About some techniques of improving numerical solutions accuracy when applying BEM

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***Abstract*—**The present paper is focused on analyzing different sources of errors that appear when using Boundary Element Method (BEM) to solve problems, and illustrating their influence on the numerical solutions accuracy. The study is made considering the errors that appear when BEM is used to solve a problem of compressible fluid flow. Analytical checking is used by referring to cases when the problem can be exactly solved, and so, the numerical solutions are compared with analytical solutions in order to check their accuracy. Some techniques to minimize the errors in order to get better numerical results are presented.

***Keywords*—** accuracy, boundary element method, compressible fluid flow, numerical solution, singular kernels.

I. Introduction

In the last period the incredible development of numerical methods led to the possibility of solving difficult problems and simulating physical phenomena, often thousands or millions of times over.

In general, problems described by complex mathematical models can not be solved exactly, and thus, more often one has to apply numerical methods to solve them. Computational methods and approximate solutions become necessary, but numerical methods don't offer the exact answer to a given problem because simplifications have to be made when applying them in order to get a solution for a certain mathematical model. Generally they can only tend to a solution, getting closer to it with each iteration.

The boundary element method (BEM) is one of the numerical methods that have been developed for finding computational solutions of partial differential equations (PDEs). It is quite similar to the finite element method, seems easier to use but has a quite restricted range of application to PDEs, because it requires the PDE to first be reformulated as an integral equation. But, when can be used, BEM has the advantage that the mesh need to cover only the boundaries of the domain, so it is very efficient.

When applying BEM two big steps have to be made: first one has to find an equivalent boundary integral representation for the mathematical model of the problem, which is usually a singular boundary integral equation, and then to solve this equation for finding the numerical solution of the problem.

For obtaining the boundary integral equation two main techniques can be used ([1],[2],[3]): the direct technique and the indirect formulation. The boundary integral equation is then solved with different kinds of boundary elements, so using different approximation models for the unknowns and for the geometry.

The problem to solve is so reduced to a linear system of equations. After solving this system the nodal values of the unknowns are found and then the quantities of interest are deduced.

For solving integral equations other techniques exist to: method of successive approximation, orthogonal polynomials, or Krylov subspaces. In case of solving singular boundary integral equations or more general, singular boundary integro-differential equations, approximate solutions can be obtained by using the collocation method as in [4] and [5].

No numerical method is useful without a computer program which permits quickly calculations, because thousands of iterations are usually necessary for obtaining a numerical solution closer to the real solution of the problem.

And, because a computer program is valuable only if it offers good results, it is very important to test the boundary element programs if possible by making analytical checking. It is also important to know the different sources of errors which frequently appear for trying to avoid them, and if it is not possible for trying to minimize them.

In general, there are many types of error sources in boundary element programs, which can be summarized as follows: wrong kernels which are picked up from the literature; integration scheme which are essential, especially for weak and near singular integrals; matrix condition; discretization; etc.

In the herein paper we will discuss some of the most important sources of errors which frequently happened in boundary element programming, considering the special case of applying the BEM to solve the problem of a compressible fluid flow around an obstacle.

Manuscript received October 15, 2008: Revised version received December 10, 2008

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II. Boundary Integral Equation

For a better understanding, a short presentation of the problem is necessary.

A uniform, steady, potential fluid flow of subsonic velocity $U_\infty \bar{i}$, pressure p_∞ and density ρ_∞ is perturbed by the presence of a fixed body of known boundary, assumed at the begging to be smooth and closed. We want to find out the perturbed motion, and the fluid action on the body.

This problem was studied using other techniques by other authors too, and also by the aid of BEM. When the BEM was applied the integral formulation was deduced in terms of potential or stream function and so for finding the velocity field, the pressure, the lift coefficient, *et*, the derivatives of the mentioned functions have to be evaluated. As regarding the numerical solution accuracy such an action introduces new errors and so a formulation that avoids differentiations is preferred.

Using dimensionless variables the velocity field and pressure are considered given by:

$$\overline{V_1} = U_\infty \bar{i} + \boldsymbol{U}_\infty \overline{\boldsymbol{V}}\ ,\ p_1 = p_\infty + \boldsymbol{\rho}_\infty \boldsymbol{U}_\infty^2 \boldsymbol{p}$$

where $\overline{V}$, and p are the dimensionless perturbation velocity and pressure

The following change of coordinates $x=X$, $y=\beta Y$, $u=\beta U$, $v=V$, leads to the following mathematical model of the problem:

$$\begin{cases} \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} = 0 \\ \dfrac{\partial v}{\partial x} - \dfrac{\partial u}{\partial y} = 0 \end{cases} \tag{1}$$

with the boundary condition:

$$(\beta + u) n_x + \beta^2 v n_y = 0 \; on \, C, \tag{2}$$

where u,v are the components of the perturbation velocity, $\beta = \sqrt{1-M^2}$, M being Mach number for the unperturbed motion, n_x, n_y are the new components of the normal vector outward the fluid. It is also required that the perturbation velocity vanishes at infinity: $\lim\limits_{\infty}(u,v) = 0$.

Using both direct and indirect techniques, singular boundary integral equations, formulated in terms of velocity, are obtained in [8] for the above problem.

