Analysis of perturbed flows of a second-order fluid using a 1D hierarchical model

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Abstract—The aim of this paper is to analyze the unsteady flow of a non-Newtonian incompressible second-order fluid in a straight rigid axisymmetric tube with circular cross-section of constant radius. To study this problem, we use the 1D nine-director Cosserat theory approach which reduces the exact three-dimensional equations to a system depending only on time and on a single spatial variable. From this one-dimensional system we obtain the relationship between mean pressure gradient and volume flow rate over a finite section of the tube. Attention is focused on some numerical simulation of steady/unsteady flows for specific mean pressure gradient and on the analysis of perturbed flows.

Keywords—Cosserat theory, perturbed flow, unsteady flow, second-order fluid, mean pressure gradient, volume flow rate.

I. INTRODUCTION

Let us consider the constitutive equation for viscoelastic fluids of differential type (also called Rivlin-Ericksen fluids) with complexity n. The extra stress tensor for these fluids has the representation (see *e.g.* Colemann and Noll [9])

$$\boldsymbol{\sigma} = \boldsymbol{S}(\boldsymbol{A}_1, \boldsymbol{A}_2, \cdots, \boldsymbol{A}_n) \tag{1}$$

where $\boldsymbol{S}$ is an isotropic tensor function and $\boldsymbol{A}_1, \boldsymbol{A}_2, \cdots, \boldsymbol{A}_n$ are the first n Rivlin-Ericksen tensors (see Rivlin and Ericksen [23]), given by the recurrence formula ($n = 2, 3, \cdots$):

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$$\boldsymbol{A}_1 = \nabla\boldsymbol{\vartheta} + \left(\nabla\boldsymbol{\vartheta}\right)^T, \tag{2}$$

$$\boldsymbol{A}_n = \frac{d}{dt}\left(\boldsymbol{A}_{n-1}\right) + \boldsymbol{A}_{n-1}\nabla\boldsymbol{\vartheta} + \left(\nabla\boldsymbol{\vartheta}\right)^T\boldsymbol{A}_{n-1}, \tag{3}$$

here $\boldsymbol{\vartheta}$ is the three-dimensional velocity field, and $\frac{d}{dt}(\cdot)$ denoting the material time derivative defined by the expression:

$$\frac{d}{dt}\left(\boldsymbol{A}_{n-1}\right) = \frac{\partial \boldsymbol{A}_{n-1}}{\partial t} + \boldsymbol{\vartheta}\cdot\nabla\boldsymbol{A}_{n-1}.$$

For viscoelastic fluids of differential type, with complexity $n = 2$, the extra stress tensor (1) reduces to

$$\boldsymbol{\sigma} = \mu\boldsymbol{A}_1 + \alpha_1\boldsymbol{A}_2 + \alpha_2\boldsymbol{A}_1^2, \tag{4}$$

where μ is the constant fluid viscosity, α_1 and α_2 are material coefficients usually called the normal stress moduli and the kinematic first two Rivlin-Ericksen tensors $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$ are given by $(2)-(3)$. The Cauchy stress tensor for an incompressible and homogeneous Rivlin-Ericksen fluid of differential type, with complexity $n = 2$, is given by (see [9])

$$\boldsymbol{T} = -p\boldsymbol{I} + \mu\boldsymbol{A}_1 + \alpha_1\boldsymbol{A}_2 + \alpha_2\boldsymbol{A}_1^2 \tag{5}$$

where p is the pressure and $-p\boldsymbol{I}$ is the spherical part of the stress due to the constraint of incompressibility. The 3D fluid dynamic model associated to the constitutive equation (5) has been studied by several authors (see *e.g.* [1],[10],[16]) under different perspectives. The thermodynamics and stability of the fluids related with the Cauchy stress tensor (5) have been studied in detail by Dunn and Fosdick [13], who showed that if the fluid is to be compatible with thermodynamics in the sense that all motions of the fluid meet the Clausius-Duhem inequality and the assumption that the specific Helmholtz free energy of the fluid is a minimum in equilibrium, then

$$\mu \geqslant 0, \quad \alpha_1 \geqslant 0, \quad \alpha_1 + \alpha_2 = 0. \tag{6}$$

