega + \beta)] > 0.$$

It follows that any solution of (18) has negative real part, which contradicts the fact that $z = \theta^2$. This shows that all solutions of the characteristic (15) have negative real parts for any time delay $\tau \geq 0$. □

By using the similar technique, the following theorem can be shown.

Theorem 5. *If* $0 < p < 1$ *and* $N_2 \in (0, C/\sigma)$,

$$I_2 - S_2/(1+\omega I_2) > 0$$

and

$$\eta I_2 - \sigma S_2/(1+\omega I_2) > 0$$

then the endemic equilibrium point E_2 *is locally asymptotically stable for any time delay* $\tau \geq 0$.

Proof: The characteristic equation of (5) at $(S^*, I^*, R^*) = E_2 = (S_2, I_2, R_2)$ becomes

$$(\lambda+\sigma)[\lambda^2 + (\eta + A_2)\lambda + \eta A_2 - B_2(\lambda+\sigma)e^{-\lambda\tau}] = 0, \quad (19)$$

where $A_2 = \beta I_2/(1+\omega I_2) + \sigma$ and $B_2 = \beta S_2/(1+\omega I_2)^2$.

Now, we consider

$$\lambda^2 + (\eta + A_2)\lambda + \eta A_2 - B_2(\lambda+\sigma)e^{-\lambda\tau} = 0 \quad (20)$$

For $\tau = 0$, we have

$$\lambda^2 + (\eta + A_2 - B_2)\lambda + \eta A_2 - \sigma B_2 = 0.$$

Observe that

$$\eta + A_2 - B_2 = \eta + \sigma + \beta\frac{(I_2 - \frac{S_2}{1+\omega I_2})}{1+\omega I_2} > 0$$

and

$$\eta A_2 - \sigma B_2 = \eta\sigma + \beta\frac{(\eta I_2 - \frac{\sigma S_2}{1+\omega I_2})}{1+\omega I_2} > 0.$$

It follows that any solutions of (20) have negative real parts for $\tau = 0$. If (20) has a purely imaginary solution $\lambda = \pm i\theta$ for some $\theta > 0$ and $\tau > 0$, then we have

$$-\theta^2 + \eta A_2 = \sigma B_2 \cos\theta\tau + B_2\theta\sin\theta\tau \quad (21)$$

and

$$\theta(\eta + A_2) = B_2\theta\cos\theta\tau - \sigma B_2 \sin\theta\tau. \quad (22)$$

Squaring both sides of (21) and (22), then adding, we have

$$\theta^4 + (\eta^2 + A_2^2 - B_2^2)\theta^2 + \eta^2 A_2^2 - \sigma^2 B_{21}^2 = 0.$$

Letting $z = \theta^2$, we then have

$$z^2 + (\eta^2 + A_2^2 - B_2^2)z + \eta^2 A_2^2 - \sigma^2 B_2^2 = 0 \quad (23)$$

We observe that

$$\eta^2 + A_2^2 - B_2^2 = \eta^2 + [(\frac{\beta}{1+\omega I_2}(I_2 + \frac{S_2}{1+\omega I_2}) + \sigma) * (\frac{\beta}{1+\omega I_2}(I_2 - \frac{S_2}{1+\omega I_2}) + \sigma)] > 0$$

and

$$\eta^2 A_2^2 - \sigma^2 B_2^2 = [\eta(\frac{\beta I_2}{1+\omega I_2} + \sigma) + \frac{\beta\sigma S_2}{(1+\omega I_2)^2}] * [\frac{\beta I_2}{1+\omega I_2}(\eta I_2 - \frac{\sigma S_2}{1+\omega I_2}) + \eta\sigma] > 0$$

Therefore, all solutions of (23) have negative real parts, which contradicts the fact that $z = \theta^2$ This shows that all solutions of (20) have negative real parts for $\tau \geq 0$. □

Theorem 6. *If* $p = 1$, *then the endemic equilibrium point* E_3 *is locally asymptotically stable.*

Proof: If $p=1$, the characteristic equation at $E_3=(0,I_3,R_3)$ becomes

$$(\lambda+\sigma)\left[\lambda^2+\left(\eta+\sigma+\frac{\beta I_3}{1+\omega I_3}\right)\lambda+\eta\left(\frac{\beta I_3}{1+\omega I_3}+\sigma\right)\right]=0. \quad (24)$$

It is easy to see that any solutions of (24) have negative real parts. □

B. Global Stability

We now consider the following equation with time delay

$$\dot{u}=\frac{au_\tau}{1+\omega u_\tau}-cu, u(\theta)=\Phi(\theta)\ge 0,\theta\in[-\tau,0),\Phi(0)>0, \quad (25)$$

where $u=u(t),u_\tau=u(t-\tau)$. The parameters a,c and ω are positive constants and $\tau\ge 0$. Note that (25) always has a trivial equilibrium $u=0$. Moreover, if $a>c$, then (25) has also a unique positive equilibrium $u^*=a-c/\omega c$. For (25), we have the following result.