We first consider in this paper the boundary integral equation obtained with a direct technique. The singular boundary equation is written in terms of a new unknown, G, given by relation: $G = (\beta + u) n_y - v n_x$, and has the following form:

$$G(\bar{x}_0) + 2 \oint_C{}^{/} \left[u^* n_x^0 + v^* n_y^0\right] G ds + \\ +2 \oint_C{}^{/} M^2 \frac{n_x n_y}{n_x^2 + \beta^2 n_y^2} \left(v^* n_x^0 - u^* n_y^0\right) G ds = 2\beta n_y^0 \tag{3}$$

where $\bar{x}_0 \in C$, is a point on the boundary, $u^*(\bar{x}, \bar{x}_0)$, $v^*(\bar{x}, \bar{x}_0)$, are the fundamental solutions, and $\bar{n}^0 = \bar{n}(\bar{x}_0)$.

The sign " $\oint_C{}^{/}$ " denotes the principal value in the Cauchy sense of an integral ([15]). For simplifying the writing we shall not use the prim sign to specify that an integral must be understand in its Cauchy sense.

Some of the error sources that can appear when applying BEM can be avoided quite easy and they don't represent the goal of this paper. For example to avoid wrong kernels picked up from the literature, symbolic computation programs could be used to perform verifying task.

We will focus our attention on the following types of errors: boundary discretization (with respect to approximation of geometry, approximation of unknown function, type of boundary element, refinement of the mesh), computation of the coefficients matrix, type of boundary elements formulation, matrix condition and system solving.

III. Influence of Domain Discretization on Numerical Solution

As we have mentioned before the accuracy of the numerical solution depends on the boundary discretization, not only on the refinement of the mesh but also on the type of boundary element used.

For proving this we use different types of boundary elements to solve the boundary integral equation of the mentioned problem and different number of nodes for the boundary discretization.

First we consider the case of constant boundary elements and then linear isoparametric boundary elements.

For the computation of the coefficients matrix the analytical calculation was preferred in order to eliminate errors arising from numerical integration schemes. For the singular kernels analytical expressions have been found too using the Cauchy Principal Value of an integral.

A. *Errors when using constant boundary elements*

The boundary C is divided into N linear segments, noted L_j, $j = 1, N$ with extremes in $\bar{x}_j, \bar{x}_{j+1}$, $j = \overline{1, N}$, $\bar{x}_{N+1} = \bar{x}_1$, situated on C. The unknown function G is constant on each L_j, and equal with the value taken in the midpoint of the segment, $\bar{x}_j^0$, $j \in \{1, 2, ..., N\}$.

We obtain for $G_i = G\left(\bar{x}_i^0\right)$ the following linear algebraic system:

$$\frac{1}{2}G_i + \sum_{j=1}^{N} A_{ij} G_j = B_i\,,\ i = 1,\ldots,N\ , \qquad (4)$$

The coefficients A_{ij} of matrix A contain boundary integrals of the kernels. They can be calculated numerically, with a computer, but they can also be calculated analytically, even if they arise from non singular or singular kernels.

For eliminating the errors the last possibility is chosen because it is an exact method for evaluating integrals.

The expressions of these coefficients are presented in paper [11], and they depend only on the coordinates of the nodes chosen for the boundary discretization.

After solving system (4) we can evaluate the perturbing velocity on the boundary by the following relations:

$$\beta + u_i = \frac{\beta^2 n_y\left(\bar{x}_i^0\right) G_i}{n_x^2\left(\bar{x}_i^0\right) + \beta^2 n_y^2\left(\bar{x}_i^0\right)},$$

$$v_i = -\frac{n_x\left(\bar{x}_i^0\right) G_i}{n_x^2\left(\bar{x}_i^0\right) + \beta^2 n_y^2\left(\bar{x}_i^0\right)} \qquad (5).$$

1) Mesh Refinement Influence - Case of nonlifting obstacles

The method can be implemented into a computer code which offers numerical values for the velocity field and for the local pressure coefficient on the boundary.

An efficient way to check BEM programs is to use analytical checking, so we have considered a simple case when exact solution exists, in order to compare the numerical solution obtained with the exact one.

For the case of the circular obstacle and an incompressible fluid flow an exact solution of the problem is known ([9]). The exact solution for the velocity and for the local pressure coefficient on the boundary gives in this case:

$$u = -\cos 2\theta,\ v = -2\sin\theta,\ cp = -1 + 2\cos 2\theta\,. \qquad (6)$$

With two computer codes made in C programming language, we found exact and numerical nodal values for the components of the velocity and for the local pressure coefficient. The results are represented in the following graphics: Fig.1- Fig.5.

In order to check the sensitivity of the mesh size distribution on the accuracy of the converged solution, the mesh is refined, considering different number of nodes.

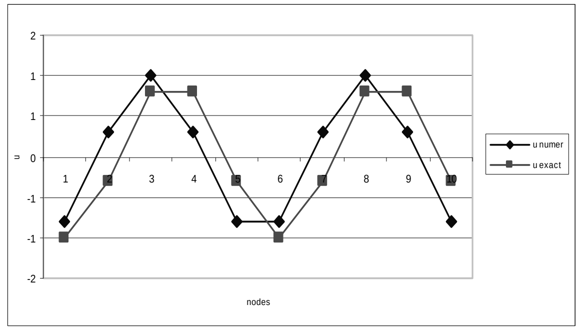

Fig.1 Exact and numerical values of *u* for 10 nodes on the boundary.