Later, Fosdick and Rajagopal [15], based on the experimental observation, showed that for many non-Newtonian fluids of current rheological interest the reported values for α_1 and α_2 do not satisfy the restriction $(6)_{2,3}$, relaxed that assumption. Also, they showed that for arbitrary values of $\alpha_1 + \alpha_2$, with $\alpha_1 < 0$, a fluid filling a compact domain and adhering to the boundary of the domain exhibits an anomalous behavior not expected on real fluids. The condition $(6)_3$ simplifies substantially the mathematical model and the corresponding analysis. The fluids characterized by (6) are known as second-grade fluids as opposed to the general second-order fluids. It should also be added that the use of Clausius-Duheim inequality is the subject matter of much controversy (see *e.g.* Coscia and Galdi [10]). A possible simplification to a three-dimensional model for an incompressible viscoelastic fluid inside a domain is to consider the evolution of average flow quantities using simpler one-dimensional models. Usually, in the case of flow in a tube, the classical 1D models are obtained by imposing additional assumptions and integrating both the equations of conservation of linear momentum and mass over the cross section of the tube. Here, we introduce a 1D model for non-Newtonian Rivlin-Ericksen fluids of second-order, based on the nine-director approach developed by Caulk and Naghdi [8]. This theory includes an additional structure of directors (deformable vectors) assigned to each point on a space curve (Cosserat curve), where a three-dimensional system of equations is replaced by a one-dimensional system depending on time and on a single spatial variable. The use of directors in continuum mechanics goes back to Duhen [12] who regards a body as a collection of points together with associated directions. Theories based on such a model of an oriented medium were further developed by the French scientist Eugène and François Cosserat [11] and have also been used by several authors in studies of rods, plates and shells (see *e.g.* Ericksen and Truesdell [14], Truesdell and Toupin [26], Green *et al.* [19], [20] and Naghdi [22]). An analogous hierarchial theory for unsteady and steady flows has been developed by Caulk and Naghdi [8] in straight pipes of circular cross-section and by Green and Naghdi [21] in channels. The same theory was applied to unsteady viscous fluid flow in curved pipes of circular and elliptic cross-section by Green *et al.* [18]. Recently, this theory has been applied to haemodynamics by Robertson *et al.* [25] and Carapau *et al.* [2]. Also by Carapau and Sequeira [3], [4], [6], and by Carapau [7] considering non-Newtonian fluids. This theory it was validated on the special case of a uniform tube of constant radius for Newtonian fluid (see [8]), and also for non-Newtonian fluids (see [2], [3]). The relevance of using a theory of directed curves is not in regarding it as an approximation to three-dimensional equations, but rather in their use as independent theories to predict some of the main properties of the three-dimensional problems. Advantages of the director theory include: (i) the theory incorporates all components of the linear momentum; (ii) it is a hierarchical theory, making it possible to increase the accuracy of the model; (iii) there is no need for closure approximations; (iv) invariance under superposed rigid body motions is satisfied at each order and (v) the wall shear stress enters directly in the formulation as a dependent variable. This paper deals with the study of the initial boundary value problem for an incompressible homogeneous second-order fluid model in a straight circular rigid tube with constant radius, where the fluid velocity field, given by the director theory, can be approximated by the following finite series (see [8]):

$$\boldsymbol{\vartheta} = \boldsymbol{v} + \sum_{N=1}^{k} x_{\alpha_1} \dots x_{\alpha_N} \boldsymbol{W}_{\alpha_1 \dots \alpha_N}, \tag{7}$$

with

$$\boldsymbol{v} = v_i(z,t)\boldsymbol{e}_i, \quad \boldsymbol{W}_{\alpha_1 \dots \alpha_N} = W^i_{\alpha_1 \dots \alpha_N}(z,t)\boldsymbol{e}_i, \tag{8}$$

(latin indices subscript take the values $1, 2, 3$; greek indices subscript $1, 2$, and the usual summation convention is employed over a repeated index). Here, $\boldsymbol{v}$ represents the velocity along the axis of symmetry z at time t, $x_{\alpha_1} \dots x_{\alpha_N}$ are the polynomial weighting functions with order k (this number identifies the order of hierarchical theory and is related to the number of directors), the vectors $\boldsymbol{W}_{\alpha_1 \dots \alpha_N}$ are the director velocities which are symmetric with respect to their indices and $\boldsymbol{e}_i$ are the associated unit basis vectors. Using this director theory, the 3D system of equations governing the fluid motion is replaced by a system which depends only on a single spatial and time variables, as previously mentioned. From this new system, we obtain the unsteady relationship between mean pressure gradient and volume flow rate. Here in this work we will extend the results obtain on [3] and [5]. Attention is focused on some numerical simulation of steady/unsteady flows for specific mean pressure gradient and on the analysis of perturbed flows.

II. GOVERNING EQUATIONS OF MOTION

Let x_i $(i = 1, 2, 3)$ be the rectangular cartesian coordinates and for convenience set $x_3 = z$. We consider a homogeneous fluid moving within a circular straight and impermeable tube, the domain Ω (see Fig.1) contained in $\mathbb{R}^3$. Also, let us consider the surface scalar function $\phi(z,t)$, that is related with the cross-section of the tube by the following relationship

$$\phi^2(z,t) = x_1^2 + x_2^2. \tag{9}$$

The boundary $\partial\Omega$ is composed by, the proximal cross-section Γ_1, the distal cross-section Γ_2 and the lateral wall of the tube, denoted by Γ_w.

