# Two Random Interfaces of Statistical Mechanics Models

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***Abstract*—**We consider the limiting statistical properties of fluctuations of statistical mechanics models. The two random interfaces of one-dimensional statistical physics models is modeled and investigated in the present paper. The two random interfaces are constructed by assuming that there is a specified value of the large area in the intermediate region of the two random interfaces, and the two random interfaces have fixed endpoints. When the inverse temperature is large enough, we show that the limiting distributions of the two random interfaces of the model convergence to a Gaussian distribution.

***Keywords*—**Two random interfaces, Gaussian distribution, probability measure, fluctuation.

## I. Introduction

In recent years, some research work has been done to investigate the statistical properties of the random interfaces for some statistical physics models, for example see Refs. [1-7]. The problem of description of shapes of interfaces is a well-known problem of statistical mechanics, in details that, the aim of the research is to investigate the asymptotical behavior of the corresponding sequence of probability measure describing the statistical properties of these interfaces. In this paper, we consider the statistical properties of the two random paths model. This work originates in an attempt to describe the fluctuations of the interfaces in two random interfaces models (e.g. one-dimensional two random interfaces S.O.S. model), see [8-9]. In Ref. [1], the statistical properties of random walks and the interface of Widom-Rowlinson model (conditioned by fixing a large area under their paths and conditioned by fixing the terminating point) are considered, and the central limit theorem for these conditional distributions is proved. In [7], the interfaces of supercritical Ising model on the lattice fractal---the Sierpinski carpet is studied . So it is interesting for us to consider the similar problems arising in describing the fluctuations of two random interfaces models. In the first part of the present paper, with the conditions `fixed area' of the intermediate layer and `fixed end points' in a two random paths model, we study the limiting properties of the two random paths, see [10].

At each site $x$ of the one dimensional lattice $Z$, we attach two variables of `heights' $\omega_x^1, \omega_x^2 \in Z$, therefore the configurations of the random paths model on a horizontal set $L_X = \{x_0, x_0+1, ..., x_0+L\} \subset Z$ (with the length of $L$) are represented by sets of heights $\{\omega^1, \omega^2\} = \{\omega_x^1, \omega_x^2\}_{x \in L_X}$, for the simplicity, we assume $x_0 = 0$. In this paper, we study a two random paths model, the Hamiltonian of the model on the horizontal set of $L_X$ is given by

$$H_L(\omega^1, \omega^2) = \sum_{\langle xy \rangle : x, y \in L_X} \left( \left| \omega_x^1 - \omega_y^1 \right|^{\alpha_1} + \left| \omega_x^2 - \omega_y^2 \right|^{\alpha_2} \right)$$

where the constants $\alpha_1, \alpha_2 > 0$, and the sum extends over nearest-neighbor pairs in $L_X$. For the two random paths models, let $\Omega_L = \left\{ \left(\omega^1, \omega^2\right) : \omega_x^1, \omega_x^2 \in Z, x \in L_X \right\}$ be the corresponding configuration space. The partition function of this system is

$$Z_{L,\beta} = \sum_{\left(\omega^1, \omega^2\right) \in \Omega_L} \exp\left[ -\beta H_L(\omega^1, \omega^2) \right]$$

where $\beta$ is a positive parameter called an inverse temperature. The corresponding Gibbs probability distribution on $\Omega_L$ is given by

$$P_{L,\beta}\left(\omega^1, \omega^2\right) = \left(Z_{L,\beta}\right)^{-1} \exp\left[ -\beta H_L(\omega^1, \omega^2) \right].$$

Next we define the paths of the two random paths model as followings, for $t \in [0,1]$

$$X_L^{\omega^1}\left(\frac{j}{L}\right) = \omega_j^1, \ j \in L_X$$

$$X_L^{\omega^1}(t) = (j+1-Lt)\, X_L^{\omega^1}\left(\frac{j}{L}\right) + (Lt-j)\, X_L^{\omega^1}\left(\frac{j+1}{L}\right), \quad j \le Lt \le j+1,$$

and $X_L^{\omega^2}\left(\frac{j}{L}\right)$, $X_L^{\omega^2}(t)$ are defined similarly as above definitions.

For $x \in L_X$ and $x \ge 1$, let $\xi_x = \omega_x^1 - \omega_{x-1}^1$, $\eta_x = \omega_x^2 - \omega_{x-1}^2$, so we have $\omega_x^1 = \sum_{i=0}^{x} \xi_x$ and $\omega_x^2 = \sum_{i=0}^{x} \eta_x$, where let $\xi_0 = 0$, $\eta_0 = 0$. Let $\xi = \{\xi_x, x \in L_X\}$, $\eta = \{\eta_x, x \in L_X\}$, then rewrite above partition function as

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$$Z_{L,\beta}=\sum_{\xi,\eta}\exp\left[-\beta H_L(\xi,\eta)\right]$$

where $H_L(\xi,\eta)$ is the Hamiltonian function for $(\xi,\eta)$. Then we have the corresponding Gibbs probability distribution $P_{L,\beta}(\xi,\eta)$ for the partition function $Z_{L,\beta}$. Thus we have the corresponding paths $X_L^{\xi}\left(\frac{j}{L}\right), X_L^{\xi}(t), X_L^{\eta}\left(\frac{j}{L}\right), X_L^{\eta}(t)$.