Lemma 7. *If $a>c$, then the positive equilibrium $u^*=a-c/\omega c$ of (25) is globally asymptotically stable; if $a<c$, then the trivial equilibrium $(0,0)$ of (25) is globally asymptotically stable.*

Proof: See [9]. □

Proposition 8. *Any solution $(S(t),I(t),R(t))$ of system (2) satisfies*

$$\limsup_{t\to+\infty}(S(t)+I(t)+R(t))\le\frac{C}{\sigma}.$$

Proof: Since $\dot{N}(t)=\dot{S}(t)+\dot{I}(t)+\dot{R}(t)$, we have

$$\limsup_{t\to+\infty}N(t)\le\frac{C}{\sigma}$$ □

Proposition 9. *Any solution $(S(t),I(t),R(t))$ of (2) with initial conditions in (3) is defined on $[0,+\infty)$ and remains positive for all $t\ge 0$.*

Proof: Define

$K=\{\varphi=(\varphi_1,\varphi_2.\varphi_3)\in C\,|\,\frac{c}{\sigma}\ge\varphi_1\ge 0,\varphi_2\ge 0,\varphi_3\ge 0\}$. From Proposition 8, we see that K attracts all solutions of (2). For any $\varphi=(\varphi_1,\varphi_2.\varphi_3)\in K$, let $(S(t),I(t),R(t))$ be the solution of (2) with the initial function φ. We claim that for any $t\ge 0$, $S(t)\le C/\sigma$. In fact, if there is a $t_1>0$ such that $S(t_1)\le C/\sigma$ and $\dot{S}(t_1)\ge 0$, then we have that

$$\dot{S}(t_1)=(1-p)C-\frac{\beta S(t_1)I(t_1-\tau)}{1+\omega I(t_1-\tau)}-\sigma S(t_1)$$
$$\le-\frac{\beta S(t_1)I(t_1-\tau)}{1+\omega I(t_1-\tau)}\le 0.$$

Here, we have used $S(t_1)>C/\sigma$. This is a contradiction to $\dot{S}(t_1)>0$. The claim is proved. Hence, K is positively invariant with respect to the system (2). □

Theorem 10. *If $p=0$ and $R_0<1$ then the disease-free equilibrium point E_0 is globally asymptotically stable.*

Proof: Let $(S(t),I(t),R(t))$ be any positive solution of (2). When $p=0$, $(S(t),I(t),R(t))$ is the solution of system (2) with the initial conditions (3). By Proposition 8, we have $\limsup_{t\to+\infty}S(t)\le C/\sigma$. If $R_0<1$ then $\beta C/\sigma<\eta$. We may choose $\varepsilon>0$ sufficiently small satisfying

$$\beta(\frac{C}{\sigma}+\varepsilon)<\eta. \quad (26)$$

Hence, for $\varepsilon>0$ sufficiently small satisfying (26), there is a $T_1>0$ such that if $t>T_1$, then $S(t)\le C/\sigma+\varepsilon$. When $p=0$, we derive from the second equation of (2), for $t>T_1+\tau$, that

$$\dot{I}(t)\le\frac{\beta(\frac{C}{\sigma}+\varepsilon)I_\tau}{1+\omega I_\tau}-\eta I(t)$$

Consider the following auxiliary equation

$$\dot{u}(t)=[\beta(C/\sigma+\varepsilon)u_\tau/(1+\omega u_\tau)]-\eta u(t).$$

From inequality (26), $\beta(C/\sigma+\varepsilon)<\eta$ and by Lemma 7, it follows that

$$\lim_{t\to+\infty}u(t)=0$$

By comparison we find that $\limsup_{t\to+\infty}I(t)=0$. Hence there is a, T_2, $T_2>T_1+\tau$, such that if $t>T_2$ then $I(t)<\varepsilon$. We find, from the third equation of (2), that, for $t>T_2$,

$$\dot{R}(t)=\gamma I(t)-\sigma R(t)\le\gamma\varepsilon-\sigma R(t)$$

By comparison it follows that

$$\limsup_{t\to+\infty}R(t)=0$$

When $p=0$, we find, from the first equation of (2), for $t>T_2+\tau$, that

$$\dot{S}(t)\ge C-\sigma S(t)-\frac{\beta S(t)\varepsilon}{1+\omega\varepsilon}.$$

By comparison, it follows that

$$\liminf_{t\to+\infty}S(t)\ge\frac{C(1+\omega\varepsilon)}{\sigma+(\beta+\omega\sigma)\varepsilon}.$$

Letting $\varepsilon\to 0$, we obtain

$$\liminf_{t\to+\infty}S(t)\ge C/\sigma.$$

Therefore,

$$\lim_{t\to+\infty}S(t)=C/\sigma.$$

If $p=0$ and $R_0<1$, then E_0 is locally asymptotically stable. We conclude that E_0 is globally asymptotically stable. □

Permanence (or persistence) is an important property of dynamical systems and other systems in epidemiology, biology, ecology, and so on. A more basic and important question to ask is whether or not those involved populations will be alive and well in the long run.

