Parameter Identification of a Two Degrees of Freedom Mechanical System

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***Abstract*—A method for identifying parameters of a linear mechanical system is proposed. The method is based on ERA/OKID identification method developed by Juang [1]. This procedure has been tested on a light-damped mechanical linear apparatus.**

***Index Terms*—Markov Parameters, Eigenvalues Realization Algorithm, Observer/Kalman Filter Identification.**

I. Introduction

SYSTEM identification is the art of determining a model of a dynamical process by combining information obtained from experimental data with that derived from an a priori knowledge of the physical behavior of the system. In general, the system or the physical process to be modeled can be of any kind, even though applied system identification usually considers only deterministic processes.

System identification is a discipline that can be studied at different levels. From the basic point of view, the purpose of identification is to determine just how many states or modes are needed to construct a model of the system. Once past this stage, one can begin an higher level system identification. On the other hand, the most refined level of identification is the parametric identification.

Between these two extremes there are many system identification techniques whose purpose is to model the dynamic behavior of a physical system without determining its equations of motion. In general, there is a wide range of identification techniques and the choice of the technique to be used needs to be decided from time to time. In the field of parametric identification there are basically three possible approaches: time-domain analysis, frequency-domain analysis and bifurcations analysis. In the case of time-domain analysis, one should find an approximate analytical expression, written in terms of unknown parameters, and compare it with experimentally measured data. In this case we need to measure only one output signal and then is possible to perform an efficient identification procedure. I wish you the best of success.

II. Mathematical Background

A. State-Space model

The equations of motion for a finite-dimensional linear-dynamic system are a set of n_2 second-order differential equations, where n_2 is the number of independent coordinates. Let $\mathbf{M}$, $\mathbf{R}$ and $\mathbf{K}$ be the mass, damping and stiffness matrices, respectively. The equations of motion can be expressed in matrix notation as:

$$\mathbf{M}\,\ddot{\mathbf{x}}(t) + \mathbf{R}\,\dot{\mathbf{x}}(t) + \mathbf{K}\,\mathbf{x}(t) = \mathbf{F}(t) \tag{1}$$

where $\ddot{\mathbf{x}}(t)$, $\dot{\mathbf{x}}(t)$ and $\mathbf{x}(t)$ are vectors of generalized acceleration, velocity and displacement, respectively, and $\mathbf{F}(t)$ is the forcing function.

On the other hand, if the response of the dynamic system is measured by the m output quantities in the output vector $\mathbf{y}(t)$, then the output equations can be written in a matrix form as follows:

$$\mathbf{y}(t) = \mathbf{C_a}\,\ddot{\mathbf{x}}(t) + \mathbf{C_v}\,\dot{\mathbf{x}}(t) + \mathbf{C_d}\,\mathbf{x}(t) \tag{2}$$

where $\mathbf{C_a}$, $\mathbf{C_v}$ and $\mathbf{C_d}$ are output influence matrices for acceleration, velocity and displacement, respectively. These output influences matrices describe the relation between the vectors $\ddot{\mathbf{x}}(t)$, $\dot{\mathbf{x}}(t)$, $\mathbf{x}(t)$ and the measurement vector $\mathbf{y}(t)$.

Let $\mathbf{z}(t)$ be the state vector of the system:

$$\mathbf{z}(t) = \begin{bmatrix} \mathbf{x}(t) \\ \dot{\mathbf{x}}(t) \end{bmatrix} \tag{3}$$

If the excitations of the dynamic system is measured by the r input quantities in the input vector $\mathbf{u}(t)$, the equations of motions and the set of output equations can both be respectively rewritten in terms of the state vector as follows:

$$\dot{\mathbf{z}}(t) = \mathbf{A_c}\,\mathbf{z}(t) + \mathbf{B_c}\,\mathbf{u}(t) \tag{4}$$

$$\mathbf{y}(t) = \mathbf{C}\,\mathbf{z}(t) + \mathbf{D}\,\mathbf{u}(t) \tag{5}$$

where $\mathbf{A_c}$ is the state matrix, $\mathbf{B_c}$ is the state influence matrix, $\mathbf{C}$ is the measurement influence matrix and $\mathbf{D}$ is the direct transmission matrix. These matrix can be computed in this way:

$$\mathbf{A_c} = \begin{bmatrix} \mathbf{O} & \mathbf{I} \\ -\mathbf{M}^{-1}\,\mathbf{K} & -\mathbf{M}^{-1}\,\mathbf{R} \end{bmatrix} \tag{6}$$

$$\mathbf{B_c} = \begin{bmatrix} \mathbf{O} \\ \mathbf{M}^{-1}\,\mathbf{B_2} \end{bmatrix} \tag{7}$$

$$\mathbf{C} = \begin{bmatrix} \mathbf{C_d} - \mathbf{C_a}\,\mathbf{M}^{-1}\,\mathbf{K} & \mathbf{C_v} - \mathbf{C_a}\,\mathbf{M}^{-1}\,\mathbf{R} \end{bmatrix} \tag{8}$$

$$\mathbf{D} = \mathbf{C_a}\,\mathbf{M}^{-1}\,\mathbf{B_2} \tag{9}$$

where $\mathbf{B_2}$ is an influence matrix characterizing the locations and the type of inputs according to this equation:

$$\mathbf{F}(t) = \mathbf{B_2}\,\mathbf{u}(t) \tag{10}$$

The equations (4) and (5) constitute a continuous-time state-space model of a dynamical system.

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On the other hand, consider a discrete-time state-space model of a dynamical system:

$$\mathbf{z}(k+1) = \mathbf{A}\,\mathbf{z}(k) + \mathbf{B}\,\mathbf{u}(k) \tag{11}$$

$$\mathbf{y}(k) = \mathbf{C}\,\mathbf{z}(k) + \mathbf{D}\,\mathbf{u}(k) \tag{12}$$

Because experimental data are discrete in nature, the equations (11) and (12) form the basis for the system identification of linear, time-invariant, dynamical systems. The state matrix $\mathbf{A}$ and the influence matrix $\mathbf{B}$ of the discrete-time model can be computed from the analogous matrices $\mathbf{A_c}$, $\mathbf{B_c}$ of the continuous-time model sampling the system at equally spaced intervals of time:

$$\mathbf{A} = e^{\mathbf{A_c}\,\Delta t} \tag{13}$$

$$\mathbf{B} = \int_0^{\Delta t} e^{\mathbf{A_c}\,\tau} d\tau\ \mathbf{B_c} \tag{14}$$

where Δt is a constant interval.

B. System Markov Parameters

Solving the equations (11) and (12) with zero initial conditions and in terms of previous inputs yields:

$$\mathbf{z}(k) = \sum_{j=1}^{k} \mathbf{A}^{j-1}\,\mathbf{B}\,\mathbf{u}(k-j) \tag{15}$$

$$\mathbf{y}(k) = \mathbf{C}\sum_{j=1}^{k} \mathbf{A}^{j-1}\,\mathbf{B}\,\mathbf{u}(k-j) + \mathbf{D}\,\mathbf{u}(k) \tag{16}$$

To observe the response to a pulse in one of the input variables, consider the following input vector:

$$\mathbf{u}(k=0) = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \quad , \quad \mathbf{u}(k>0) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{17}$$

When the substitution is performed for each input, the results can be assembled into a sequence of pulse-response matrix $\mathbf{Y}_k$ with dimesion m by r as follows:

$$\begin{array}{l} \mathbf{Y}_0 = \mathbf{D}\,,\ \mathbf{Y}_1 = \mathbf{C}\,\mathbf{B}\,,\ \mathbf{Y}_2 = \mathbf{C}\,\mathbf{A}\,\mathbf{B}\,,\ \ldots \\ \ldots,\ \mathbf{Y}_k = \mathbf{C}\,\mathbf{A}^{k-1}\,\mathbf{B} \end{array} \tag{18}$$

The constant matrices in the sequence are known as system Markov parameters. The Markov parameters are commonly used as the basis for identifying mathematical models for linear dynamical systems starting from experimental data. Indeed, it is clear that the matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{D}$ are embedded in the Markov parameter sequence. Since Markov parameter sequence is simply the pulse response of the system, they must be unique for a given system.