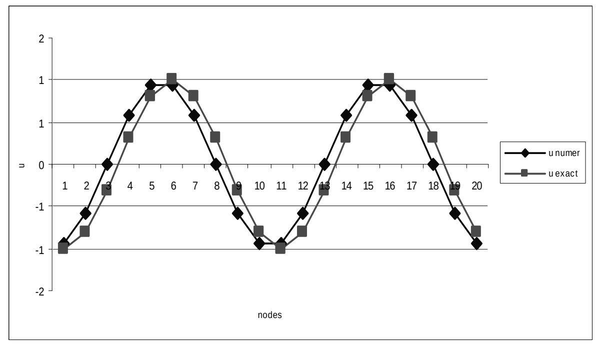

Fig.2 Exact and numerical values of *u* for 20 nodes on the boundary.

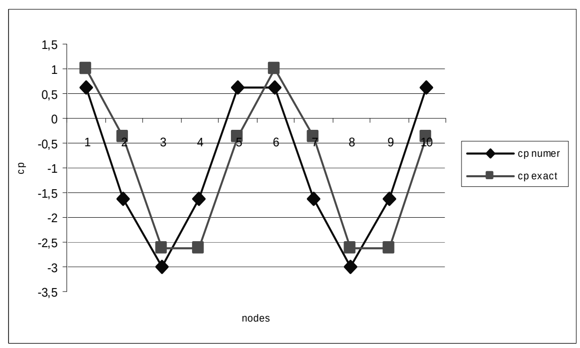

Fig.3 Exact and numerical values of *cp* for 10 nodes on the boundary.

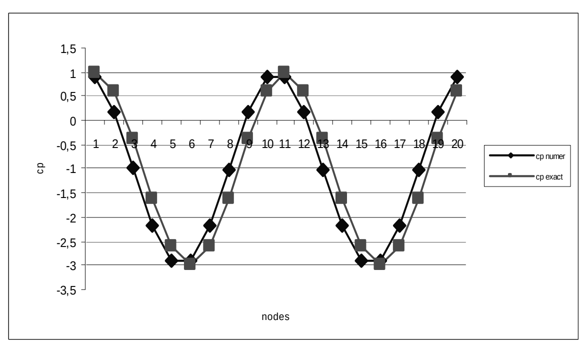

Fig.4 Exact and numerical values of *cp* for 20 nodes on the boundary.

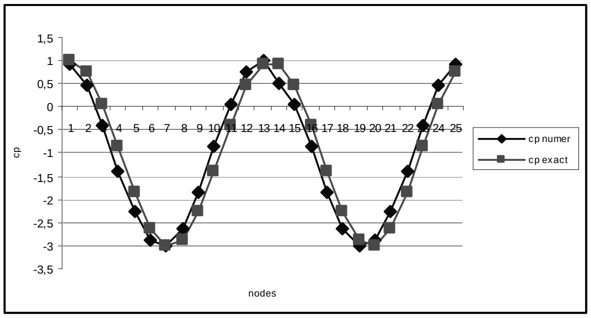

Fig.5 Exact and numerical values of *cp* for 25 nodes on the boundary.

The influence of the mesh size on the numerical solution can be better studied from the following graphic, where the maximum error that appears when using different number of nodes for the boundary discretization, is considered.

Errors are considered not only for the local pressure coefficient, but for the components of the velocity too. We can notice that as the number of nodes grows, the error is smaller, fact that proves the convergence of the method.

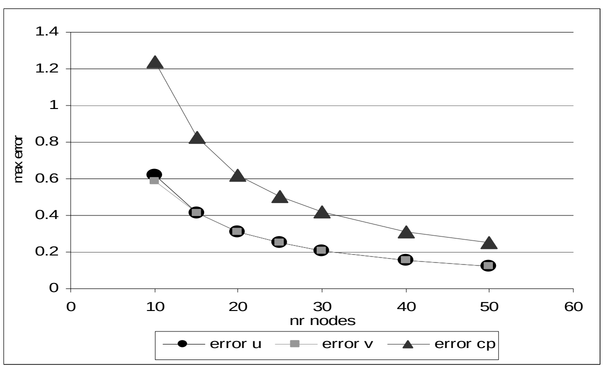

Fig.6 Influence of mesh refinement on errors.

We can also observe that if the number of nodes is bigger than a certain value the error decreases but not in the same manner as if the number of nodes is under that value. It decreases slower. Because the computational effort is bigger when more nodes are considered for the boundary discretization this effort is sometimes not justified and so, for obtaining an efficient numerical solution and a good ration accuracy computational effort, we have to choose a number of nodes in the range 35-45.

2) Mesh Refinement Influence - Case of lifting obstacles

We consider another particular obstacle, with a cusped trailing edge, for which the problem has an exact solution - a Jukovsky profile.

In this case the Kutta-Jukovsky condition in terms of the unknown G gives the relation: $G(P_s)+G(P_i)=0$, where P_S, P_i are adjacent to the trailing edge, situated on the upper, respective on the lower boundary. So, for obtaining the unique solution of the problem, we have to complete system (4), with the above condition, and to introduce a new variable, λ – the regularization variable. We finally get the system:

$$\lambda+\frac{1}{2}G_i+\sum_{j=1}^{N}A_{ij}G_j=B_i,\ i=1,2,3,\ldots N$$

$$G_1+G_N=0, \tag{7}$$

for the unknowns $G_1, G_2, \ldots G_N, \lambda$.

After finding G the velocity is obtained with relations (5).