From above definitions, $\xi=\{\xi_x, x\in L_X\}$ and $\eta=\{\eta_x, x\in L_X\}$ can be seen as the sequences of i.i.d. random variables respectively. So, the two random paths model has two independent random *SOS* paths, that is, the model corresponds to the ensemble of two independent self-avoiding paths in $[0,L]\times Z$ starting from $(0,0)$ and ending at sites $z$ in the line $\{x=L\}$ (where $z=(x,y)$ ), which do not go back in the horizontal direction. Next we introduce the generating function of the height of the endpoints for one `step', that is, for a fixed $x\in L_X$, let

$$Q(\mu,\nu)=\sum_{\xi_x,\eta_x}e^{\beta\mu\xi_x+\beta\nu\eta_x-\beta(|\xi_x|+|\eta_x|)}\Big/\sum_{\xi_x,\eta_x}\exp\left[-\beta H_L(\xi_x,\eta_x)\right]$$

where $Q(\mu,\nu)$ is independent of $x$ and $-\infty<\xi_x,\eta_x<+\infty$. Due to the independence of the random variables $\{\xi_x, x\in L_X\}$ and $\{\eta_x, x\in L_X\}$, thus

$$Q(\mu,\nu)^L=\sum_{\xi,\eta}\exp\left[\beta\mu\overline{\xi}+\beta\nu\overline{\eta}\right]\exp\left[-\beta H_L(\xi,\eta)\right]\Big/Z_{L,\beta}$$

where $\overline{\xi}=\sum_{x=1}^{L}\xi_x$ and $\overline{\eta}=\sum_{x=1}^{L}\eta_x$. For $(\mu,\nu)\in R\times R$, we define

$$\varphi(\mu,\nu)=\lim_{L\to 0}\frac{1}{L}\times$$
$$\ln\left(\sum_{\xi,\eta}\exp[\beta\mu\overline{\xi}+\beta\nu\overline{\eta}]\exp[-\beta H_L(\xi,\eta)]\Big/Z_{L,\beta}\right)$$

by the Refs. [1][3], it is known that the limit exists if $(\mu,\nu)$ is in some neighborhood of the origin.

The aim of this paper is to study the asymptotes of fluctuations of the two random paths conditioned by fixing a large area between the two random paths. Denote by $a_L^{\xi}$, $a_L^{\eta}$ representing the areas under the paths $X_L^{\xi}\left(\frac{j}{L}\right)$, $X_L^{\eta}\left(\frac{j}{L}\right)$ respectively, and denote by $a_L^{\eta-\xi}=a_L^{\eta}-a_L^{\xi}$ representing the area of the intermediate layer between the two random paths. For a real $\overline{\zeta}_0$ and $0\le s\le 1$, assume that

$$F\left(\overline{\zeta}_0,\beta,s\right)=\frac{d}{d\zeta_0}\varphi\left(-(1-s)\zeta_0,(1-s)\zeta_0\right)\Big|_{\zeta_0=\overline{\zeta}_0},$$

$$\frac{1}{\beta}\int_0^1 F\left(\overline{\zeta}_0,\beta,s\right)ds=a \tag{1}$$

where $a>0$ is some constant. Above (1) is the important condition of the present paper, we will use this condition to fulfill our proof in the followings. Then we state the main results of this paper.

**Theorem 1** Assume that for some $\delta(\beta)>0$ and $a>0$, there exists a real $\overline{\zeta}_0$ satisfying above condition (1) and $\left|\overline{\zeta}_0\right|<\delta(\beta)$, then the process

$$Y_L(t)=\frac{1}{\sqrt{L}}\left\{X_L^{\eta}(t)-X_L^{\xi}(t)-\frac{L}{\beta}\int_0^t F\left(\overline{\zeta}_0,\beta,s\right)ds\right\},$$

under $P_{L,\beta}\left(\cdot\left|a_L^{\eta-\xi}=\lfloor aL\rfloor\right.\right)$, converges weakly to the process

$$Y(t)=\frac{1}{\beta}\int_0^t\sqrt{\varphi''\left(-(1-s)\overline{\zeta}_0,(1-s)\overline{\zeta}_0\right)}dB(s)$$

conditioned that $\int_0^1 Y(t)dt=0$, where $\{B(s)\}_{s\ge 0}$ is the one dimensional standard Brownian motion, and $\lfloor aL\rfloor$ is the integer part of $aL$.

**Remark 1** In Theorem 1, the model is only conditioned by fixing a large area between the two random paths and having the same starting endpoints. The results can also be proved similarly for the two random paths with fixed value of area and the two same endpoints.