Using the definition of Markov parameters, the equation (16) can be rewritten as:

$$\mathbf{y}(k) = \sum_{j=0}^{k} \mathbf{Y}_j\,\mathbf{u}(k-j) \tag{19}$$

From this equation it can be seen that the contribution to the output at time step k by the input applied at time step k and at the previous time steps are weighted by the pulse response sequence. For this reason the pulse response sequence is also known as the weighting sequence and this input-output description is called the weighting sequence description. If the system is asymptotically stable, the summation in equation (19) can have a finite approximation because in this case the weighting sequence may be truncated after a finite number of time steps.

The weighting sequence uses the pulse response sequence to describe the input-output relationship instead of using the state description. It does not require a state equation as an intermediate step to compute outputs from given inputs. The advantage of the weighting sequence is that the dimension of $\mathbf{Y}_k$ is determined by the number of inputs and outputs only, regardless of the number of independent coordinates in the state equation for the state-space description.

The input-output description of a system describes only the relationship between the input and the output under the assumption that the initial condition is zero or that the system is in the condition of a steady state. This description does not reveal the behavior inside the system, such as the interaction between the physical parameters. Consequently, the input-output description may not characterize a system completely. In practice, one may be interested in only the system modal parameters including frequencies, dampings and modes shapes. In these cases, steady-state tests provide enough informations for a test engineer to extract the modal parameters from the input-output description. A steady-state test can be done by first allowing the transient response to decay.

C. State-Space Observer Model

A state-space model for a linear system describe the system input and output via a the state vector $\mathbf{z}(k)$. However the state vector is, in general, not accessible for direct measurement. A state estimator $\mathbf{G}$, also known as an observer, can be used to provide an estimate of the system state from input and output measurements. Introducing the state estimator the equations of motions and the output equations of the system can be rewritten as follows:

$$\mathbf{z}(k+1) = \bar{\mathbf{A}}\,\mathbf{z}(k) + \bar{\mathbf{B}}\,\mathbf{v}(k) \tag{20}$$

$$\mathbf{y}(k) = \mathbf{C}\,\mathbf{z}(k) + \mathbf{D}\,\mathbf{u}(k) \tag{21}$$

where:

$$\mathbf{v}(k) = \begin{bmatrix} \mathbf{u}(k) \\ \mathbf{y}(k) \end{bmatrix} \tag{22}$$

The state matrix $\bar{\mathbf{A}}$ and the influence matrix $\bar{\mathbf{B}}$ can be computed using the observer matrix $\mathbf{G}$ in this way:

$$\bar{\mathbf{A}} = \mathbf{A} + \mathbf{G}\,\mathbf{C} \tag{23}$$

$$\bar{\mathbf{B}} = \begin{bmatrix} \mathbf{B} + \mathbf{G}\,\mathbf{D} & -\mathbf{G} \end{bmatrix} \tag{24}$$

The equations (20) and (21) constitute a discrete-time state-space observer model of a dynamical system. These equations are identical in form to the equations (10) and (11) but the eigenvalues of $\bar{\mathbf{A}}$ are moved by a consequence of the additional term $\mathbf{G}\,\mathbf{C}$ and the number of columns in $\bar{\mathbf{B}}$ is increased relative to those of $\mathbf{B}$ by the number of outputs m. Because the matrix $\mathbf{G}$ can be arbitrarily chosen, $\bar{\mathbf{A}}$ may be made as asymptotically stable as desired.

It can be showed that the the matrix $\mathbf{G}$ can be interpreted in terms of an observer. By using this observer matrix the system discrete-time state-space model can be written in terms of the observer state vector $\hat{\mathbf{z}}(k)$ and the estimated output $\hat{\mathbf{y}}(k)$ as follows:

$$\hat{\mathbf{z}}(k+1) = \bar{\mathbf{A}}\,\hat{\mathbf{z}}(k) + \bar{\mathbf{B}}\,\mathbf{v}(k) \tag{25}$$

$$\hat{\mathbf{y}}(k) = \mathbf{C}\,\hat{\mathbf{z}}(k) + \mathbf{D}\,\mathbf{u}(k) \tag{26}$$