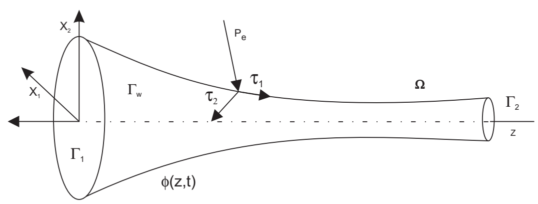

Fig. 1: General fluid domain Ω with the tangential components of the surface traction vector τ_1, τ_2 and p_e, where $\phi(z,t)$ denote the radius of the domain surface along the axis of symmetry z at time t.

Consider the motion of an incompressible fluid without body forces inside a straight circular tube. The equations of motion, stating the conservation of linear momentum and mass are given by (in $\Omega \times (0,T)$)

$$\begin{cases} \rho\Big(\dfrac{\partial \boldsymbol{\vartheta}}{\partial t} + \boldsymbol{\vartheta}\cdot\nabla\boldsymbol{\vartheta}\Big) = \nabla\cdot\boldsymbol{T}, \\ \nabla\cdot\boldsymbol{\vartheta} = 0, \\ \boldsymbol{T} = -p\boldsymbol{I} + \mu\boldsymbol{A}_1 + \alpha_1\boldsymbol{A}_2 + \alpha_2\boldsymbol{A}_1^2, \; \boldsymbol{t}_w = \boldsymbol{T}\cdot\boldsymbol{\eta}, \end{cases} \tag{10}$$

with the initial condition

$$\boldsymbol{\vartheta}(x,0) = \boldsymbol{\vartheta}_0(x) \;\text{ in }\; \Omega, \tag{11}$$

and the homogeneous Dirichlet boundary condition

$$\boldsymbol{\vartheta}(x,t) = 0 \;\text{ on }\; \Gamma_w \times (0,T), \tag{12}$$

where $\boldsymbol{\vartheta} = \vartheta_i\boldsymbol{e}_i$ is the velocity field and ρ is the constant fluid density. Equation $(10)_1$ represents the balance of linear momentum and $(10)_2$ is the incompressibility condition. In equation $(10)_3$, $\boldsymbol{t}_w$ denotes the stress vector on the surface whose outward unit normal is $\boldsymbol{\eta} = \eta_i\boldsymbol{e}_i$. The kinematical first two Rivlin-Ericksen tensors $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$ are given by $(2)-(3)$. If $\alpha_1 = \alpha_2 = 0$ in $(10)_3$ the classical Navier-Stokes system are recovered.

The components of the outward unit normal to the surface $\phi(z,t)$ are

$$\eta_1 = \frac{x_1}{\phi\sqrt{1+\phi_z^2}}, \eta_2 = \frac{x_2}{\phi\sqrt{1+\phi_z^2}}, \eta_3 = -\frac{\phi_z}{\sqrt{1+\phi_z^2}}, \tag{13}$$

where the subscript variable denotes partial differentiation. Since equation (9) defines a material surface, the velocity field $\boldsymbol{\vartheta}$ must satisfy the kinematic condition

$$\frac{d}{dt}\big(\phi^2(z,t) - x_1^2 - x_2^2\big) = 0,$$

i.e.

$$\phi\phi_t + \phi\phi_z\vartheta_3 - x_1\vartheta_1 - x_2\vartheta_2 = 0 \tag{14}$$

on the boundary (9).

Averaged quantities such as flow rate and average pressure are needed to study 1D models. Consider $S(z,t)$ as a generic axial section of the tube at time t defined by the spatial variable z and bounded by the circle defined in (9) and let $A(z,t)$ be the area of this section $S(z,t)$. Then, the volume flow rate Q is defined by

$$Q(z,t) = \int_{S(z,t)} \vartheta_3(x_1,x_2,z,t)da, \tag{15}$$

and the average pressure $\bar{p}$, by

$$\bar{p}(z,t) = \frac{1}{A(z,t)}\int_{S(z,t)} p(x_1,x_2,z,t)da. \tag{16}$$

III. ONE-DIMENSIONAL APPROACH

Using the director theory approach (7) with $k = 3$, it follows (see [8]) that the approximation of the velocity field $\boldsymbol{\vartheta}(x_1,x_2,z,t) = \vartheta_i\boldsymbol{e}_i$, with nine directors, is given by