Considering the Jukovsky profile represented in Fig.7, for which the complex coordinate of a point on it is given in [16] by:

$$z(\theta)=\tilde{z}(\theta)+\frac{b^2}{\tilde{z}(\theta)}, \tag{8}$$

with $\tilde{z}(\theta)=\cos\theta+i\sin\theta+x_0+iy_0$, $b=0.8$, $y_0=0.189$, $x_0=b-\sqrt{1-y_0^2}$, other programs are made for evaluating the numerical values of the perturbation velocity and the exact ones setting M=0.

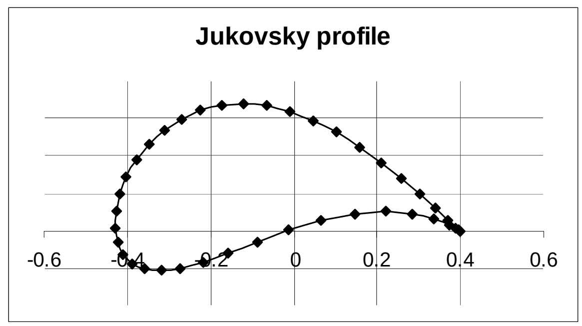

Fig.7. Jukovsky profile.

For analyzing the accuracy of the numerical solution we compared it with the exact solution through the local pressure coefficient. For the boundary discretization we have used 40 nodes.

Fig.8 shows that the convergence of the method is excellent except near the trailing edges. Even in this region the numerical solutions obtained are still finite. From the same figure we can also see the difference between the numerical values on the upper and the lower surface, so the appearance of the lifting force.

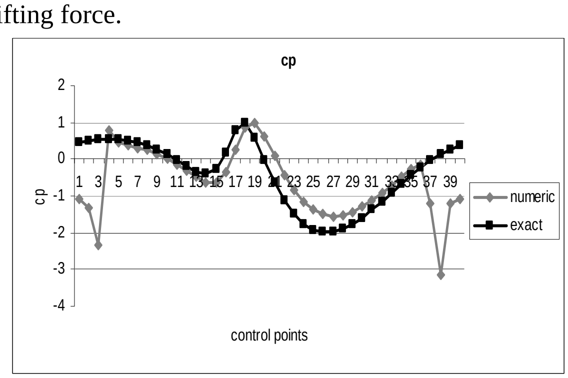

Fig.8. Exact and numerical local pressure coefficient for 40 nodes on the boundary.

The influence of Mach number upon the convergence can be studied too. It can be shown that as M grows the convergence is slower.

In case of a Jukovsky profile the sensitivity of the mesh size distribution on the accuracy of the numerical solution can be observed from Fig.9 that shows the maximum errors for different number of nodes: 25, 40 and 45. We have used the following notations: ER, if all nodes are considered, and ERR if nodes near the trailing edge are eliminated.

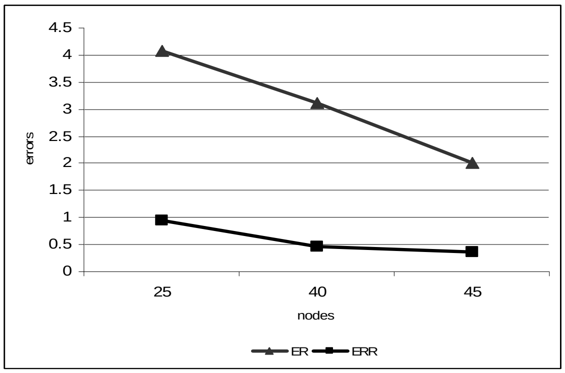

Fig.9. Illustration of the difference between maximum errors in case of a uniform distribution of nodes.

It appears reasonable to expect better results in the vicinity of the trailing edge by using more control points in this area, and so for improving the numerical solution's accuracy we use a non uniform mesh with more nodes in the vicinity of the trailing edge. The errors are smaller in these situations, as we can see from the following graphics, Fig. 10, and Fig.11. We have noted ERNUM (ERRNUM) the maximum errors in case of a non uniform mesh.

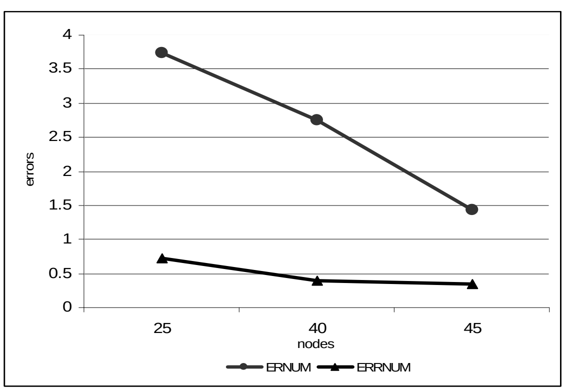

Fig.10. Illustration of the difference between maximum errors in case of a non uniform distribution of nodes.

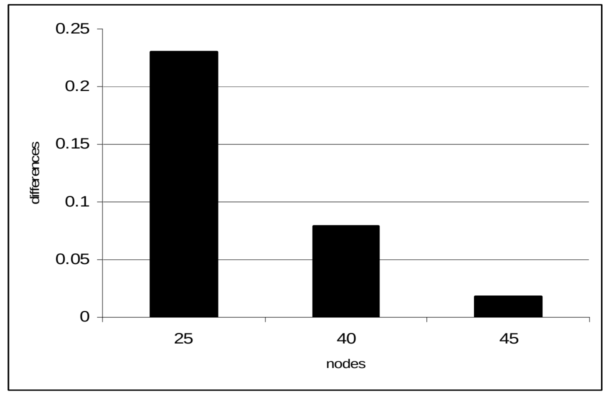

Fig. 11. Differences between maximum errors ($\left|ERR - ERRNUM\right|$), for different number of nodes:

Better results in the vicinity of the trailing edge can be obtained by using other kinds of boundary elements such as linear or curved ones.