where $\bar{\mathbf{A}}$ and $\mathbf{v}(k)$ are identical to those defined in the equations (22) and (23). Defining the state estimation error in this way:

$$\mathbf{e}(k) = \mathbf{z}(k) - \hat{\mathbf{z}}(k) \tag{27}$$

The equations governing the estimation error can be written as:

$$\mathbf{e}(k+1) = \bar{\mathbf{A}}\;\;\mathbf{e}(k) \tag{28}$$

If $\bar{\mathbf{A}}$ is asymptotically stable, the for large k the estimated state $\hat{\mathbf{z}}(k)$ tends to the true state $\mathbf{z}(k)$. Theoretically, one would choose the gain matrix $\mathbf{G}$ to make the state estimation error diminish as quickly as possible. In the presence of process and measurements noises, under ideal conditions, the quickest observer is the Kalman filter. The observer equation is necessary for a system which has uncertainties and which contains output noises and/or has unknown initial conditions.

D. Observer Markov Parameters

Since the equations (20) and (21) are similar to the equations (10) and (11), a parameter sequence equivalent to that in the equations (18) can be defined as follows:

$$\begin{aligned}&\bar{\mathbf{Y}}_0 = \mathbf{D}\,,\ \bar{\mathbf{Y}}_1 = \bar{\mathbf{C}}\,\bar{\mathbf{B}}\,,\ \bar{\mathbf{Y}}_2 = \bar{\mathbf{C}}\,\bar{\mathbf{A}}\,\bar{\mathbf{B}}\,,\ \dots\\ &\dots,\ \bar{\mathbf{Y}}_k = \bar{\mathbf{C}}\,\bar{\mathbf{A}}^{k-1}\,\bar{\mathbf{B}}\end{aligned} \tag{29}$$

The constant matrices in this sequence are defined as observer Markov parameters. The generic matrix $\bar{\mathbf{Y}}_k$ of the observer Markov parameter sequence can be rewritten as follows:

$$\bar{\mathbf{Y}}_k = \begin{bmatrix} \bar{\mathbf{Y}}_k^{(1)} & -\bar{\mathbf{Y}}_k^{(2)} \end{bmatrix} \tag{30}$$

where $\bar{\mathbf{Y}}_k^{(1)}$ and $\bar{\mathbf{Y}}_k^{(2}$ are defined as:

$$\bar{\mathbf{Y}}_k^{(1)} = \mathbf{C}\,(\mathbf{A} + \mathbf{G}\,\mathbf{C})^{k-1}\,(\mathbf{B} + \mathbf{G}\,\mathbf{D}) \tag{31}$$

$$\bar{\mathbf{Y}}_k^{(1)} = \mathbf{C}\,(\mathbf{A} + \mathbf{G}\,\mathbf{C})^{k-1}\,\mathbf{G} \tag{32}$$

The observer Markov parameters can be used as the basis for computing system Markov parameters. Indeed, it is clear from the equations (29) and (30) that the matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$ and $\mathbf{G}$ are embedded in the observer Markov parameter sequence.

The extra freedom inherent in the observer Markov parameter can be exploited to develop various identification algorithms. Consider the case where $\mathbf{G}$ corresponds to a dead-beat observer gain. The observer Markov parameters become identically zero after a finite number of terms. The matrix $\mathbf{G}$ thus chosen become optimal in the sense that the number of computed Markov parameters is the minimum number needed to describe the system input-output relationship. For lightly damped structures, this means that the system can be by a small number of observer Markov parameters instead of an otherwise large number of the usual system Markov parameters. For this reason, they are the natural parameters to be identified to characterize the system of interest.

