A weighted curvature flow for planar curves

Sheng-Gwo Chen, Mei-Hsiu Chi, Ying-Jen, Lin, and Jyh-Yang Wu

Abstract**—**In this paper we shall discuss a weighted curvature flow for a regular curve in the 2D Euclidean space. The weighted curvature flow for planar curves is a generalization of the well-know curvature flow discussed by Gage, Hamilton and Grayson. Under the suitable weighted curvature flow, convex curves will remain convex under the deformation process. However, the curve may not converge to a round point for general weights. Indeed, for a nonnegative weight function $\omega(u)$ with k isolated zeros, a curve will converge to a limiting k-polygon. The weighted curvature flow will have many useful properties which have applications to image processing as the usual curvature flow does. We shall also present some numerical simulations to illustrate how curves deform under the weighted curvature flow with different weight functions $\omega(u)$.

Keywords**—**A weighted curvature flow, convex curves, regular k-polygons.

I. Introduction

It is well known that the curvature flow can be used to smooth shape. In this note, we discuss a generalized version of the curvature flow. Indeed, we shall modify the curvature flow by adding a nonnegative weight function. Thus, we call it the weighted curvature flow. As shown in Gage, Hamilton (1986) and Grayson (1987), the curvature flow will shrink an embedding planar curve to a point, becoming round in the limit without developing singularities.

Our weighted curvature flow may not shrink a planar curve to a point, instead a k-polygon when the weight function has k isolated zeros. For the positive weight function, the weighted curvature flow will still shrink a planar curve to a point in finite time. However, it may not become round in the limit in the case that the weight function is not constant.

The weighted curvature flow still has some nice properties which are useful for the deformation of shape. For instance, the inclusion order preservation, the length, area and the total curvature decreasing property, and the convexity preserving property.

This paper is organized as follows. In section two we discuss the existence theorem for the weighted curvature flow. We present some basic properties for the weighted curvature flow in section three. In section four, a discrete approximation of the weighted curvature flow is given. This provides a framework for the numerical simulation. In section five we give some computational results to illustrate the properties of the weighted curvature flow discussed in sections two and three.

II. A Weighted Curvature Flow for Regular Curves

The general form of deformation of a regular curve in the 2D Euclidean space can be described as

$$\frac{\partial}{\partial t}\alpha(u,t) = a(u,t)\vec{T} + b(u,t)\vec{N} \qquad (1)$$

with the initial smooth curve $\alpha(u,0)$ where $\alpha(u,t)$ is the position vector of the curve, $\vec{T}$ is the tangent, $\vec{N}$ is the inward normal, u is the path parameter, t is the time parameter of the deformation, and a,b are arbitrary smooth functions. By a reparametrization of the parameter u, Equation (1) can be reduced to the form

$$\frac{\partial}{\partial t}\alpha(u,t) = c(u,t)\vec{N} \qquad (2)$$

for a smooth function c. This form of deformation contains some well-known methods in image processing, such as the prairie fire model of Blum(1973).

The curvature flow discussed by Gage, Hamilton (1986) and Grayson (1987) is a special case of Equation (2):

$$\frac{\partial}{\partial t}\alpha(u,t) = k(u,t)\vec{N} \qquad (3)$$

where the function $k(u,t)$ is the signed curvature of the planar curve $\alpha(u,t)$. The results of Gage, Hamilton (1986) and Grayson (1987) about the curvature flow (3) cab be stated as

Theorem 1. Let $\alpha(u,0)$ be a parametrized simple closed smooth planar curve. Then there exists some $T>0$, and a smooth family of curves $\alpha(u,t)$ with $t<T$ such that $\alpha(u,t)$ satisfies Equation (3) and it converges to a "round" point as $t \to T$.

In this paper we consider a weighted version of Equation (3):

$$\frac{\partial}{\partial t}\alpha(u,t) = \omega(u)k(u,t)\vec{N} \qquad (4)$$

where the nonnegative smooth function $\omega(u)$ is called a weight function. When $\omega(u)$ is the constant function 1, Equation (4) reduced to Equation (3). Equation (4) is also useful in image processing. When we want to fix certain part of the shapes under the deformation, then we can take value 0 for

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the weighted function $\omega(u)$ at that part of the shape boundary curve.

Using the techniques in the proof of Theorem 1, one can prove the following existence result.

Theorem 2. Let $\alpha(u,0)$ be a parametrized simple closed smooth planar curve. Then there exists some $T>0$, and a smooth family of curves $\alpha(u,t)$ with $t<T$ such that $\alpha(u,t)$ satisfies Equation (4). Moreover, when the weighted function $\omega(u)$ has k isolated zeros at u_i, $i=1,2,...,k$, the limiting shape as $t\to T$ is a k-polygon with vertice $\alpha(u_i,0)$, $i=1,2,...,k$

III. Some Basic Properties for the Weighted Curvature Flow

In this section we shall give some basic properties about the weighted curvature flow (4). The weighted curvature flow is also a special case of the reaction-diffusion process. The maximum principle for parabolic differential equations (see Protter and Weinberger 1984) implies that disjoint curves will remain disjoint. Therefore, two shapes, one inside another, will never meet during the deformation.