The fluid flow around profiles with cusped trailing edges is treated in paper [4] using BEM too, but the fluid is considered incompressible and the potential and stream functions are used in the boundary integral formulation, not the velocity as in this approach.

B. Errors when using linear boundary elements

Another important influence on the accuracy of the numerical solution is due to the type of the boundary element used to solve the boundary integral equation, because it contains both the boundary geometry approximation and the unknown function approximation. This aspect is outline in this paper by making a comparison between the numerical solutions obtained by using constant and linear boundary elements, considering in each situation the same number of nodes for the boundary discrteization. Again the direct boundary formulation is considered.

In case of using linear isoparametric boundary elements, the geometry and the unknown are local approximated by linear models that use the same base functions. The contour C is approximated with a polygonal line, as in the first case, but the unknown G has a linear variation on each of the boundary elements. As before we reduce by discretization the integral equation to an algebraic system and the solutions of this system are then used to calculate the perturbation velocity and the pressure coefficient over the obstacle.
The linear boundary element uses the relations:

$$x = x_i^1\varphi_1 + x_i^2\varphi_2,\ y = y_i^1\varphi_1 + y_i^2\varphi_2,$$
$$G = G_i^1\varphi_1 + G_i^2\varphi_2, \tag{9}$$

where φ_1, φ_2 are shape functions given by:

$$\varphi_1(t) = 1 - t,\ \varphi_2(t) = t,\ t \in [0,1], \tag{10}$$

and G_i^1, G_i^2 are the nodal values of the unknown at the extremes of boundary element L_i. We finally get the following system of equations:

$$\sum_{j=1}^{N} C_{ij} G_j = B_i, i = \overline{1, N} \quad . \tag{11}$$

All the coefficients that occur in system (11) can be computed analytically and their expression can be found in [12]. So no errors are introduced at this stage, as in case of constant boundary elements.

After solving the systems we can evaluate the perturbing velocities and the pressure coefficients in the same manner as in case of constant boundary elements.

This approach was implemented in a computer code which offers numerical solutions for the perturbation velocity and for the local pressure coefficient, for different types of obstacles.

In order to study the errors we consider the case of the circular obstacle and an incompressible flow.

The exact solution of the problem, the numerical one obtained with constant boundary elements, and the numerical solutions obtained with linear boundary elements are compared in Fig.12.

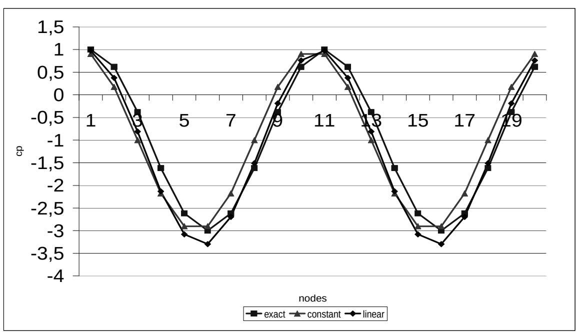

Fig.12 Exact and numerical solutions of the local pressure coefficient for 20 nodes.

As we can see, the calculated and the analytical values of the pressure coefficient are very close, in both cases, but a higher degree of accuracy is obtained by the use of linear elements.

We can also observe that a small number of elements (20) is sufficient for obtaining satisfactory results. As expected, better results are obtained when using more nodes or curved boundary elements which allow a better approximation of the geometry.

IV. INTEGRATION SCHEME INFLUENCE ON NUMERICAL SOLUTION

Another source of errors in applying BEM is represented by different techniques of integrating the kernels, specially the singular ones. Coefficients obtained from them are dominant and situated near the principal diagonal of the system's matrix and so they play an important role in getting an accurate solution.

We consider in this study the boundary integral equation obtained when an indirect technique with sources distribution is applied to solve the problem. In this case the boundary of the obstacle is considered to be a continuous distribution of sources of unknown intensities.

If denoting by *f* the intensitie of the sources, persumed to satisfay a hölder condition on the boundary, the singular boundary integral equivalent to the mathematical model with partial differential equations of the problem, is:

$$\begin{aligned} &\left(n_x^{0^2} + \beta^2 n_y^{0^2}\right) f(\bar{x}_0) + \\ &+ \frac{1}{\pi} \int_C f(\bar{x}) \frac{(x - x_0) n_x^0 + \beta^2 (y - y_0) n_y^0}{\left|\bar{x} - \bar{x}_0\right|^2} ds = 2\beta n_x^0 \end{aligned} \tag{12}$$

with the same notations as before.

When solving the above equation with constant and linear boundary elements we follow the same ideas as in case of equation (3), and conclusions are similar.

As we mentioned before if higher order boundary elements are used we expect better results because of the better approximation of the geometry and of the unknown function.

In paper [11] the above equation is solved using quadratic boundary elements. The geometry and the unknown, have local a quadratic variation, on each of the boundary elements.