Solving the equations (25) and (26) with zero initial conditions in terms of the previous inputs and outputs yields:

$$\hat{\mathbf{x}}(k) = \sum_{j=1}^{k} \bar{\mathbf{A}}^{j-1}\,\bar{\mathbf{B}}\,\mathbf{v}(k-j) \tag{33}$$

$$\hat{\mathbf{y}}(k) = \mathbf{C}\sum_{j=1}^{k} \bar{\mathbf{A}}^{j-1}\,\bar{\mathbf{B}}\,\mathbf{v}(k-j) + \mathbf{D}\,\mathbf{u}(k) \tag{34}$$

Using the definition of the observer Markov parameter, equation (34) can be rewritten as:

$$\hat{\mathbf{y}}(k) = \sum_{j=1}^{p} \bar{\mathbf{Y}}_j\,\mathbf{v}(k-j) + \mathbf{D}\,\mathbf{u}(k) \tag{35}$$

provided that $\bar{\mathbf{Y}}_k$ can be neglected for $k > p$. This is equivalent to making $\bar{\mathbf{A}} = \mathbf{A} + \mathbf{G}\,\mathbf{C}$ sufficiently stable with a proper choice of $\mathbf{G}$ such that $\bar{\mathbf{A}}^p$ can be neglected. In this case, the estimated output $\hat{\mathbf{y}}(k)$ closely approaches the measured output $\mathbf{y}(k)$ for $k > 0$ because the estimation error $\varepsilon(k)$ approaches zero.

Equation (35) is commonly called the linear difference model for multi-input/multi-output, linear, time-invariant systems. This is also often referred to as the ARX model, where AR refers to the AutoRegressive part, related to output data, and X refers to the eXogeneous part, related to input data. This form is commonly used in developing recursive system identification techniques.

The ARX model is also an input-output description of a system similar to the weighting sequence description expressed with the equation (19). The ARX model describe only the relationship between the input and the output under the assumption that the initial condition is zero or that the system is in the condition of a steady state. If this assumption is not satisfied, the ARX model is not valid. In practice, if the test for direct input and output measurements is sufficiently long to allow the transient response to decay, then the error due to a nonzero initial condition becomes negligible.

E. Computation of Markov Parameters

Now consider the equation (11) and (12). Assuming zero initial conditions, the set of these equations for a sequence different time $k = 0\,, 1\,, \dots, l-1$ can be grouped in a matrix form to yield:

$$\mathbf{y} = \mathbf{Y}\,\mathbf{U} \tag{36}$$

where:

$$\mathbf{Y} = \begin{bmatrix} \mathbf{Y}_0 & \mathbf{Y}_1 & \mathbf{Y}_2 & \dots & \mathbf{Y}_{l-1} \end{bmatrix} \tag{37}$$

$$\mathbf{y} = \begin{bmatrix} \mathbf{y}(0) & \mathbf{y}(1) & \mathbf{y}(2) & \dots & \mathbf{y}(l-1) \end{bmatrix} \tag{38}$$

$$\mathbf{U} = \begin{bmatrix} \mathbf{u}(0) & \mathbf{u}(1) & \dots & \mathbf{u}(l-1) \\ \mathbf{0} & \mathbf{u}(0) & \dots & \mathbf{u}(l-2) \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{u}(0) \end{bmatrix} \tag{39}$$

Equation (36) is a matrix representation of the relationship between the input and output time histories. The matrix $\mathbf{y}$ is a $[m,l]$ output data matrix, where m is the number of the outputs and l is the number data samples. The matrix $\mathbf{Y}$ is a $[m,r\,l]$ matrix which contains all the Markov parameters to be determined. The matrix $\mathbf{U}$ is an $[r\,l,l]$ block upper triangular input matrix. It is square in the case of a single input system, and otherwise ha more rows than columns. Inspection of equation (36) indicates that there are $[m,r\,l]$ unknowns in the Markov parameters matrix but only $[m,l]$ equations. For the case where $r > 1$ the solution for $\mathbf{Y}$ is not unique. However, it is known that for a finite-dimensional linear system, $\mathbf{Y}$ must be unique. The matrix $\mathbf{Y}$ can only be uniquely determined from this set of equations for $r = 1$. Even in this case, if the input has zero initial value, or the input signals are not rich enough in frequency content, or l is too large, the matrix $\mathbf{U}$ become ill-conditioned and its inverse cannot be accurately computed.