$$\begin{aligned} \boldsymbol{\vartheta} \;=\; & \Big[x_1(\xi + \sigma(x_1^2+x_2^2)) - x_2(\omega + \eta(x_1^2+x_2^2))\Big]\boldsymbol{e}_1 \\ + & \Big[x_1(\omega + \eta(x_1^2+x_2^2)) + x_2(\xi + \sigma(x_1^2+x_2^2))\Big]\boldsymbol{e}_2 \\ + & \Big[v_3 + \gamma(x_1^2+x_2^2)\Big]\boldsymbol{e}_3 \end{aligned} \tag{17}$$

where $\xi, \omega, \gamma, \sigma, \eta$ are scalar functions of the spatial variable z and time t. The physical significance of these scalar functions in (17) is the following: γ is related to transverse shearing motion, ω and η are related to rotational motion (also called swirling motion) about $\boldsymbol{e}_3$, while ξ and σ are related to transverse elongation.

Using the boundary condition (12), the velocity field (17) on the surface (9) is given by

$$\xi + \phi^2\sigma = 0, \;\; \omega + \phi^2\eta = 0, \;\; v_3 + \phi^2\gamma = 0. \tag{18}$$

The incompressibility condition $(10)_2$ applied to the velocity field (17), can be written as

$$(v_3)_z + 2\xi + \left(x_1^2 + x_2^2\right)\left(\gamma_z + 4\sigma\right) = 0. \tag{19}$$

For equation (19) to hold at every point in the fluid, the velocity coefficients must satisfy the conditions

$$(v_3)_z + 2\xi = 0, \ \ \gamma_z + 4\sigma = 0. \tag{20}$$

Taking into account $(18)_{1,3}$ these separate conditions (20) reduce to

$$(v_3)_z + 2\xi = 0, \ \ \left(\phi^2 v_3\right)_z = 0. \tag{21}$$

Moreover, replacing the velocity field (17) in condition (14) defined at the boundary (9), we get

$$\phi_t + \left(v_3 + \phi^2\gamma\right)\phi_z - \left(\xi + \phi^2\sigma\right)\phi = 0. \tag{22}$$

Now, let us consider a flow in a rigid tube, i.e.

$$\phi = \phi(z), \tag{23}$$

without swirling motion ($\omega = \eta = 0$). From (23) and (18) we verify that the kinematic condition (22) is satisfied identically. Conditions (15), (17), $(18)_3$ and $(21)_2$ imply that the volume flow rate Q is a function of time t, given by

$$Q(t) = \frac{\pi}{2}\, \phi^2(z) v_3(z,t). \tag{24}$$

Then, for a flow in a rigid tube without rotation, with volume flow rate (24) and conditions $(18)_{1,3}$ and $(21)_1$, the velocity field (17) becomes

$$\begin{aligned} \boldsymbol{\vartheta} \ &= \ \Big[x_1\big(1 - \frac{x_1^2 + x_2^2}{\phi^2}\big)\frac{2\phi_z Q(t)}{\pi\phi^3}\Big]\boldsymbol{e}_1 && (25)\\ &+ \ \Big[x_2\big(1 - \frac{x_1^2 + x_2^2}{\phi^2}\big)\frac{2\phi_z Q(t)}{\pi\phi^3}\Big]\boldsymbol{e}_2 \\ &+ \ \Big[\frac{2Q(t)}{\pi\phi^2}\big(1 - \frac{x_1^2 + x_2^2}{\phi^2}\big)\Big]\boldsymbol{e}_3. && (26)\end{aligned}$$

Also, from Caulk and Naghdi [8] the stress vector (see $(10)_3$) on the lateral surface Γ_w can be given by

$$\begin{aligned} \boldsymbol{t}_w \ &= \ \Big[\frac{1}{\phi(1+\phi_z^2)^{1/2}}\Big(\tau_1 x_1\phi_z - p_e x_1 - \tau_2 x_2(1+\phi_z^2)^{1/2}\Big)\Big]\boldsymbol{e}_1 \\ &+ \ \Big[\frac{1}{\phi(1+\phi_z^2)^{1/2}}\Big(\tau_1 x_2\phi_z - p_e x_2 + \tau_2 x_1(1+\phi_z^2)^{1/2}\Big)\Big]\boldsymbol{e}_2 \\ &+ \ \Big[\frac{1}{(1+\phi_z^2)^{1/2}}\Big(\tau_1 + p_e\phi_z\Big)\Big]\boldsymbol{e}_3 && (27)\end{aligned}$$

where τ_1, τ_2 and p_e are the tangential components of the surface traction vector.