**Theorem 2** Let $\varphi'_{\mu}(\mu,\nu)=\frac{\partial}{\partial\mu}\varphi(\mu,\nu)$, and let

$$F_{\mu}\left(\overline{\zeta}_0,\beta,s\right)=-\varphi'_{\mu}\left(-(1-s)\overline{\zeta}_0,(1-s)\overline{\zeta}_0\right).$$

With the same conditions of Theorem 1, the probability distribution of the random process $-X_L^{\xi}(t)/L$, under $P_{L,\beta}\left(\cdot\left|a_L^{\eta-\xi}=\lfloor aL\rfloor\right.\right)$, converges weakly to the corresponding probability distribution concentrated on the function

$$Y_1(t)=\frac{1}{\beta}\int_0^t\sqrt{F_{\mu}\left(\overline{\zeta}_0,\beta,s\right)}ds.$$

## II. Estimation for the Fluctuations of the Two Random Paths Model

In this section, we begin discussing the area between the two random paths. Then we show the some results about the weak convergence of random vector of the two random interfaces for the model. Now we define the areas of $a_L^{\xi}$, $a_L^{\eta}$, $a_L^{\eta-\xi}$ as followings,

$$a_L^{\xi}=\sum_{x=1}^{L}\omega_x^1/L=\sum_{x=1}^{L}(1-x/L)\xi_x,$$

$$a_L^{\eta}=\sum_{x=1}^{L}\omega_x^2/L=\sum_{x=1}^{L}(1-x/L)\eta_x,$$

$$a_L^{\eta-\xi}=a_L^{\eta}-a_L^{\xi}=\sum_{x=1}^{L}(1-x/L)(\eta_x-\xi_x).$$

By the independence of $\{\xi_x, x\in L_X\}$ and $\{\eta_x, x\in L_X\}$, the generation function of the area $a_L^{\eta-\xi}$ is defined by

$$Q_{a_L^{\eta-\xi}}(\zeta) = \sum_{\eta,\xi} \exp\left\{\beta\zeta a_L^{\eta-\xi}\right\} \exp\left\{-\beta H_L(\xi,\eta)\right\} \Big/ Z_{L,\beta} = \prod_{x=1}^{L} Q\left(-\zeta(1-x/L), \zeta(1-x/L)\right).$$

Let $q$ be a natural number, and let $\{t_i, 1 \le i \le q\}$ be any set of real numbers, such that $0 < t_1 < ... < t_q \le 1$. Set a random vector as

$$\hat{X}_L^{(q)}(t_1,...,t_q) = \left(a_L^{\eta-\xi}, \omega^2_{\lfloor t_1 L\rfloor} - \omega^1_{\lfloor t_1 L\rfloor}, ..., \omega^2_{\lfloor t_q L\rfloor} - \omega^1_{\lfloor t_q L\rfloor}\right).$$

Then for $\underline{\zeta} = (\zeta_0, \zeta_1, ..., \zeta_q) \in R^{q+1}$, we have

$$\sum_{\eta,\xi} e^{\beta\underline{\zeta}\hat{X}_L^{(q)}(t_1,...,t_q)} e^{-\beta H_L(\xi,\eta)} \Big/ Z_{L,\beta} = \prod_{x=1}^{L} Q\left(-\zeta_L(x;\underline{\zeta}), \zeta_L(x;\underline{\zeta})\right)$$

where $\zeta_L(x;\underline{\zeta}) = \zeta_0(1-x/L) + \sum_{i=1}^{q} \zeta_i 1_{[0,Lt_i]}(x)$. For the real $\bar{\zeta}_0$ defined in (1) and some small constant $\alpha > 0$, let $\underline{\zeta} \in R^{q+1}$ satisfy the following conditions

$$D_{\alpha,\bar{\zeta}_0} = \left\{\zeta : -\alpha < \zeta_0 < \bar{\zeta}_0 + \alpha, |\zeta_i| < \alpha, i = 1,...,q\right\}$$

Next we introduce the corresponding quadratic form, a $(q+1) \times (q+1)$ matrix $V_L(\underline{\zeta})$ denote by

$$V_L(\underline{\zeta}) = \frac{1}{\beta^2 L} Hess \ln\left(\sum_{\eta,\xi} e^{\beta\underline{\zeta}\hat{X}_L^{(q)}(t_1,...,t_q)} e^{-\beta H_L(\xi,\eta)} \Big/ Z_{L,\beta}\right),$$

where $V_L(\underline{\zeta})$ is analytic in $D_{\alpha,\bar{\zeta}_0}$. Assume that $\underline{\zeta} \in D_{\alpha,\bar{\zeta}_0}$, and according to the definition of $V_L(\underline{\zeta})$, then uniformly in $\underline{\zeta}$ and $\underline{y} = (y_0,...,y_q) \in R^{q+1}$ such that $|\underline{y}| = 1$, we have

$$\underline{y}.V_L(\underline{\zeta})\underline{y} \to \underline{y}.V(\underline{\zeta})\underline{y}, \quad \text{as } L \to \infty$$

where

$$V(\underline{\zeta}) = \frac{1}{\beta^2} Hess \int_0^1 \ln Q\left(-\zeta(s), \zeta(s)\right) ds,$$

and $\zeta(s) = \zeta_0(1-s)\sum_{i=1}^{q} \zeta_i 1_{[0,t_i]}(s)$, for $0 \le s \le 1$. Let $\hat{P}_L^{(q)}$ be the probability distribution of $\hat{X}_L^{(q)}(t_1,...,t_q)$ under $P_{L,\beta}$, and $\hat{P}_{L,\underline{\zeta}}^{(q)}$ be given by