For obtaining the discret equation the boundary is divided into *N* unidimensional quadratic boundary elements, each of them with three nodes: two extreme nodes and an interior one. For describing the local geometry and the local behavior of the unknown f, on a boundary element, we use a quadratic model, with the same set of basic functions, noted N_1, N_2, N_3, given by:

$$N_1(\xi) = \frac{\xi(\xi - 1)}{2},\ N_2(\xi) = 1 - \xi^2,\ N_3(\xi) = \frac{\xi(\xi + 1)}{2},\ \xi \in [-1, 1]. \tag{13}$$

We used two systems of notation: a global and a local one (global, f_j is the value of *f* at node number *i*, and local $f_l^i, l = \overline{1,3}, i = \overline{1, N}$ is the value of f at node number *l* of element number *i*).

$$\left(n_x^{j^2} + \beta^2 n_y^{j^2}\right) f(\bar{x}_j) + \frac{1}{\pi} \sum_{i=1}^{N} \left(\sum_{l=1}^{3} a_{ij}^l f_l^i \right) = 2\beta n_x^j \tag{14}$$

Where

$$a_{ij}^l = \int_{-1}^{1} N_l \frac{\left([N]\{x^i\} - x_j\right) n_x^j + \beta^2 \left([N]\{y^i\} - y_j\right) n_y^j}{\left|[N]\{\bar{x}\} - \bar{x}_j\right|^2} J(\xi) d\xi \tag{15}$$

$$[N] = (N_1\ N_2\ N_3).$$

Denoting by:

$$m_i = x_1^i + x_3^i - 2x_2^i,\ n_i = x_3^i - x_1^i,\ u_{ij} = x_2^i - x_j,$$
$$M_i = y_1^i + y_3^i - 2y_2^i,\ N_i = y_3^i - y_1^i,\ U_{ij} = y_2^i - y_j,$$
$$a_i = \frac{m_i^2 + M_i^2}{4},\ aa_i = \frac{n_i^2 + N_i^2}{4},\ b_i = \frac{m_i n_i + M_i N_i}{2},$$
$$c_{ij} = aa_i + m_i u_{ij} + M_i U_{ij},$$
$$d_{ij} = n_i u_{ij} + N_i U_{ij},\ e_{ij} = u_{ij}^2 + U_{ij}^2,\ i, j = \overline{1,2N}, \tag{16}$$

we get for $\bar{x} \in L_i$:

$$\left\|\bar{x} - \bar{x}_j\right\|^2 = a_i\xi^4 + b_i\xi^3 + c_{ij}\xi^2 + d_{ij}\xi + e_{ij} \underset{not}{=} N_{ij}(\xi), \tag{17}$$

and the following expression for the coefficients $a_{ij}^l, l = 1,2,3$:

$$a_{ij}^1 = \frac{1}{4}\int_{-1}^{1}\frac{A_{ij}\xi^4 + B_{ij}\xi^3 + C_{ij}\xi^2 + D_{ij}\xi}{a_i\xi^4 + b_i\xi^3 + c_{ij}\xi^2 + d_{ij}\xi + e_{ij}} J(\xi)d\xi,$$

where:

$$\begin{aligned} A_{ij} &= m_i n_x^j + \beta^2 M_i n_y^j, \\ B_{ij} &= (n_i - m_i)n_x^j + \beta^2(N_i - M_i)n_y^j, \\ C_{ij} &= (2u_{ij} - n_i)n_x^j + \beta^2(2U_{ij} - N_i)n_y^j, \\ D_{ij} &= -2u_{ij}n_x^j - 2\beta^2 U_{ij}n_y^j \end{aligned} \tag{18}$$

$$a_{ij}^2 = \int_{-1}^{1}\frac{-A_{ij}\xi^4 - B_{ij}\xi^3 - C_{ij}\xi^2 + B_{ij}\xi + D_{ij}}{a_i\xi^4 + b_i\xi^3 + c_{ij}\xi^2 + d_{ij}\xi + e_{ij}} J(\xi)d\xi,$$

where:

$$\begin{aligned} A_{ij} &= \frac{m_i n_x^j + \beta^2 M_i n_y^j}{2},\ B_{ij} = \frac{n_i n_x^j + \beta^2 N_i n_y^j}{2}, \\ C_{ij} &= \frac{(2u_{ij} - m_i)n_x^j + \beta^2(2U_{ij} - M_i)n_y^j}{2}, \\ D_{ij} &= -u_{ij}n_x^j - \beta^2 U_{ij}n_y^j \end{aligned} \tag{19}$$

$$a_{ij}^3 = \frac{1}{4}\int_{-1}^{1}\frac{A_{ij}\xi^4 + B_{ij}\xi^3 + C_{ij}\xi^2 + D_{ij}\xi}{a_i\xi^4 + b_i\xi^3 + c_{ij}\xi^2 + d_{ij}\xi + e_{ij}} J(\xi)d\xi,$$

where:

$$\begin{aligned} A_{ij} &= m_i n_x^j + \beta^2 M_i n_y^j, \\ B_{ij} &= (n_i + m_i)n_x^j + \beta^2(N_i + M_i)n_y^j, \\ C_{ij} &= (2u_{ij} + n_i)n_x^j + \beta^2(2U_{ij} + N_i)n_y^j, \\ D_{ij} &= 2u_{ij}n_x^j + 2\beta^2 U_{ij}n_y^j \end{aligned} \tag{20}$$

Returning to the global system of notation, and noting $B_j = 2\pi\beta n_x^j$ we finaly obtain the following linear algebraic system with $2N$ equations and $2N$ unknowns:

$$[A]\{f\} = \{B\},\ A \in M_{2N}(R),\ \{f\},\{B\} \in R^{2N} \tag{21}$$

The coefficients of matrix A depend on the nodes coordinates and on the following integrals:

$$I_{ij}^k = \int_{-1}^{1}\frac{\xi^k}{N_{ij}(\xi)} J(\xi)d\xi, \quad i, j = \overline{1,2N}, k = 0,1,2,3,4, \tag{22}$$

where $J(\xi) = \sqrt{4a_i\xi^2 + 2b_i\xi + aa_i}$.