As in the curvature flow, the weighted one is also a curve-shortening flow. From now on we assume the nonnegative weighted function $\omega(u)$ can has only isolated zeros.

Theorem 3 *(Length and Area Decreasing Property)*:

Let a smooth family of planar curves $\alpha(u,t)$ satisfy Equation (4). Then the length function $L(t)=L(\alpha(\cdot,t))$ of the curve $\alpha(\cdot,t)$ and the area function $A(t)=A(\alpha(\cdot,t))$ bounded by the curve $\alpha(\cdot,t)$ are decreasing. That is, one has $L(t)<L(0)$ and $A(t)<A(0)$ for $0<t<T$.

The total curvature is a measurement of complexity for a curve and it is given by

$$\bar{k}(t)=\int_0^L |k(s,t)|ds\ . \qquad (5)$$

As for the Equation (3), we still have

Theorem 4. (*Total Curvature Decreasing Property*)

Let a smooth family of planar curves $\alpha(u,t)$ satisfy Equation (4). Then the total curvature stratifies $\bar{k}(t)<\bar{k}(0)$ for $0<t<T$.

Furthermore, if the initial curve $\alpha(u,0)$ is convex (i.e., the signed curvature $k(u,0)$ is nonnegative.), it will remain so during the weighted curvature flow.

This can be proved using the methods developed in Protter and Weinberger (1984).

Theorem 5. (*Convexity Preserving Property*)

Let a smooth family of planar curves $\alpha(u,t)$ satisfy Equation (4). If the signed curvature $k(u,0)$ is nonnegative, then $k(u,t)$ is still nonnegative for $0<t<T$.

IV. Discrete Approximation for the Weighted Curvature Flow

$$\vec{t}_i^{\,'}=\omega_1\frac{\vec{t}_i-\vec{t}_{i-1}}{\|p_i-p_{i-1}\|}+\omega_2\frac{\vec{t}_{i+1}-\vec{t}_i}{\|p_{i+1}-p_i\|}. \qquad (7)$$

Note that the vector $\vec{t}_i^{\,'}$ may not be normal to the tangent vector $\vec{t}_i$ due to the discrete effect. Hence we can define the signed curvature κ_i of the discrete curve C at p_i by the formula

$$\vec{t}_i^{\,'}-(\vec{t}_i{}'\cdot\vec{t}_i)\vec{t}_i=\kappa_i\vec{n}_i\,. \qquad (8)$$

From these discussions, we can also define the derivative of a function f or a vector field V on the discrete curve C by

$$\begin{aligned} f'(p_i) &= \omega_1\frac{f(p_i)-f(p_{i-1})}{\|p_i-p_{i-1}\|} \\ &\quad+\omega_2\frac{f(p_{i+1})-f(p_i)}{\|p_{i+1}-p_i\|} \end{aligned} \qquad (9)$$

And

$$V'(p_i)=\omega_1\frac{V(p_i)-V(p_{i-1})}{\|p_i-p_{i-1}\|}+\omega_2\frac{V(p_{i+1})-V(p_i)}{\|p_{i+1}-p_i\|}.$$

Indeed, when we know how to differentiate functions and vector fields on a discrete curve C, we can develop a differential theory on C.From the experience given in Chen, Wu (2004, 2005) and Chen, Chi and Wu (2006), we shall use the centroid weights for the weights ω_1 and ω_2. Namely, for the discrete curve $C=\{p_i\in R^3 : i=1,2,\cdots,k\}$, we have at the point p_i

$$\omega_1=\frac{\dfrac{1}{\|p_i-p_{i-1}\|^2}}{\left(\dfrac{1}{\|p_i-p_{i-1}\|^2}+\dfrac{1}{\|p_{i+1}-p_i\|^2}\right)} \qquad (10)$$

and

$$\omega_2=\frac{\dfrac{1}{\|p_{i+1}-p_i\|^2}}{\left(\dfrac{1}{\|p_i-p_{i-1}\|^2}+\dfrac{1}{\|p_{i+1}-p_i\|^2}\right)}. \qquad (11)$$

Now we can approximate the weighted curvature flow by the discrete curvature flow:

$$\frac{\partial}{\partial t} p_i(t) = \omega(p_i)\kappa_i \vec{n}_i \qquad (12)$$

where $\omega(p_i) = \omega(u_i)$ is now viewed as a function defined on the discrete curve C. In the smooth case, Grayson showed that when $\omega = 1$, embedded plane curves will flow to a "round point". We shall give in section five the numerical simulation of our discrete curvature flow for some simple closed discrete plane curves for nonnegative weight function ω with only isolated zeros.