$$\hat{P}_{L,\underline{\zeta}}^{(q)}(\underline{z}) = e^{\beta\underline{\zeta}\underline{z}} \hat{P}_L^{(q)}(\underline{z}) \Big/ E_{L,\beta}\left(e^{\beta\underline{\zeta}\hat{X}_L^{(q)}(t_1,...,t_q)}\right)$$

for all $\underline{\zeta} \in D_{\alpha,\bar{\zeta}_0}$ and $\underline{z} \in Z_L^{(q)} = (L^{-1}Z) \times Z^q$. Denote by $\hat{E}_{L,\underline{\zeta}}^{(q)}(\cdot)$ the corresponding expectation function for $\hat{X}_L^{(q)}(t_1,...,t_q)$. By the uniform boundedness of the family of analytical functions $V_L(\underline{\zeta})$ for all $L$ and all $\underline{\zeta}$ in $D_{\alpha,\bar{\zeta}_0}$, according to Lemma 2.6 and Proposition 2.7 in [1], we have the following Lemma 1 and Lemma 2.

**Lemma 1** Let $\underline{\zeta}_L, \underline{\zeta} \in D_{\alpha,\bar{\zeta}_0}$, and $\underline{\zeta}_L \to \underline{\zeta}$ as $L \to \infty$. Then the random vector

$$\hat{Y}_L^{(q)}(t_1,...,t_q) = \frac{1}{\sqrt{L}}\left(\hat{X}_L^{(q)}(t_1,...,t_q) - \hat{E}_{L,\underline{\zeta}_L}^{(q)} \hat{X}_L^{(q)}(t_1,...,t_q)\right)$$

converges weakly to a Gaussian random vector $\hat{Y}^{(q)}(t_1,...,t_q)$ of which covariance matrix is given by $V(\underline{\zeta})$.

Let $g_{\underline{\zeta}}$ be the density function of the Gaussian vector $\hat{Y}^{(q)}(t_1,...,t_q)$ given in Lemma 1.

**Lemma 2** Let $Z_L^{(q)} = (L^{-1}Z) \times Z^q$, then for each $\underline{z}_L \in Z_L^{(q)}$ and $\underline{\zeta}_L \in D_{\alpha,\bar{\zeta}_0}$, define

$$\underline{y}_L = \frac{1}{\sqrt{L}}\left(\underline{z}_L - \hat{E}_{L,\underline{\zeta}_L}^{(q)} \hat{X}_L^{(q)}(t_1,...,t_q)\right).$$

Then we have

$$L^{(q+3)/2} \hat{P}_{L,\underline{\zeta}_L}^{(q)}(\underline{z}_L) - g_{\underline{\zeta}_L}(\underline{y}_L) \to 0, \quad \text{as} \quad L \to \infty$$

uniformly in $\underline{z}_L \in Z_L^{(q)}$ and $\underline{\zeta}_L \in D_{a,\bar{\zeta}_0}$.

## III. Proof of the Main Results

In this section, we discuss the limiting properties of the random vector $\hat{X}_L^{(q)}(t_1,...,t_q)$ defined in Section 2, and show the proofs of Theorem .

**Proof of Theorem 1**. In Section 2, the random vector $\hat{X}_L^{(q)}(t_1,...,t_q)$ is given. First we consider the convergence of the finite-dimensional distribution of the random vector $\hat{Y}_L^{(q)}(t_1,...,t_q)$ defined in Lemma 1. Let $\underline{\zeta}_L^0, \underline{\zeta}^0$ be a special sequence in $D_{\alpha,\underline{\zeta}_L^0}$, such that

$$\underline{\zeta}_L^0 = \left(\underline{\zeta}_{L,0}, 0, ...0\right), \quad \underline{\zeta}^0 = \left(\bar{\zeta}_0, 0, ...0\right)$$

where $\bar{\zeta}_0$ is defined in (1), and $\underline{\zeta}_{L,0}$ satisfies the following condition

$$\frac{d}{d\zeta_0} \ln Q_{a_L^{\eta-\varepsilon}}(\zeta_0)\Big|_{\zeta_0 = \zeta_{L,0}} = \lfloor aL \rfloor, \tag{2}$$

by (1)(2), it can be proved that $\underline{\zeta}_L^0 \to \underline{\zeta}^0$ as $L \to \infty$. Let

$$\varphi_L(\underline{\zeta}; t_1,...,t_q) = \frac{1}{L} \ln\left(\sum_{\xi,\eta} e^{\beta\underline{\zeta}\cdot\hat{X}_L^{(q)}(t_1,...,t_q)} e^{-\beta H_L(\xi,\eta)} \Big/ Z_{L,\beta}\right)$$

and denote by

$$\varphi^{(q)}(\zeta; t_1,...,t_q) = \lim_{L\to\infty} \varphi_L(\zeta; t_1,...,t_q)$$

for $\underline{\zeta} \in D_{\alpha,\underline{\zeta}_L^0}$. By the uniform boundedness of $Hess_{\underline{\zeta}}\varphi_L$, we have

$$\hat{E}_{L,\underline{\zeta}_L^0}^{(q)} \hat{X}_L^{(q)}(t_1,...,t_q) = \left(\lfloor aL \rfloor, \hat{E}_{L,\underline{\zeta}_L^0}^{(q)}\left(\omega^2_{\lfloor t_1 L\rfloor} - \omega^1_{\lfloor t_1 L\rfloor}\right), ..., \hat{E}_{L,\underline{\zeta}_L^0}^{(q)}\left(\omega^2_{\lfloor t_q L\rfloor} - \omega^1_{\lfloor t_q L\rfloor}\right)\right)$$

$$= \frac{L}{\beta}\left(\nabla_{\underline{\zeta}}\varphi_L\right)\left(\underline{\zeta}_L^0;t_1,...,t_q\right)$$

$$= \frac{L}{\beta}\left(\nabla_{\underline{\zeta}}\varphi^{(q)}\right)\left(\underline{\zeta}^0;t_1,...,t_q\right)+o(1).$$