The treatment of singularities represents one of the most important sources of errors in BEM.

For integrals of singular kernels, evaluated in [13] using different techniques, we have considered in this paper two methods: the method of truncation the interval, and a regularization method.

The nonsingular integrals can be evaluated with a usual numerical method. For treating singularities one can use different efficient techniques ([1], [15]) such as: the non-linear coordinate transformation, the use of element subdivision, singularity isolation using Taylor expansion, etc.

For the singular integrals we have considered here two techniques: the method of truncation the interval and a regularization method.

Using the method of truncation the interval, the singularity that appeares is isolated, and the integral is calculated on an interval where the integrand has no singularities. So, it becomes an usual integral, and a computer can be used for its evaluation.

So, the singular integrals (22) are evaluated as follows:

a) If $j = 2i - 1$, I_{ij}^k has a singularity for $\xi = -1$ so it will be evaluated on $[-1+\varepsilon,1]$, where ε is a verry small positive number.

b) If $j = 2i$, the singularity appeares for $\xi = 0$ and so it will be evaluated on $[-1,-\varepsilon]\bigcup[\varepsilon,1]$.

c) If $j = 2i + 1$, the singularity appeares for $\xi = 1$, and so it will be evaluated on $[-1,1-\varepsilon]$.

After solving the system (21), so after we find the values of f for the $2N$ nodes choosen for the discretization of the boundary we may also compute the velocity for these nodes. Denoting by u,v the components of the velocity we deduce:

$$\begin{aligned} u(\bar{x}_j) &= -\frac{1}{2} f_j n_x^j - \frac{1}{2\pi}\sum_{i=1}^{N}\left(f_1^i b_{ij}^1 + f_2^i b_{ij}^2 + f_3^i b_{ij}^3\right), \\ v(\bar{x}_j) &= -\frac{1}{2} f_j n_y^j - \frac{1}{2\pi}\sum_{i=1}^{N}\left(f_1^i c_{ij}^1 + f_2^i c_{ij}^2 + f_3^i c_{ij}^3\right), \end{aligned} \tag{23}$$

where the coefficients depend on the nodes coordinates and on the same integrals as the system's coefficients. For eample:

$$b_{ij}^1 = \frac{m_i}{4} I_{ij}^4 + \frac{n_i - m_i}{4} I_{ij}^3 + \frac{2u_{ij} - n_i}{4} I_{ij}^2 - \frac{u_{ij}}{2} I_{ij}^1,$$

and analogus for the others.

If $j \neq 2i-1, 2i, 2i+1$ we have usual integrals, and if $j = 2i-1$, $j = 2i$, $j = 2i+1$ the integrals are singular, but they can be evaluated using the truncation method as in case of the singular coefficients of the matrix.

We compare the exact solution obtained for an incompressible ideal fluid and a circular obstacle with the numerical one obtained with the truncation method in Fig.13.

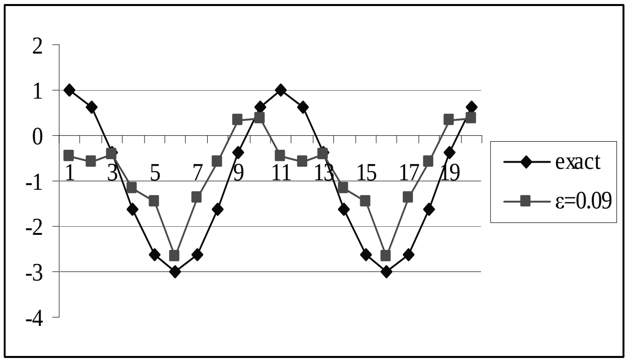

Fig. 13. Exact and numerical solutions in case of truncation method.

As we can see from the above graphic this method doesnt leed to a verry good numerical solution even if quadratic boundary elements are used for the boundary discretization, so it is not recomanded for this types of singular integrals. The error between the exact solution and the numerical one is quite big and it is bigger than that obtained when we use linear and constant boundary elements for solving the problem. The truncation parameter, ε, is set to be 0.09, and 20 nodes are used on the boundary.

The last method applied is the regularization method, the one that made the subject of paper [13], and was inspired by the work of M Bonnet([1]).

The regularization method uses new coordinates and new modified shape functions It is based on Taylor expansion. Taylor polynomials suitable chosen allow us to simplify some factors and so to get modified shape functions, in fact, new integrands. This method offers the best results for this problem.