Instead of satisfying the momentum equation $(10)_1$ pointwise in the fluid, we impose the following integral conditions

$$\int_{S(z,t)} \Big[\nabla\cdot\boldsymbol{T} - \rho\Big(\frac{\partial\boldsymbol{\vartheta}}{\partial t} + \boldsymbol{\vartheta}\cdot\nabla\boldsymbol{\vartheta}\Big)\Big]da = 0, \tag{28}$$

$$\int_{S(z,t)} \Big[\nabla\cdot\boldsymbol{T} - \rho\Big(\frac{\partial\boldsymbol{\vartheta}}{\partial t} + \boldsymbol{\vartheta}\cdot\nabla\boldsymbol{\vartheta}\Big)\Big]x_{\alpha_1}\ldots x_{\alpha_N}da = 0, \tag{29}$$

where $N = 1, 2, 3$.

Using the divergence theorem and integration by parts, equations (28) – (29) for nine directors, can be reduced to the four vector equations:

$$\frac{\partial\boldsymbol{n}}{\partial z} + \boldsymbol{f} = \boldsymbol{a}, \tag{30}$$

$$\frac{\partial\boldsymbol{m}^{\alpha_1\ldots\alpha_N}}{\partial z} + \boldsymbol{l}^{\alpha_1\ldots\alpha_N} = \boldsymbol{k}^{\alpha_1\ldots\alpha_N} + \boldsymbol{b}^{\alpha_1\ldots\alpha_N}, \tag{31}$$

where $\boldsymbol{n}$, $\boldsymbol{k}^{\alpha_1\ldots\alpha_N}$, $\boldsymbol{m}^{\alpha_1\ldots\alpha_N}$ are resultant forces defined by

$$\boldsymbol{n} = \int_S \boldsymbol{T}_3 da, \ \ \boldsymbol{k}^{\alpha} = \int_S \boldsymbol{T}_\alpha da, \tag{32}$$

$$\boldsymbol{k}^{\alpha\beta} = \int_S \Big(\boldsymbol{T}_\alpha x_\beta + \boldsymbol{T}_\beta x_\alpha\Big)da, \tag{33}$$

$$\boldsymbol{k}^{\alpha\beta\gamma} = \int_S \Big(\boldsymbol{T}_\alpha x_\beta x_\gamma + \boldsymbol{T}_\beta x_\alpha x_\gamma + \boldsymbol{T}_\gamma x_\alpha x_\beta\Big)da, \tag{34}$$

$$\boldsymbol{m}^{\alpha_1\ldots\alpha_N} = \int_S \boldsymbol{T}_3 x_{\alpha_1}\ldots x_{\alpha_N}da. \tag{35}$$

The quantities $\boldsymbol{a}$ and $\boldsymbol{b}^{\alpha_1\ldots\alpha_N}$ are inertia terms defined by

$$\boldsymbol{a} = \int_S \rho\Big(\frac{\partial\boldsymbol{\vartheta}}{\partial t} + \boldsymbol{\vartheta}\cdot\nabla\boldsymbol{\vartheta}\Big)da, \tag{36}$$

$$\boldsymbol{b}^{\alpha_1\ldots\alpha_N} = \int_S \rho\Big(\frac{\partial\boldsymbol{\vartheta}}{\partial t} + \boldsymbol{\vartheta}\cdot\nabla\boldsymbol{\vartheta}\Big)x_{\alpha_1}\ldots x_{\alpha_N}da, \tag{37}$$

and $\boldsymbol{f}$, $\boldsymbol{l}^{\alpha_1\ldots\alpha_N}$, which arise due to surface traction on the lateral boundary, are defined by

$$\boldsymbol{f} = \int_{\partial S} \sqrt{1+\phi_z^2}\, \boldsymbol{t}_w ds, \tag{38}$$

$$\boldsymbol{l}^{\alpha_1\ldots\alpha_N} = \int_{\partial S} \sqrt{1+\phi_z^2}\, \boldsymbol{t}_w\, x_{\alpha_1}\ldots x_{\alpha_N}ds. \tag{39}$$

The equation for the mean pressure gradient as a function of the volume flow rate will be obtain using the results quantities (32) – (39) on equations (30) – (31).

IV. NUMERICAL RESULTS RELATED WITH VOLUME FLOW RATE

Let us consider the system $(10)-(12)$ where the normal stress coefficients α_1 and α_2 do not satisfy the restriction $(6)_{2,3}$. We consider the case of a straight circular rigid walled tube with constant radius, i.e. $\phi = ct$. Replacing the results $(32)-(39)$ obtained by the nine-director model into equations $(30)-(31)$, we get the unsteady relationship

$$\bar{p}_z(z,t) = -\frac{8\mu}{\pi\phi^4}Q(t) - \frac{4\rho}{3\pi\phi^2}\Big(1 + 6\frac{\alpha_1}{\rho\phi^2}\Big)\dot{Q}(t), \qquad (40)$$

were the notation $\dot{Q}$ is used for time differentiation. Flow separation occurs when the axial component τ_1 of the stress vector on the lateral surface (*cf.* (27)) is in the direction of the flow, i.e. $\tau_1 > 0$. The expression for the wall shear stress τ_1 is given by

$$\tau_1 = \frac{4\mu}{\pi\phi^3}Q(t) + \frac{\rho}{6\pi\phi}\Big(1 + 24\frac{\alpha_1}{\rho\phi^2}\Big)\dot{Q}(t). \qquad (41)$$