Proposition 11. *Suppose that* $p=0$ *and* $R_0>1$, *then for any solution* $(S(t),I(t),R(t))$ *of system (2) with initial conditions (3), we have that*

$$\lim_{t\to+\infty}\inf \mathrm{S(t)} \geq \frac{\mathrm{C}(\sigma+\mathrm{C}\omega)}{\mathrm{C}(\omega\sigma+\beta)+\sigma^2} \coloneqq \mathrm{d}_1,$$
$$\lim_{t\to+\infty}\inf \mathrm{I(t)} \geq \frac{\beta\mathrm{C}-\sigma\eta}{\eta(\omega\sigma+\beta)}\mathrm{e}^{-\eta\tau} \coloneqq \mathrm{d}_2,$$
$$\lim_{t\to+\infty}\inf \mathrm{R(t)} \geq \frac{\gamma \mathrm{d}_2}{\sigma} \coloneqq \mathrm{d}_3.$$

Proof: Let $(S(t),I(t),R(t))$ be any solution of (2). When $p=0$, $(S(t),I(t),R(t))$ is the solution of (2) with the initial conditions (3). By Proposition 8, it follows that $\lim_{t\to+\infty}\sup I(t)\leq \frac{C}{\sigma}$. Hence, for $\varepsilon>0$ sufficiently small, there is a $T_1>0$ such that if $t>T_1$, $\mathrm{I(t)}<C/\sigma+\varepsilon$. When $p=0$, we therefore find, from the first equation of (2), for $t>T_1+\tau$, that

$$\dot{S}(t)\geq C-\left(\sigma+\frac{\beta\left(\frac{C}{\sigma}+\varepsilon\right)}{1+\omega\left(\frac{C}{\sigma}+\varepsilon\right)}\right)S(t).$$

Thus,

$$\lim_{t\to+\infty}\inf S(t)\geq \frac{C(\sigma+C\omega)}{C(\omega\sigma+\beta)+\sigma^2}\coloneqq d_1.$$

We now show that

$$\lim_{t\to+\infty}\inf\, I(t)\geq d_2.$$

For $t\geq 0$, define a differentiable function

$$V_1(t)=I(t)+\beta S_1\int_{t-\tau}^{t}\frac{I(u)}{1+\omega I(u)}du. \tag{27}$$

Calculating the derivative of $V_1(t)$ along the solutions of (2) when $p=0$, we obtain

$$\dot{V}_1(t)=\frac{\beta I_\tau}{1+\omega I_\tau}(S(t)-S_1)+\left(\frac{\beta S_1}{1+\omega I(t)}-\eta\right)I(t). \tag{28}$$

For any $0<q<1$, we have that

$$qI_1<I_1=\frac{\beta C-\sigma\eta}{\eta(\omega\sigma+\beta)}$$
$$S_1<\frac{C}{\frac{qI_1\beta}{1+qI_1\omega}+\sigma}.$$

Let $[qI_1\beta/(1+qI_1\omega)]+\sigma\coloneqq k_1$. There is a constant $\rho\geq 1$ sufficiently large such that $S_1<(C/k_1)(1-e^{-k_1\rho\tau})\coloneqq S_1^{\Delta}$.

We now claim that it is impossible that $I(t)\leq qI_1$ for all $t\geq\rho\tau$. Otherwise, if $I(t)\leq qI_1$ for all $t\geq\rho\tau$, when $p=0$, we have, from the first equation of (2), that, for $t\geq\rho\tau+\tau$,

$$\dot{S}(t)\geq C-\left(\sigma+\frac{\beta qI_1}{1+\omega qI_1}\right)S(t). \tag{29}$$

It then follows that, for $t>\rho\tau+\tau$,

$$\frac{dS(t)}{dt}+[\sigma+\beta qI_1/(1+\omega qI_1)]S(t)\geq C.$$

We have

$$S(t)>C\left[1-e^{-k_1(t-\rho\tau-\tau)}\right]/k_1.$$

Hence, for $t>2\rho\tau+\tau$,

$$S(t)>\frac{C}{k_1}(1-e^{-k_1\rho\tau})=S_1^{\Delta}>S_1. \tag{30}$$

Noting that $I(t)\leq qI_1<I_1$, it follows from (28) and (30) that, for $t>2\rho\tau+\tau$,

$$\dot{V}_1(t)\geq\frac{\beta I_\tau}{1+\omega I_\tau}(S(t)-S_1)+\left(\frac{\beta S_1}{1+\omega I_1}-\eta\right)I(t).$$

Consider

$$\frac{\beta S_1}{1+\omega I_1}-\eta=\frac{\beta\left(\frac{\eta+\omega C}{\sigma\omega+\beta}\right)}{1+\frac{\omega}{\eta}\left(\frac{\beta C-\sigma\eta}{\sigma\omega+\beta}\right)}-\eta$$
$$=\frac{\eta(\eta\beta+\omega\beta C)}{\eta\beta+\omega\beta C}-\eta=0,$$

We then have

$$\dot{V}_1(t)>\frac{\beta I_\tau}{1+\omega I_\tau}(S_1^{\Delta}-S_1). \tag{31}$$