If $\eta \in [-1,1]$ is the value of the local coordinate ξ, for $\bar{x} = \bar{x}_j$, we have:

$$\left\|\bar{x} - \bar{x}_j\right\|^2 = \left\|\sum_{l=1}^{3}\left(N_l(\xi) - N_l(\eta)\right)\bar{x}_l\right\|^2 \tag{24}$$

with $N_l(\xi) - N_l(\eta) = (\xi - \eta)\hat{N}_l(\xi,\eta)$. (25)

Based on this technique we obtain the expressions for the modified shape functions. Denoting by $\rho = \xi - \eta$ we have:

$$\hat{N}_1(\rho,\eta) = \eta - \frac{1}{2} + \frac{1}{2}\rho, \quad \hat{N}_2(\xi,\eta) = -2\eta - \rho,$$

$$\hat{N}_3(\xi,\eta) = \eta + \frac{1}{2} + \frac{1}{2}\rho \tag{26}$$

Comparisons between the numerical solution obtained with this method, when we use for the discretization 20 nodes, and the exact solution, are performed in Fig.14.

As we can see it shows a verry good agreement, because the calculated and analytical values are very close.

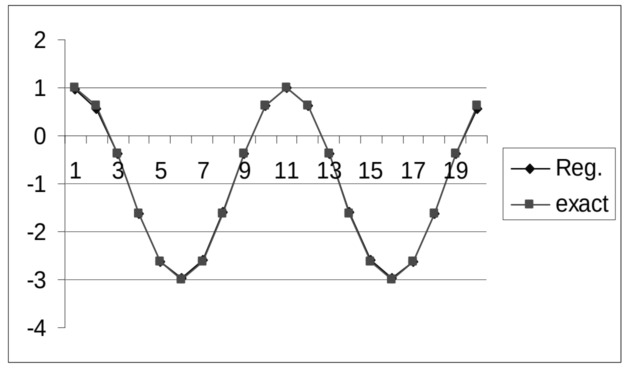

Fig.14 Exact and numerical solutions in case of regularization method

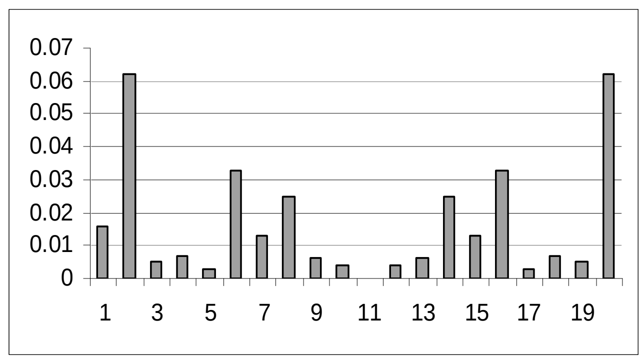

Fig.15. Error distribution in case of regularization methods and 20 nodes on the boundary.

As we can see form the above graphics the accuracy of the numerical solution is very good in case of quadratic boundary elements when the regularization method for integrating the singular kernels.

V. Influence of Different Types of Boundary Element Formulations

The accuracy of numerical solution is influenced by the technique used to obtain the integral formulation of the problem.

Comparison studies between the exact solution and the numerical solutions obtained by considering three different types of boundary element formulations, in fact different techniques of obtaining the singular boundary integral equation: with a direct technique, by applying an indirect technique with sources distribution and with vortex distribution are made in papers [11], [14], [15].

In case of approximating the boundary with a continuous distribution fundamental solutions of vortex type, having the unknown intensity $g(\bar{x})$, the integral equation is solved in [14] using constant boundary elements and in [15] using linear isoparametric boundary elements.

In [14] it was shown that the numerical solution obtained with the direct method is closer to the exact one than the others, by considering in all cases the same obstacle, the circular one, and an incompressible fluid, the same number of nodes for the boundary discretization, and the analytical calculus for the system coefficients.

In case of using the indirect technique, for obtaining the singular boundary integral equation, and isoparametric linear boundary elements to solve it, a comparison between the numerical solutions deduced in case of sources, respective, vortex distributions, shows that the indirect technique with vortex distribution leads to better results in both situations (when using constant [14], or linear boundary elements[15]). Different number of nodes are also considered, and the analytical calculus too. Tests were made for the circular obstacle and for the incompressible fluid flow. Of course by using higher order boundary elements the numerical solution better agrees with the exact one.

VI. Conclusions

This paper briefs out some aspects about the errors that appear when solving problems of partial differential equations by using the boundary element method, through a concrete case: the problem of the subsonic compressible fluid flow around obstacles. Conclusions regarding how to minimize these errors have been formulated.

Taking into account that the boundary integral equation itself is a statement of the exact solution to the problem posed, we can say that errors arise in principal due to discretization and numerical approximations, due to our inability to carry out the required integrations in closed form.

A refined mesh of the boundary offers a very good numerical solution even for lower order boundary elements.

When the number of nodes is bigger than a certain value the error decreases but not so quickly as if the number of nodes is less than that value. It decreases slowly.

The computational effort is bigger when more nodes are used and so, for obtaining an efficient numerical solution and a good ration accuracy computational effort, we have to choose for the number of nodes a certain range of values.

The accuracy of numerical solution is better in case of higher order elements.

The effectiveness of the BEM is clearly dependent on the implementation of efficient and accurate integration procedures to evaluate boundary integrals of the singular kernels. If the numerical integration procedure is made sufficiently sophisticated (by using for example curved boundary elements and continuously varying distributions of functions over the boundary) than the errors so introduced can be very small indeed.

Numerical integration is, of course, always a much more stable and precise process than numerical differentiation and neither the direct nor the indirect BEM require any differentiation of numerical quantities whatsoever, and this recommends the BEM as an efficient numerical technique for solving boundary values problems of partial differential equations.

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