Integrating equation (40), over a finite section of the tube, between z_1 and position z_2 ($z_1 < z_2$), we get the mean pressure gradient

$$\begin{aligned} G(t) &= \frac{\bar{p}(z_1,t) - \bar{p}(z_2,t)}{z_2 - z_1} \\ &= \frac{8\mu}{\pi\phi^4}Q(t) + \frac{4\rho}{3\pi\phi^2}\Big(1 + 6\frac{\alpha_1}{\rho\phi^2}\Big)\dot{Q}(t). \qquad (42) \end{aligned}$$

Now, let us consider the following dimensionless variables

$$\hat{z} = \frac{z}{\phi}, \;\; \hat{t} = \omega_0\, t, \;\; \hat{Q} = \frac{2\rho}{\pi\phi\mu}\, Q, \;\; \hat{\bar{p}} = \frac{\phi^2\rho}{\mu^2}\, \bar{p}, \qquad (43)$$

where ϕ is the characteristic radius of the tube and ω_0 is the characteristic frequency for unsteay flow. Substituting the new variables (43) into equation (40), we obtain

$$\hat{\bar{p}}_{\hat{z}} = -4\hat{Q}(\hat{t}) - \frac{2}{3}\Big(1 + 6\mathcal{W}_e\Big)\mathcal{W}_0^2\dot{\hat{Q}}(\hat{t}), \qquad (44)$$

where $\mathcal{W}_0 = \phi\sqrt{\rho\omega_0/\mu}$ is the *Womersley number* and $\mathcal{W}_e = |\alpha_1|/(\rho\phi^2)$ is a viscoelastic parameter, also called the *Weissenberg number* (see *e.g.* Galdi *et al.* [17]). The dimensionless number $\mathcal{W}_0$ is the most commonly used parameter to reflect the unsteady pulsatility of the flow. Integrating (44) over a finite section of the tube between $\hat{z}_1$ and $\hat{z}_2$, we get the relationship between mean pressure gradient and volume flow rate given by

$$\hat{G}(\hat{t}) = 4\hat{Q}(\hat{t}) + \frac{2}{3}\Big(1 + 6\mathcal{W}_e\Big)\mathcal{W}_0^2\dot{\hat{Q}}(\hat{t}). \qquad (45)$$

Moreover, the dimensionless form of equation (41) is

$$\hat{\tau}_1 = \frac{\phi^2\rho}{\mu^2}\,\tau_1 = 2\hat{Q}(\hat{t}) + \frac{1}{12}\Big(1 + 24\mathcal{W}_e\Big)\mathcal{W}_0^2\dot{\hat{Q}}(\hat{t}). \qquad (46)$$

If we consider $\mathcal{W}_e = 0$ on above equations (45) and (46), we recover the results obtained by Caulk and Naghdi (see [8]) to the case of a viscous fluid inside a circular straight tube. Next, solving equation (45), we can compute in time the volume flow rate $\hat{Q}$ in terms of the mean pressure gradient $\hat{G}$.

Flow under constant mean pressure gradient

In the particular case of a constant mean pressure gradient $\hat{G}(\hat{t}) = \hat{G}_0$ the system converges toward a steady state solution. In Fig.2 this steady state volume flow rate is obtained solving the time dependent problem but, if we are not interested in the behaviour during the initial transient phase, the steady (asymptotic) value of the volume flow rate can be obtained directly from (45) setting $\dot{\hat{Q}}(\hat{t}) = 0$, since at constant pressure gradient we have $\dot{Q}(\hat{t}) \to 0$ as $\hat{t} \to \infty$. Therefore the steady solution is characterized by

$$\hat{Q} = \hat{G}_0/4, \qquad (47)$$

which is in excellent agreement with the numerical results shown in Fig.2. From Fig.2, we can realize that there is

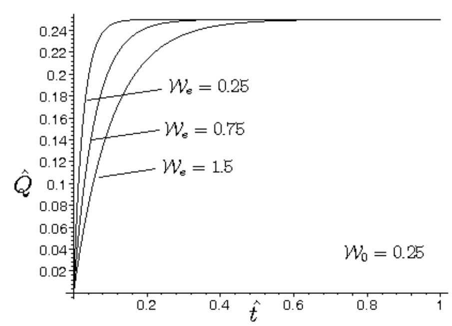

Fig. 2: Time evolution of the volume flow rate given by (45) with constant mean pressure gradiente, fixed Womersley number ($\mathcal{W}_0 = 0.25$) and different values of the Weissenberg number ($\mathcal{W}_e = (0.25, 0.75, 1.5)$).

no qualitative difference between solutions for different values of Weissenberg number, except from the fact that the corresponding curves becomes less dense as the Weissenberg number increases. Several numerical tests have also been performed for other values of Womersley number (fixed or not) and Weissenberg number (fixed or not) showing similar results.