Setting

$$\underline{j}=\min_{\theta\in[-\tau,0]}I(\theta+2\rho\tau+2\tau),$$

we claim that $I(t)\geq\underline{j}$ for all $t>2\rho\tau+\tau$. Otherwise, if there is a $T\geq 0$ such that $I(t)\geq\underline{j}$ for $2\rho\tau+\tau\leq t\leq 2\rho\tau+2\tau+T$, $I(2\rho\tau+2\tau+T)=\underline{j}$ and $\dot{I}(2\rho\tau+2\tau+T)\leq 0$, it follows from inequality (30) and the second equation of (2) when $p=0$ that, for $t_1=2\rho\tau+2\tau+T$,

$$\dot{I}(t_1)=\frac{\beta S(t_1)I(t_1-\tau)}{1+\omega I(t_1-\tau)}-\eta I(t_1)$$

or

$$\dot{I}(t_1)>\left[\frac{\beta S_1^{\Delta}}{1+\omega I_1}-\eta\right]\underline{j}>0.$$

This is a contradiction. Hence, $I(t)\geq\underline{j}$ for all $t\geq 2\rho\tau+\tau$. Accordingly, for $t\geq 2\rho\tau+2\tau$, it follows from inequality (31) that

$$\dot{V}_1(t)>\frac{\beta\underline{j}}{1+\omega\underline{j}}(S_1^{\Delta}-S_1),$$

which yields $V_1(t)\to+\infty$ as $t\to+\infty$. It follows from (27) that there is a $T_2>0$ such that if $t>T_2$,

$$V_1(t)\leq\frac{C}{\sigma}+\frac{C\beta S_1\tau}{\sigma+C\omega}.$$

A contradiction occurs. Hence, the claim is proved, that is, it is impossible that $I(t)\leq qI_1$ for all $t\geq\rho\tau$.

By the claim, we are left to consider two possibilities. First, $I(t)\geq qI_1$ for all t sufficiently large. Second, $I(t)$ oscillates about qI_1 for all t sufficiently large. We now show that $I(t)\geq qd_2$ for all t sufficiently large. The conclusion is obvious for the first case. For the second case, let $t_1<t_2$ be sufficiently large such that

$$I(t_1)=I(t_2)=qI_1,\ \ I(t)<qI_1,\qquad t_1<t<t_2.$$

If $t_2-t_1\leq\tau$, when $p=0$, it follows from the second equation of (2) that $\dot{I}(t)>-\eta I(t)$, which yields, for $t_1<t<t_2$,

$$I(t) > I(t_1)e^{-\eta(t-t_1)} > qI_1e^{-\eta\tau} = q\left[\frac{\beta C-\sigma\eta}{\eta(\sigma\omega+\beta)}\right]e^{-\eta\tau} = qd_2.$$

If $t_2 - t_1 > \tau$, we obtain $I(t) \geq qd_2$ for $t \in [t_1, t_1+\tau]$.

We now claim that $I(t) \geq qd_2$ for all $t \in [t_1+\tau, t_2,]$. Otherwise, there is a $T_1^* > 0$ such that $I(t) \geq qd_2$ for $t_1 \leq t \leq t_1+\tau+T_1^*$, $I(t_1+\tau+T_1^*) = qd_2$ and $\dot{I}(t_1+\tau+T_1^*) \leq 0$. On the other hand, when $p = 0$, it follows from the second equation of (2), for $t_0 = t_1+\tau+T_1^*$, that

$$\dot{I}(t_0) = \frac{\beta S(t_1+\tau+T_1^*)I(t_1+T_1^*)}{1+\omega I(t_1+T_1^*)} - \eta I(t_1+\tau+T_1^*) \geq qd_2\left[\frac{\beta S_1^{\Delta}}{1+\omega I_1} - \eta\right] > 0,$$

a contradiction. Hence, $I(t) \geq qd_2$ for $t \in [t_1, t_2]$. Since the interval $[t_1, t_2]$ is chosen arbitrarily, we conclude that $I(t) \geq qd_2$ for all t sufficiently large for the second case. Since $0 < q < 1$ is chosen arbitrarily, we have that

$$\liminf_{t\to+\infty} I(t) \geq d_2.$$

We now show that $\liminf_{t\to+\infty} R(t) \geq d_3$. Since $I(t) \geq d_2$, we have $I(t) \geq d_2 - \varepsilon$ for large t and for any sufficiently small $\varepsilon > 0$. Thus the third equation of (2) gives

$$\frac{dR(t)}{dt} \geq \gamma(d_2-\varepsilon) - \sigma R(t)$$

for large t, which implies that

$$\liminf_{t\to+\infty} R(t) \geq \gamma d_2/\sigma. \qquad \square$$

By using the similarly techniques, the following propositions can be shown.