V. Computational results

From figure 1 to figure 4, we test the curvature flows (12) with different weight functions. In figure 1 and 2, we used the constant weight function, i.e. $w(t) = 1$. In our simulations, the graph of curve will deform to a circle. In figure 3 and 4, we chose the weight function by $w(t) = |\sin(6\pi t)|$, six fixed points and $w(t) = |\sin(12\pi t)|$, twelve fixed points. Under these weight functions, the simple curves will deform to a n-polygon, where n is the number of fixed points.

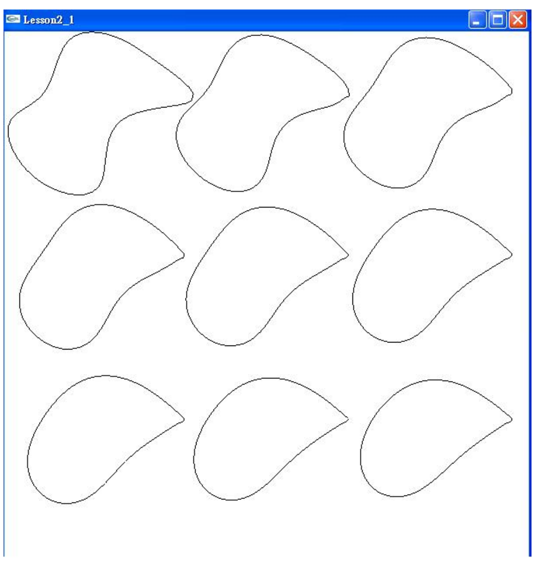

Fig 1. $w(t) = 1$

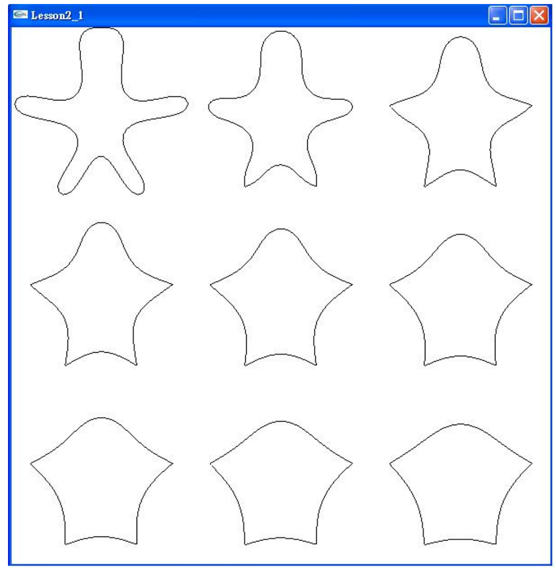

Fig 2. $w(t) = 1$

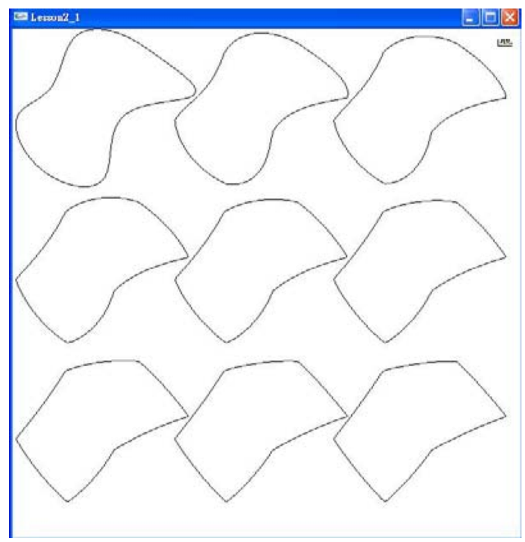

Fig. 3 $w(t) = |\sin(6\pi \cdot t)|$

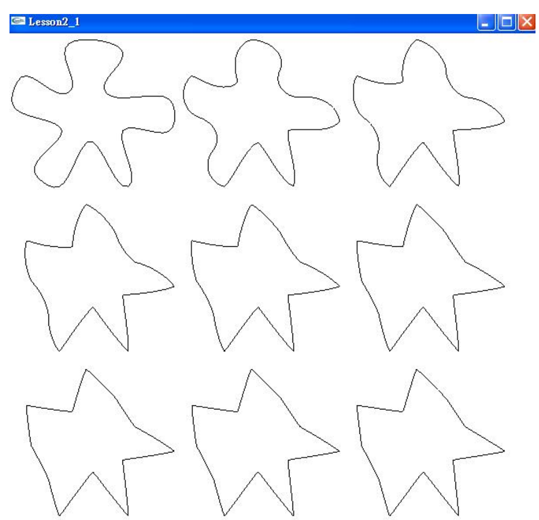

Fig4. $w(t) = |\sin(12\pi \cdot t)|$

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