By Lemma 2, we have for $-\infty < a_j < b_j < \infty, 1 \le j \le q$,

$$\lim_{L\to\infty}\hat{P}_L^{(q)}\left(y_j \in \left[a_j,b_j\right],\ 1\le j\le q \middle| z_0 = \lfloor aL \rfloor\right)$$

$$= \lim_{L\to\infty}\hat{P}_{L,\underline{\zeta}_L^0}^{(q)}\left(y_j \in \left[a_j,b_j\right],\ 1\le j\le q \middle| z_0 = \lfloor aL \rfloor\right)$$

$$= \frac{\int_{[a_1,b_1]\times\ldots\times[a_q,b_q]} g_{\underline{\zeta}^0}\left(0,y_1,...,y_q\right)dy_1\cdots dy_q}{\int_{R^q} g_{\underline{\zeta}^0}\left(0,y_1,...,y_q\right)dy_1\cdots dy_q}.$$

According to Lemma 1, let

$$\hat{Y}^{(q)}\left(t_1,...,t_q\right) = \left(Y_0, Y(t_1),...,Y(t_q)\right)$$

be a Gaussian random vector with distribution density $g_{\underline{\zeta}}\left(y_0,...,y_q\right)$. Then its covariance matrix is given by

$$E\left[Y\left(t_j\right)Y\left(t_K\right)\right] = \frac{1}{\beta^2}\int_0^{t_j\wedge t_k}\varphi''\left(-(1-s)\overline{\zeta}_0,(1-s)\overline{\zeta}_0\right)ds$$

$$E\left[Y_0Y\left(t_j\right)\right] = \frac{1}{\beta^2}\int_0^{t_j}\varphi''\left(-(1-s)\overline{\zeta}_0,(1-s)\overline{\zeta}_0\right)ds$$

$$E\left[Y_0^2\right] = \frac{1}{\beta^2}\int_0^{1}\varphi''\left(-(1-s)\overline{\zeta}_0,(1-s)\overline{\zeta}_0\right)ds$$

for $j,k = 1,...,q$, where $a \wedge b = \min\{a,b\}$. This means that $\left\{Y_0,\{Y(t)\}_{t\in[0,1]}\right\}$ is a Gaussian random process with covariance matrix given above for every $q \ge 1$. In above proof, we suppose that $\omega^2_{\lfloor t_iL\rfloor} - \omega^1_{\lfloor t_iL\rfloor} = X_L^\eta\left(\frac{\lfloor Lt_i\rfloor}{L}\right) - X_L^\xi\left(\frac{\lfloor Lt_i\rfloor}{L}\right)$, for $i = 1,...,q$. Similarly to Ref. [1], the above argument is also true if we replace $X_L^\eta\left(\frac{\lfloor Lt_i\rfloor}{L}\right) - X_L^\xi\left(\frac{\lfloor Lt_i\rfloor}{L}\right)$ with $X_L^\eta(t_i) - X_L^\xi(t_i)$ for every $1 \le i \le q$. Then the distribution of $\hat{X}_L^{(q)}\left(t_1,...,t_q\right)$, under $P_{L,\beta}\left(\cdot\middle|\alpha_L^{\eta-\xi} = \lfloor aL\rfloor\right)$, converges weakly to the corresponding distribution of the Gaussian random vector $\hat{Y}_L^{(q)}\left(t_1,...,t_q\right)$.

Secondly, the tightness of above conditional distribution of the random process $Y_L(t)$ should be discussed, see [10]. Following the similar argument of Section 3 in Ref. [1], we can prove a sufficient condition for the tightness of the considered process $Y_L(t)$. Together with the first part of this proof, this completes the proof of Theorem 1.

**Remark 2** According to the arguments of [1], and with the results of Theorem 1, the probability distribution of the random process

$$\left(X_L^\eta(t) - X_L^\xi(t)\right)/L$$

under $P_{L,\beta}\left(\cdot\middle|\alpha_L^{\eta-\xi} = \lfloor aL\rfloor\right)$, converges weakly to the corresponding distribution concentrated on the function $\frac{1}{\beta}\int_0^1 F\left(\overline{\zeta}_0,\beta,s\right)ds$.

## IV. The Extension of the Main Result

In this section, we give the extension results of Theorem 1, and show the proofs of Theorem 2 and Corollary 1.