Proposition 12. *If $0 < p < 1$ and $N_2 \in (0, C/\sigma)$, then for any solution $(S(t), I(t), R(t))$ of system (2) with initial conditions (3), we have that*

$$\liminf_{t\to+\infty} S(t) \geq \frac{(1-p)C(\sigma+C\omega)}{C(\sigma\omega+\beta)+\sigma^2} := e_1,$$
$$\liminf_{t\to+\infty} I(t) \geq \frac{1}{\alpha}(C-\sigma N_2)e^{-\eta\tau} := e_2,$$
$$\liminf_{t\to+\infty} R(t) \geq \frac{\gamma e_2}{\sigma} := e_3.$$

Proposition 13. *Suppose that $p = 1$, then for any solution $(S(t), I(t), R(t))$ of system (2) with initial conditions (3), we have that*

$$\liminf_{t\to+\infty} S(t) > 0,$$
$$\liminf_{t\to+\infty} I(t) \geq \frac{C}{\eta}e^{-\eta\tau} := l_1,$$
$$\liminf_{t\to+\infty} R(t) \geq \frac{\gamma}{\sigma}l_1 := l_2.$$

Theorem 14. *If $p = 0$ and $R_0 > 1$, then the endemic equilibrium point E_1 is globally asymptotically stable.*

Proof: We define the function

$$f(x) = \frac{x}{1+\omega x}.$$

When $p = 0$, from the first and the second equations of (2) at E_1, we have

$$C = \sigma S_1 + \beta S_1 f(I_1) \qquad (32)$$

and

$$\eta I_1 = \beta S_1 f(I_1). \qquad (33)$$

Let

$$g(x) = x - 1 - \ln x$$
$$V_{S_1}(t) = g\left(\frac{S(t)}{S_1}\right)$$
$$V_{I_1}(t) = g\left(\frac{I(t)}{I_1}\right)$$
$$V_{E_1}(t) = \int_0^{\tau} g\left(\frac{I(t-s)}{I_1}\right)ds.$$

We will study the behavior of the Lyapunov functional

$$V_{E_{1+}}(t) = \frac{1}{\beta f(I_1)}V_{S_1} + \frac{I_1}{\beta S_1 f(I_1)}V_{I_1} + V_{E_1}. \qquad (34)$$

We note that $g: R_+ \to R_{+0}$ has the strict global minimum $g(1) = 0$. Since $V_{S_1} = V_{I_1} = V_{E_1} = 0$ so that $V_{E_{1+}}(t) = 0$ if and only if $S(t)/S_1 = I(t)/I_1 = 1$ and $I(t-s)/I_1 = 1$ for all $s \in [0, \tau]$. Hence $V_{E_{1+}}(t) \geq 0$.

By Proposition 8 and Proposition11, solutions are bounded above and bounded away from zero for large time. Without loss of generality, we may assume that, when $p = 0$ the solution of system (2) satisfies these bounds for all $t \geq 0$. Thus, $V_{E_{1+}}(t)$ is defined for all $t \geq 0$.

Let

$$v_1 = \frac{S}{S_1}, \quad v_2 = \frac{I}{I_1}$$

and

$$v_3 = \frac{I_\tau}{I_1}.$$

Additionally, let

$$F(v_3) = \frac{f(I_1v_3)}{f(I_1)} = \frac{f(I_\tau)}{f(I_1)}.$$

Then

$$\frac{dV_{E_{1+}}}{dt} = \frac{1}{\beta f(I_1)}dV_{S_1} + \frac{I_1}{\beta S_1 f(I_1)}dV_{I_1} + dV_{E_1} = \frac{-\sigma}{\beta f(I_1)}\frac{(S-S_1)^2}{SS_1} + 2 - \frac{1}{v_1} + F(v_3) - \frac{v_1F(v_3)}{v_2} - v_3 + \ln v_3 - \ln v_2.$$

By adding and subtracting the quantity $\ln(v_1F(v_3))$, we obtain

$$\frac{dV_{E_{1+}}}{dt} = \frac{-\sigma}{\beta f(I_1)}\frac{(S-S_1)^2}{SS_1} - g\left(\frac{1}{v_1}\right) - g\left(\frac{v_1F(v_3)}{v_2}\right) + F(v_3) - v_3 + \ln v_3 - \ln F(v_3).$$

Since $f(I_1) > 0$ and $g: R_+ \to R_{+0}$. So that $-g(1/v_1) \leq 0$ and $-g(v_1F(v_3)/v_2) \leq 0$. We see that $\frac{dV_{E_{1+}}}{dt} \leq 0$ if

$$F(v_3) - v_3 + \ln v_3 - \ln F(v_3) \leq Q(v_3).$$

We consider

$$F(v_3) - v_3 + \ln v_3 - \ln F(v_3) := Q(v_3)$$

and find the critical point. We have

$$\frac{dQ(v_3)}{dv_3} = 0$$
$$\acute{F}(v_3)\left[1 - \frac{1}{F(v_3)}\right] - \left[1 - \frac{1}{v_3}\right] = 0$$

where

$$F(v_3) = \frac{v_3}{1+\omega I_1 v_3} + \frac{v_3 \omega I_1}{1+\omega I_1 v_3},$$
$$\acute{F}(v_3) = \frac{1+\omega I_1}{(1+\omega I_1 v_3)^2},$$
$$\frac{1}{F(v_3)} = \frac{1+\omega I_1 v_3}{v_3(1+\omega I_1)}.$$

Substituting $\acute{F}(v_3)$ and $F(v_3)$, we have

$$(v_3 - 1)[1 - (1+\omega I_1 v_3)^2] = 0.$$

We see $v_3 = 1$ is the solution of $Q(v_3)$ or

$$\omega I_1 v_3 (2 + \omega I_1 v_3) = 0.$$

We see $v_3 = 0$ or $v_3 = -2/\omega I_1$ is the solution of $\acute{Q}(v_3)$ but it is impossible because we are considering an endemic equilibrium point.