Flow under time mean pressure gradient

In the general situation of imposing a time dependent mean pressure gradient the theory still holds, but additional conditions must be imposed in order to get convenient solutions. We will only briefly show numerical results for specific mean pressure gradient. The case of the mean pressure gradient rising (falling, respectively) exponentially with time was study by Carapau and Sequeira (see [3]), here the 1D nine-director solution was compared with the 3D exact solution obtained by Soundalgekar [24]. Now, considering the following mean pressure gradient

$$\hat{G}(\hat{t}) = 1+ \mid sin(\hat{t}) \mid + \mid cos(2\hat{t}) \mid, \qquad (48)$$

we can observe in Fig.3 how the volume flow rate $\hat{Q}$ change with the time for different values of Womersley and Weissenberg numbers. From these results we can realize that there is no qualitative different between solutions. However, the behavior of the sinusoidal solution start to decrease the values on the peacks when we increase the Weissenberg number. Considering other values for Womersley and Weissenberg numbers we get the same solution behavior.

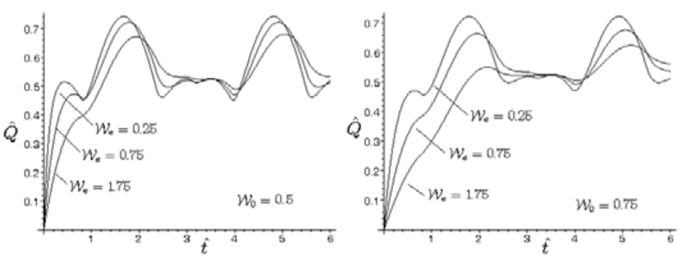

Fig. 3: Time evolution of the volume flow rate given by (45) with mean pressure gradient (49), with Womersley number´s $\mathcal{W}_0 = (0.5, 0.75)$ and Weissenberg number´s $\mathcal{W}_e = (0.25, 0.75, 1.75)$.

V. PERTURBED FLOWS

In many industrial applications involving fluid flows in specific domains it is important to determine the changes in flow characteristics induced by perturbations in the initial or boundary data, body forces or pressure drop. In fact, since it is virtually impossible to maintain an exactly constant pressure drop, one should be able to predict how much a perturbation of given magnitude in pressure drop will affect the volume flow rate. Let us consider a uniform perturbation of magnitude ε (see Fig.4) related with the mean pressure gradient

$$\hat{G}(\hat{t}) = 1+ \mid sin(\hat{t}) \mid + \mid cos(2\hat{t}) \mid . \qquad (49)$$

For each $\varepsilon > 0$, defining the quantities,

$$\hat{G}^{+}_{\varepsilon}(\hat{t}) = (1+\varepsilon)\hat{G}(\hat{t}), \;\; \hat{G}^{-}_{\varepsilon}(\hat{t}) = (1-\varepsilon)\hat{G}(\hat{t}), \qquad (50)$$

we denote by $\hat{Q}^{+}_{\varepsilon}$ and $\hat{Q}^{-}_{\varepsilon}$ the perturbed volume flow rates corresponding to $\hat{G}^{+}_{\varepsilon}$ and $\hat{G}^{-}_{\varepsilon}$, respectively.

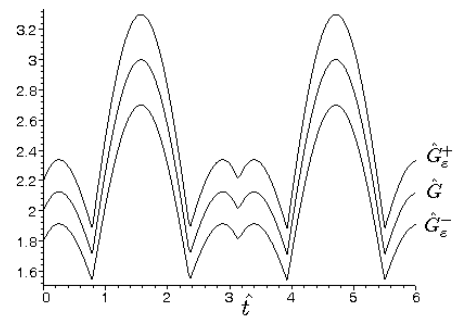

Fig. 4: Multiplicative perburbation of the mean pressure gradient (49), with magnitude $\varepsilon = 0.1$.