**Proof of Theorem 2** Let $q$ be a natural number, and let $\{t_i, 1\le i\le q\}$ be any set of real numbers, such that $0 < t_1 < ... < t_q \le 1$. Set a random vector as

$$\hat{X}_L^\xi\left(t_1,...,t_q\right) = \left(a_L^{\eta-\xi}, -\omega^1_{\lfloor t_1L\rfloor},...,-\omega^1_{\lfloor t_qL\rfloor}\right).$$

Let $\underline{\zeta}_L^0, \underline{\zeta}^0$ be a special sequence in $D_{\alpha,\underline{\zeta}_L^0}$, such that

$$\underline{\zeta}_L^0 = \left(\underline{\zeta}_{L,0},0,...0\right), \quad \underline{\zeta}^0 = \left(\overline{\zeta}_0,0,...0\right)$$

where $\overline{\zeta}_0$ is defined in (1), and $\underline{\zeta}_{L,0}$ is defined in (2). Then we have the corresponding function as following

$$\varphi_L^1\left(\underline{\zeta};t_1,...,t_q\right) = \frac{1}{L}\ln\left(\sum_{\xi,\eta} e^{\beta\underline{\zeta}\cdot\hat{X}_L^\xi(t_1,...,t_q)}e^{-\beta H_L(\xi,\eta)} \Big/ Z_{L,\beta}\right)$$

$$= \frac{1}{L}\ln\prod_{x=1}^{L} Q\left(-\zeta_L^\xi\left(x;\underline{\zeta}\right),\zeta_L^\eta\left(x;\zeta_0\right)\right)$$

where $\zeta_L^\xi\left(x;\underline{\zeta}\right) = \zeta_0(1-x/L) + \sum_{i=1}^q \zeta_i 1_{[0,Lt_i]}(x)$, $\zeta_L^\eta\left(x;\underline{\zeta}\right) = \zeta_0(1-x/L)$. For any $\underline{\zeta} = \left(\zeta_0,\zeta_1,...,\zeta_q\right)\in R^{q+1}$ satisfy the following conditions

$$D_{\alpha,\overline{\zeta}_0} = \left\{\zeta : -\alpha < \zeta_0 < \overline{\zeta}_0 + \alpha, \left|\zeta_i\right| < \alpha, i = 1,...,q\right\}.$$

Let

$$\varphi^1\left(\underline{\zeta};t_1,...,t_q\right) = \lim_{L\to\infty}\varphi_L^1\left(\underline{\zeta};t_1,...,t_q\right)$$

for $\underline{\zeta}\in D_{\alpha,\underline{\zeta}_L^0}$, and $\hat{E}^1_{L,\underline{\zeta}}(\cdot)$ is the corresponding expectation function for $\hat{X}_L^\xi\left(t_1,...,t_q\right)$. By the uniform boundedness of $Hess_{\underline{\zeta}}\varphi_L$, we have

$$\hat{E}^1_{L,\underline{\zeta}_L^0}\hat{X}_L^\xi\left(t_1,...,t_q\right) =$$

$$\left(\lfloor aL\rfloor, \hat{E}^1_{L,\underline{\zeta}_L^0}\left(-\omega^1_{\lfloor t_1L\rfloor}\right),...,\hat{E}^1_{L,\underline{\zeta}_L^0}\left(-\omega^1_{\lfloor t_qL\rfloor}\right)\right)$$

$$= \frac{L}{\beta}\left(\nabla_{\underline{\zeta}}\varphi_L^1\right)\left(\underline{\zeta}_L^0;t_1,...,t_q\right)$$

$$= \frac{L}{\beta}\left(\nabla_{\underline{\zeta}}\varphi^1\right)\left(\underline{\zeta}^0;t_1,...,t_q\right)+o(1),$$

where $1\le j\le q$, and

$$\hat{E}^1_{L,\underline{\zeta}_L^0}\left(-\omega^1_{\lfloor t_1L\rfloor}\right) = \sum_{x=1}^{L}\frac{\partial}{\partial\xi_i}\ln Q\left(-\zeta_L^\xi\left(x;\underline{\zeta}\right),\zeta_L^\eta\left(x;\zeta_0\right)\right)\Bigg|_{\underline{\zeta}=\underline{\zeta}_L^0}.$$

For the random vector $\hat{X}_L^\xi\left(t_1,...,t_q\right)$, by using the methods of Lemma 2.6 and Proposition 2.7 in [1], we can have the similar results as that of Lemma 1 and Lemma 2. Then following the steps in the proof of Theorem 1, we can prove that the probability distribution of the random process

$$\frac{1}{\sqrt{L}}\left\{-X_L^{\xi}(t)-\frac{L}{\beta}\int_0^t F_\mu\left(\overline{\zeta}_0,\beta,s\right)ds\right\}$$

under $P_{L,\beta}\left(\cdot\left|\alpha_L^{\eta-\xi}=\lfloor aL\rfloor\right.\right)$, converges weakly to some Gaussian distribution. Thus by Remark 2, the probability distribution of the random process $-X_L^{\xi}(t)/L$, under $P_{L,\beta}\left(\cdot\left|a_L^{\eta-\xi}=\lfloor aL\rfloor\right.\right)$, converges weakly to the corresponding probability distribution concentrated on the function

$$Y_1(t)=\frac{1}{\beta}\int_0^t\sqrt{F_\mu\left(\overline{\zeta}_0,\beta,s\right)}ds\ .$$

This completes the proof of Theorem 2.