When $0 < v_3 < 1$, $Q(v_3)$ is positive and when $v_3 > 1$ $Q(v_3)$ is negative, so that $v_3 = 1$ is the solution of $Q(v_3)$.

So, $F(v_3) - v_3 + \ln v_3 - \ln F(v_3) \le 0$ with equality only if $v_3 = 1$ together with the fact that $g \ge 0$ with equality only if the argument is 1. By the invariance principle [10], solutions limit to M, the largest invariant subset of $\left\{\frac{dV_{E_1+}}{dt} = 0\right\}$. We note that $\frac{\mathrm{dV_{E_{1+}}}}{\mathrm{dt}}$ is only zero if $v_1 = v_2 = v_3 = 1$. In particular, this requires that, for any solution in M, we have $S(t) = S_1, I(t) = I_1$ and $R(t) = R_1$ for all t and so M consists of the single point E_1. Thus, we see that all solutions approach the endemic equilibrium E_1. By Theorem 4, E_1 is locally asymptotically stable and by Proposition 8 and Proposition 11, when $p = 0$, the system (2) is permanent, allowing us to conclude that E_1 is, in fact, globally asymptotically stable. □

By using the similarly techniques, the following theorem can be shown.

Theorem 15. *If $0 < p < 1$ and $N_2 \in (0, C/\sigma)$, then the endemic equilibrium point E_2 is globally asymptotically stable.*

Theorem 16. *If $p = 1$, then the endemic equilibrium point E_3 is globally asymptotically stable.*

Proof: Consider

$$V_{E_3}(t) = S(t). \tag{35}$$

By Proposition 8 and Proposition 13, solutions are bounded above and bounded away from zero for large time. Without loss of generality, we may assume that, when $p = 1$, the solution of (2) satisfies these bounds for all $t \ge 0$. Thus, $V_{E_3}(t)$ is defined for all $t \ge 0$. Then we have

$$\dot{V}_{E_3}(t) = -S\left[\frac{\beta I_\tau}{1+\omega I_\tau} + \sigma\right] \le 0.$$

We obtain that $\dot{V}_{E_3}(t) = 0$ if and only if $S = 0$ or $[\beta I_\tau/(1 + \omega I_\tau)] + \sigma = 0$, but it is impossible in the case that $[\beta I_\tau/(1 + \omega I_\tau)] + \sigma = 0$. So that $\dot{V}_{E_3}(t) = 0$ if and only if $S = 0$.

By the invariance principle [10], solutions limit to M, the largest invariant subset of $\left\{\frac{dV_{E_3}}{dt} = 0\right\}$. Thus $S(t) = 0$ for all t.

When $p = 1$, from the second equation of (2) and $S = 0$, we further have that

$$\dot{I}(t) = C - \eta I(t).$$

Thus,

$$\lim_{t\to+\infty} I(t) = \frac{C}{\eta}.$$

From the third equation of (2) and $\lim_{t\to+\infty} I(t) = C/\eta$, we have

$$R(t) = \frac{\gamma C}{\sigma \eta}.$$

Hence, the invariance of M implies that $I(t) = C/\eta$ and $R(t) = \gamma C/\sigma\eta$ for all t. So M consists of the single point E_3. Thus, we see that all solutions approach the endemic equilibrium point E_3. By Theorem 6, E_3 is locally asymptotically stable and by Proposition 8 and Proposition 13, when $p = 1$ the system (2) is permanent, allowing us to conclude that E_3 is globally asymptotically stable. □

IV. Numerical Simulations

The qualitative behavior of the three population variables, namely the susceptibles $S(t)$, the infectives $I(t)$ and the recovered $R(t)$, are illustrated by numerical simulations, carried out in this section by using a Matlab program.

In Figure 1, we show a computer simulation of the system (2) subject to the initial conditions $S(0) = 4, I(0) = 3, R(0) = 3$, with parameters $p = 0, C = 2, \beta = 0.001, \omega = 0.2, \sigma = 0.1, \gamma = 0.05,$ $\alpha = 0.07$ and $\tau = 5$. In this case, we get $R_0 = 0.091$ and $p = 0$ which ensures that the disease-free equilibrium point E_0 is globally asymptotically stable.

In Figure 2, we show a computer simulation of the system (2) subject to the initial conditions $S(0) = 4, I(0) = 3, R(0) = 3$, with parameters $p = 0, C = 2, \beta = 0.02, \omega = 0.2, \sigma = 0.1, \gamma = 0.05,$ $\alpha = 0.07$ and $\tau = 5$. We have $R_0 = 1.82$ satisfying the conditions in Theorem 4, $R_0 > 1$ and $p = 0$, which ensures that E_0 will lose its stability while E_1 become globally asymptotically stable.