Now, considering the perburbation $\hat{G}^{\pm}_{\varepsilon} = (1 \pm \varepsilon)\hat{G}_0$, where $\hat{G}_0$ is a constant mean pressure gradient, for sufficiently large $\hat{t}$, after the transient period, we can use the characterization of the steady solution (see (45)) deduced by

$$\hat{Q} = \hat{G}_0/4, \qquad (51)$$

and explicitly compute the perturbed volume flow rate, using (50), as follows:

$$\hat{Q}^{\pm}_{\varepsilon} = \frac{1}{4}\hat{G}^{\pm}_{\varepsilon} = \frac{1}{4}(1 \pm \varepsilon)\hat{G}_0 = \hat{Q}(1 \pm \varepsilon). \qquad (52)$$

Normalizing the above perturbeded volume flow rate $\hat{Q}^{\pm}_{\varepsilon}$ by the unperturbed volume flow rate $\hat{Q}$, we get

$$\frac{\hat{Q}^{\pm}_{\varepsilon}}{\hat{Q}} = (1 \pm \varepsilon), \qquad (53)$$

which means that in the steady case, this kind of multiplicative perturbation acts linearly. Changing the mean pressure gradient by a factor of $(1 \pm \varepsilon)$, we changes the unperturbed volume flow rate by the same factor of $(1 \pm \varepsilon)$. In particular this shows that the steady state solution is linearly stable. Perturbations will be negligible if $(1 \pm \varepsilon) \simeq 1$, which happens when $\varepsilon \to 0$, i.e. for small changes in the mean pressure gradient.

In the case of time dependent mean pressure gradient the same ideas hold, apart from the fact that it is no longer possible to deduce exact expressions for the perturbed volume flow rates. However, we can compute the time evolution of the perburbation volume flow rate $\hat{Q}^{\pm}_{\varepsilon}$.

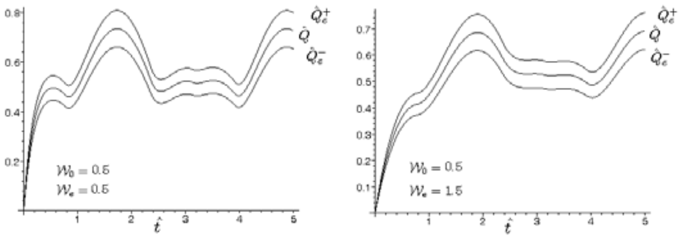

Fig. 5: Time evolution of the unperturbed volume flow rate $\hat{Q}$, and perturbeded volume flow rate $\hat{Q}^{\pm}_{\varepsilon}$, with magnitude $\varepsilon = 0.1$.

In Fig.5, we illustrate the time evolution of the volume flow rate with mean pressure gradient (49), together with the perturbed flow rates $\hat{Q}^{\pm}_{\varepsilon}$ of magnitude $\varepsilon = 0.1$, forming a strip around $\hat{Q}$ containing all perturbations of magnitude less or equal to ε. Fig.6 shows the amplitude of this strip for several values of Womersley and Weissenberg numbers, showing also that increasing the Weissenberg number reduces sensitivity to perturbations

$$|\hat{Q}^{+}_{\varepsilon} - \hat{Q}^{-}_{\varepsilon}| \tag{54}$$

with fixed Womersley number. Considering other values

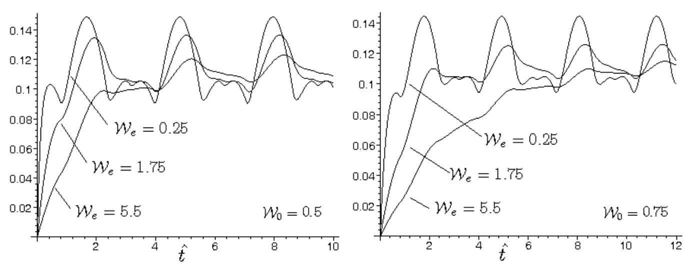

Fig. 6: Time evolution of perburbation (54) for different values of Womersley and Weissenberg numbers.

for Womersley and Weissenberg numbers we get the same solution behavior shown in Fig.6.

VI. CONCLUSIONS

The Cosserat nine-director theory has been used to derive a 1D second-order fluid model in a straight and rigid tube with circular cross-section of constant radius, as an alternative approach to predict some of the main properties of associated 3D models. Unsteady nondimensional relationship between mean pressure gradient and volume flow rate over a finite section of the tube has been obtained. For steady/unsteady mean pressure gradients we predicted some numerical results for different values of Weissenberg and Womersley numbers. Finally, we conducted numerical simulations of perturbed flows, obtaining an exact expression for the perturbed volume flow rates in the steady case, providing a first step towards stability analysis of the model. One of the possible extensions of this work is the application of this one-dimensional approach theory to study flow in curved tubes and in tubes with branches or bifurcations.

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