**Corollary 1** Suppose that the definitions and conditions of Theorem 2 hold, then the probability distribution of the random process $X_L^{\eta}(t)/L$, under $P_{L,\beta}\left(\cdot\left|\alpha_L^{\eta-\xi}=\lfloor aL\rfloor\right.\right)$, converges weakly to the corresponding probability distribution concentrated on the function

$$\frac{1}{\beta}\int_0^t F\left(\overline{\zeta}_0,\beta,x\right)dx-\frac{1}{\beta}\int_0^t F_\mu\left(\overline{\zeta}_0,\beta,x\right)dx\ .$$

Proof. The random process $\left(X_L^{\eta}(t)\left|\alpha_L^{\eta-\xi}=\lfloor aL\rfloor\right.\right)$ can be written as

$$\left(X_L^{\eta}(t)\left|\alpha_L^{\eta-\xi}=\lfloor aL\rfloor\right.\right)=\left(X_L^{\eta}(t)-X_L^{\xi}(t)\left|\alpha_L^{\eta-\xi}=\lfloor aL\rfloor\right.\right)$$
$$+\left(X_L^{\xi}(t)\left|\alpha_L^{\eta-\xi}=\lfloor aL\rfloor\right.\right).$$

For the first term of above equation, under $P_{L,\beta}\left(\cdot\left|\alpha_L^{\eta-\xi}=\lfloor aL\rfloor\right.\right)$ and by Theorem1 and Remark 2, we have that the probability distribution of the random process $\left(X_L^{\eta}(t)-X_L^{\xi}(t)\left|\alpha_L^{\eta-\xi}=\lfloor aL\rfloor\right.\right)$ converges weakly to the corresponding probability distribution of the function

$$\frac{1}{\beta}\int_0^t F\left(\overline{\zeta}_0,\beta,x\right)dx\ .$$

For the second term of above equation, under $P_{L,\beta}\left(\cdot\left|\alpha_L^{\eta-\xi}=\lfloor aL\rfloor\right.\right)$ and according to Theorem 2 and Remark 2, the probability distribution of the random process $\left(-X_L^{\xi}(t)\left|\alpha_L^{\eta-\xi}=\lfloor aL\rfloor\right.\right)$ converges weakly to the corresponding probability distribution of the function

$$\frac{1}{\beta}\int_0^t F_\mu\left(\overline{\zeta}_0,\beta,x\right)dx\ .$$

This completes the proof of Corollary 1.

## V. Two Interfaces Problems of S.O.S. Model

In this section, we discuss the relations between the two random paths model and the two interfaces S.O.S. model. The Hamiltonian $H_L^S\left(\omega^1,\omega^2\right)$ of two interfaces S.O.S. model has the same definition of $H_L^S\left(\omega^1,\omega^2\right)$ in Section 1. But the partition function of two interfaces S.O.S. model is given by

$$Z_{L,\beta}^S=\sum_{\omega_x^1,\omega_x^2,x\in L_X}\exp\left[-\beta H_L^S\left(\omega^1,\omega^2\right)\right]$$

and according to the definitions in Section 1, we have the corresponding partition function

$$Z_{L,\beta}^S=\sum_{\xi\le\eta}\exp\left[-\beta H_L^S\left(\xi,\eta\right)\right]$$

where $\xi\le\eta$ denote that $\xi_x\le\eta_x$ for all $x\in L_X$. From above definitions, for the two interfaces S.O.S. model, the two interfaces of the model don't intersect, so that, the two interfaces are not independent.

Let $Q_{l,m}\subset Z^2$ be a rectangle of side length $2l$ (horizontal size) and $2m$. For the two-dimensional Ising model (see [8][9]), by using the techniques of correlation functions for estimating the fluctuation of phase separation (or interface) line, when $\beta>\beta_c^{I\sin g}$ ($\beta_c^{I\sin g}$ is the critical point of Ising model), Higuchi and Wang proved that, with probability larger than $1-\exp\left[-c_1(\beta)\ln l\right]$ (for example see [1][7]), the interface has a height less than $c(\beta)(l\ln l)^{\frac{1}{2}}$, where $l$ large enough and $c_1(\beta)$, $c(\beta)$ are positive constants. Let $\Omega_{Q_{l,m}}$ be the configuration space of the Ising model, and $\mu_{Q_{l,m}}^{\tau}$ be the corresponding Gibbs measure with the boundary condition $\tau$, where $\tau$ is defined by

$$\tau=\begin{cases}-1 & \text{if}\quad u_2\ge m+1\\ +1 & \text{if}\quad u_2\le m\end{cases}$$

for $u=(u_1,u_2)\in Z^2$. Now we give the following Lemma which was given by Higuchi and Wang.