In Figure 3, we show a computer simulation of the system (2) with different values of R_0 subject to the initial conditions $S(0) = 4, I(0) = 3, R(0) = 3$. In Figure 3(a), we used parameters $p = 0.3, C = 2, \beta = 0.8, \omega = 1.5, \sigma = 0.03, \gamma = 0.2,$ $\alpha = 0.1$ and $\tau = 5$. In this case, we get $R_0 = 161.616 > 1$ and $E_2 = (2.754, 5.81, 38.74)$. Since $0 < p < 1$ and $N_2 = 47.304 < C/\sigma$, $I_2 - S_2/(1+\omega I_2) = 5.527 > 0$ and $\eta I_2 - \sigma S_2/(1+\omega I_2) = 1.9088 > 0$. Therefore, E_2 is globally asymptotically stable, as seen in Figure 3(a).

In Figure 3(b) we used parameters $p = 0.7, C = 2, \beta = 0.02,$ $\omega = 1.5, \sigma = 0.2, \gamma = 0.4, \alpha = 0.03$ and $\tau = 5$. In this case, we get $R_0 = 0.371 < 1$ and $E_2 = (2.853, 2.269, 4.538)$. Since $0 < p < 1$ and $N_2 = 9.66 < C/\sigma, I_2 - S_2/(1+\omega I_2) = 1.621 > 0$ and $\eta I_2 - \sigma S_2/(1+\omega I_2) = 1.29989 > 0$. Therefore, E_2 is globally asymptotically stable, as seen in Figure 3(b).

We observe that different values of R_0 ($R_0 > 1$ or $R_0 < 1$) does not effect a stability of E_2 as long as those conditions in Theorem 15 are satisfied.

In Figure 4, we show a computer simulation of the system (2) with different values of R_0 subject to the initial conditions $S(0)=4, I(0)=3, R(0)=3,$ In Figure 4(a), we used parameters $p=1, C=2, \beta=0.08, \omega=0.2, \sigma=0.1, \gamma=0.4,$ $\alpha=0.3$ and $\tau=5$. In this case we get $R_0=2>1$ and $E_3=(0,2.5,10).$ Since $p=1,$ E_3 is globally asymptotically stable, as seen in Figure 4(a). In Figure 4(b) we used parameters $p=1, C=2,$ $\beta=0.02, \omega=0.2, \sigma=0.4, \gamma=0.05, \alpha=0.07$ and $\tau=5$. In this case, we get $R_0=0.192<1$ and $E_3=(0,3.846,0.481)$. Therefore, E_3 is globally asymptotically stable, as seen in Figure 4(b).

We observe that different values of R_0 ($R_0>1$ or $R_0<1$) does not effect the stability of E_3 as long as those conditions in Theorem 16 are satisfied.

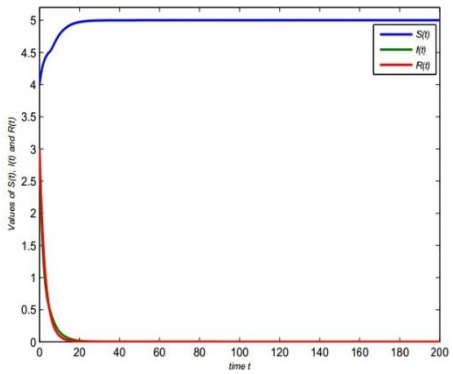

Fig. 1 Numerical simulation of the system (2) subject to the initial conditions $S(0)=4, I(0)=3, R(0)=3,$ with parameters $p=0, C=2, \beta=0.001, \omega=0.2, \sigma=0.1, \gamma=0.05,$ $\alpha=0.07$ and $\tau=5$. Hence, $R_0=0.091<1$. The solution time series tend to $E_0=(5,0,0)$, where $N=5$, as predicted in Theorem 10 in which E_0 is globally asymptotically stable.

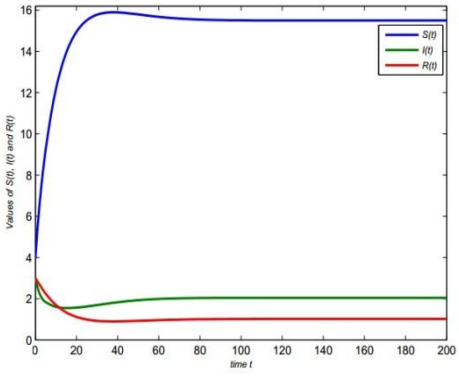

Fig. 2 Numerical simulation of the system (2) subject to the initial conditions $S(0)=4, I(0)=3, R(0)=3,$ with parameters $p=0,$ $C=2, \beta=0.02, \omega=0.2, \sigma=0.1, \gamma=0.05,$ $\alpha=0.07$ and $\tau=5$. Hence, $R_0=1.82>1$. The solution time series tend to $E_1=(15.5,2.045,1.023)$, where $N=18.568$ as predicted in Theorem 14 in which E_1 is globally asymptotically stable.