**Lemma 3 (Higuchi & Wang)** For the two-dimensional Ising model, let $Q_{l,m}$ be defined as above, $\beta>\beta_c^{I\sin g}$. For some $k(\beta)>0$, set $m=[k(\beta)l^{\frac{1}{2}}(\ln l)^{\frac{1}{2}}]$, then there are $c_1(\beta)>0$ and $l_0=l_0(\beta)>0$ independent of $Q_{l,m}$, such that for all $l>l_0$,

$$\mu_{Q_{l,m}}^{\tau}\left(\left(F_{Q_{l,m}}^{\tau}\right)^c\right)\le\exp\left[-c_1(\beta)\ln l\right]$$

where $F_{Q_{l,m}}^{\tau}$ is the event $\left\{\sigma\in\Omega_{Q_{l,m}}:\Gamma_{open}^{\tau}(\sigma)\subset Q_{l,m}\right\}$ and $\Gamma_{open}^{\tau}(\sigma)$ denote those open contours produced by the configuration $\sigma\subset Q_{l,m}$ with boundary condition $\tau$ on $Q_{l,m}$.

**Remark 3** Lemma 3 is proved for the two-dimensional Ising model, it describes the statistical properties of the interface of the Ising model. The simple case of this problem arises in the one-dimensional S.O.S. model. Through the similar arguments in the proof of Lemma 3, we can have the similar result as that of Lemma 3 for one-dimensional S.O.S. model, that is, the interface of S.O.S. model has a height less than $c(\beta)(l\ln l)^{\frac{1}{2}}$ with large probability.

In above Remark 3, we discuss the interface height for one-interface S.O.S. model. The aim of this paper is to study two random paths model and two interfaces S.O.S. model. From Section 1 to Section 3, we have studied the interface of the two random paths model conditioned on a fixed area' in the intermediate layer and fixed end points'. In this Section, by using Lemma 3 and Remark 3, we study the relations between the two random paths model and the two interfaces S.O.S. model.

In the definitions of Section 1, with the starting points $\xi_0 = 0$ and $\eta_0 = 0$, we discussed the two random paths model with the partition function of $Z_{L,\beta} = \sum_{\xi,\eta} \exp\left[-\beta H_L(\xi,\eta)\right]$. While in this Section, we modify the end points of the model. Let $H_{L_x}^{\xi,\eta}$ denote the event

$$\xi_0 = \xi_L = 0, \quad \eta_0 = \eta_L = M(\beta)(L\ln L)^{\frac{1}{2}},$$

where $M(\beta)(>4c(\beta))$ is a large positive constant. The random paths $X_L^{\xi}\left(\frac{j}{L}\right)$, $X_L^{\eta}\left(\frac{j}{L}\right)$ are defined in Section 1, and let $F_{L_x}^{\xi,\eta}$ denote the event that the random paths $X_L^{\xi}\left(\frac{j}{L}\right)$ and $X_L^{\eta}\left(\frac{j}{L}\right)$ don't intersect each other on $L_x$, then we have the following Lemma 4.

**Lemma 4** For the two random paths model defined in (2)(3)(4), there are $c_2(\beta) > 0$, $L_2 = L_2(\beta) > 0$ and $\beta_2 > 0$ such that for all $L > L_2$ and for all $\beta > \beta_2$,

$$P_{L,\beta}\left(\left(F_{L_x}^{\xi,\eta}\right)^c \middle| H_{L_x}^{\xi,\eta}\right) \le 2\exp\left[-c_2(\beta)\ln L\right]$$

The proof of Lemma 4 follows directly from Lemma 3, Remark 3 and the condition $M(\beta) > 4c(\beta)$. This lemma shows that, with large probability, the two random paths don't intersect. Let

$$\left(X_L^{\xi}(t), X_L^{\eta}(t)\right)^* = \left(\left(X_L^{\xi}(t), X_L^{\eta}(t)\right) \middle| F_{L_x}^{\xi,\eta}, H_{L_x}^{\xi,\eta}\right)$$

and $P_{L,\beta}^* = P_{L,\beta}\left(\cdot \middle| F_{L_x}^{\xi,\eta}, H_{L_x}^{\xi,\eta}\right)$ be probability distribution of the random process $\left(X_L^{\xi}(t), X_L^{\eta}(t)\right)^*$. According to the Lemma 4, we have the following corollary.

**Corollary 2** With the same conditions of Lemma 4, we have the following

$$\lim_{L\to\infty}\Big\{P_{L,\beta}\left(\left(X_L^{\xi}(t), X_L^{\eta}(t)\right)\in G \middle| H_{L_x}^{\xi,\eta}\right) - P_{L,\beta}^*\left(\left(X_L^{\xi}(t), X_L^{\eta}(t)\right)\in G\right)\Big\} = 0$$

where $G = [a_1, a_2]\times[b_1, b_2], -\infty < a_i < b_i < \infty$, for $i = 1, 2$.

From the definition of the process $\left(X_L^{\xi}(t), X_L^{\eta}(t)\right)^*$, it is known that the process $\left(X_L^{\xi}(t), X_L^{\eta}(t)\right)^*$ is a conditional two interfaces S.O.S. model (with the special fixed end points). Corollary 2 shows a limiting relation between the two random paths model and the conditional two interfaces S.O.S. model. This result is useful to study the asymptotic properties of the two interfaces S.O.S. model by using the results of two random paths model, for example, we consider the two interfaces S.O.S. model with a large fixed area between the two interfaces, etc.

## VI. Conclusion

In this paper, we studied the statistical properties of the two random paths model. Under some conditions, that there is a specified value of the large area in the intermediate region of the two random interfaces, Theorem 1 and Theorem 2 show the weak convergence of the fluctuations for the two random interfaces.


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