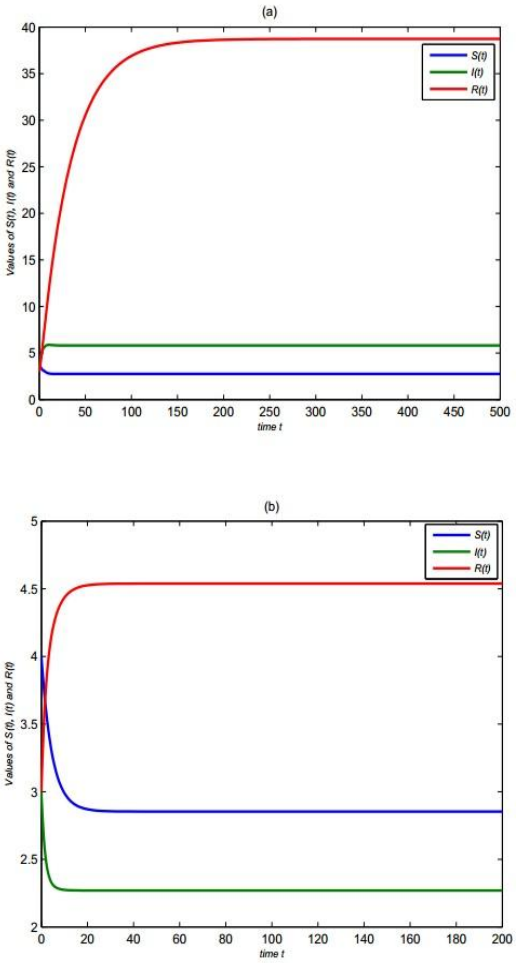

Fig. 3 Numerical simulation of the system (2) subject to the initial conditions $S(0)=4, I(0)=3, R(0)=3,$ (a) $p=0.3, C=2, \beta=0.8, \omega=1.5, \sigma=0.03, \gamma=0.2,$ $\alpha=0.1$ and $\tau=5$. such that $R_0=161.616>1$ and $N_2=47.304$. The solution curves tend to $E_2=(2.754,5.81,38.74)$ as predicted in Theorem 15 in which E_2 is globally asymptotically stable. (b) $p=0.7, C=2, \beta=0.02, \omega=1.5, \sigma=0.2, \gamma=0.4,$ $\alpha=0.03$ and $\tau=5$ such that $R_0=0.371<1$ and $N_2=9.66$ The solution curves tend to $E_2=(2.853,2.269,4.538)$ as predicted in Theorem 15 in which E_2 is globally asymptotically stable.

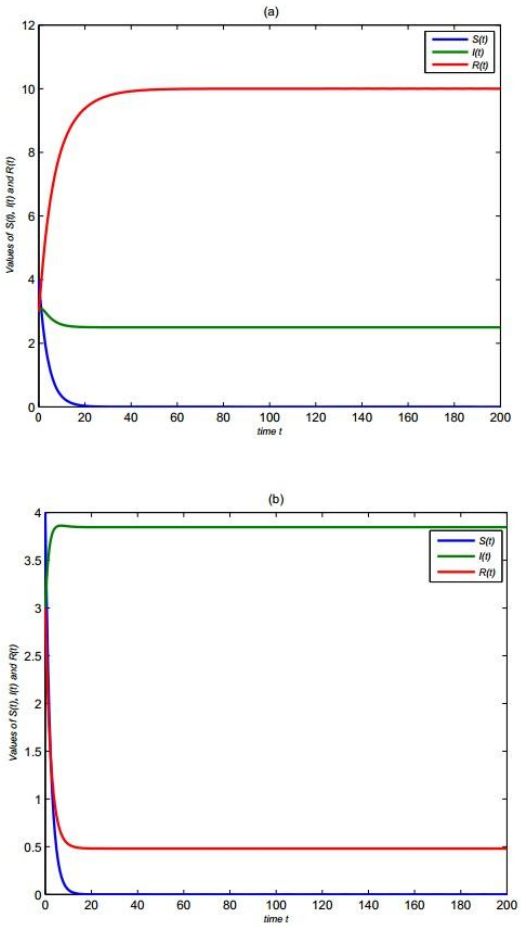

Fig. 4 Numerical simulation of the system (2) subject to the initial conditions $S(0)=4, I(0)=3, R(0)=3,$ (a) $p=1, C=2,$ $\beta=0.08, \omega=0.2, \sigma=0.1, \gamma=0.4,$ $\alpha=0.3$ and $\tau=5$ such that $R_0=2>1$ and $N=12.5$. The solution curves tend to $E_3=(0,2.5,10)$ as predicted in Theorem 16 in which E_3 is globally asymptotically stable. (b) $p=1, C=2,$ $\beta=0.02, \omega=0.2,$ $\sigma=0.4, \gamma=0.05, \alpha=0.07$ and $\tau=5$ such that $R_0=0.192<1$ and $N=4.327$. The solution curves tend to $E_3=(0,3.846,0.481)$ as predicted in Theorem 16 in which E_3 is globally asymptotically stable.

V. Conclusion

We have studied and investigated the behavior of the disease-free equilibrium point and all the endemic equilibrium points and their local and global stability. We also obtained sufficient conditions to ensure that each equilibrium point is locally or globally asymptotically stable.

The existence of E_1, E_2 or E_3 depends on the values of p, N or R_0 as described. In epidemiology, the basic reproduction number of an infection, R_0, is the average number of secondary infections generated by one primary infection in a susceptible population. In other words, it determines the number of people infected by direct or indirect contact with a single infected person. R_0 is useful because it helps us to determine whether or not an infectious disease can spread through a population.

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