Consider the case where $\mathbf{A}$ is asymptotically stable so that, for sufficient large p, $\mathbf{A}^p \simeq \mathbf{O}$. Equation (36) can the be approximated by:

$$\mathbf{y} = \hat{\mathbf{Y}}\,\hat{\mathbf{U}} \tag{40}$$

where:

$$\hat{\mathbf{Y}} = \left[\begin{array}{cccccc} \mathbf{Y}_0 & \mathbf{Y}_1 & \mathbf{Y}_2 & \dots & \mathbf{Y}_p \end{array}\right] \tag{41}$$

$$\hat{\mathbf{U}} = \begin{bmatrix} \mathbf{u}(0) & \mathbf{u}(1) & \dots & \mathbf{u}(p) & \dots & \mathbf{u}(l-1) \\ \mathbf{0} & \mathbf{u}(0) & \dots & \mathbf{u}(p-1) & \dots & \mathbf{u}(l-2) \\ \vdots & \vdots & \ddots & \vdots & \dots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{u}(0) & \dots & \mathbf{u}(l-p-1) \end{bmatrix} \tag{42}$$

Equation (40) indicate that there are more equations $m\,l$ than unknowns $m\,r\,(p+1)$, provided that the data length l is chosen greater than $r\,(p+1)$. One can conclude that the first p Markov parameters approximately satisfies the following equation:

$$\hat{\mathbf{Y}} = \mathbf{y}\,\hat{\mathbf{U}}^\dagger \tag{43}$$

The approximation error decreases as p increases.

Unfortunately, for lightly damped systems and structures, the integer p and thus the l required to make the approximation in equation (40) valid becomes impractically large in the sense that the size of the matrix $\hat{\mathbf{U}}$ is too large to solve for its pseudo-inverse $\hat{\mathbf{U}}^\dagger$ numerically. There is a method to artificially increase the damping of the system in order to allow the solution of equation (40) for the Markov parameters. The method is to introduce an observer matrix $\mathbf{G}$ to make the state-matrix $\bar{\mathbf{A}}$ as stable as desired. In this way one get the equations (20), (21) and, assuming zero initial conditions, the set of these equations for a sequence different time $k = 0, 1, \dots, l-1$ can be grouped in a matrix form to yield:

$$\mathbf{y} = \bar{\mathbf{Y}}\,\mathbf{V} \tag{44}$$

where:

$$\bar{\mathbf{Y}} = \left[\begin{array}{cccccc} \bar{\mathbf{Y}}_0 & \bar{\mathbf{Y}}_1 & \bar{\mathbf{Y}}_2 & \dots & \bar{\mathbf{Y}}_{l-1} \end{array}\right] \tag{45}$$

$$\mathbf{V} = \begin{bmatrix} \mathbf{u}(0) & \mathbf{u}(1) & \dots & \mathbf{u}(l-1) \\ \mathbf{0} & \mathbf{v}(0) & \dots & \mathbf{v}(l-2) \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{v}(0) \end{bmatrix} \tag{46}$$

Since the matrix $\mathbf{G}$ can be arbitrarily chosen, the eigenvalues of the state-matrix $\bar{\mathbf{A}}$ may be arbitrarily assigned. Therefore for sufficiently large p, we have $\bar{\mathbf{A}}^p \simeq \mathbf{O}$. Equation (44) can be approximated by:

$$\mathbf{y} = \hat{\bar{\mathbf{Y}}}\,\hat{\mathbf{V}} \tag{47}$$

where:

$$\hat{\bar{\mathbf{Y}}} = \left[\begin{array}{cccccc} \bar{\mathbf{Y}}_0 & \bar{\mathbf{Y}}_1 & \bar{\mathbf{Y}}_2 & \dots & \bar{\mathbf{Y}}_p \end{array}\right] \tag{48}$$

$$\hat{\mathbf{V}} = \begin{bmatrix} \mathbf{u}(0) & \mathbf{u}(1) & \dots & \mathbf{u}(p) & \dots & \mathbf{u}(l-1) \\ \mathbf{0} & \mathbf{v}(0) & \dots & \mathbf{v}(p-1) & \dots & \mathbf{v}(l-2) \\ \vdots & \vdots & \ddots & \vdots & \dots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{v}(0) & \dots & \mathbf{v}(l-p-1) \end{bmatrix} \tag{49}$$

From the equation (47) one can conclude that the first p Markov parameters approximately satisfies the following equation:

$$\hat{\bar{\mathbf{Y}}} = \mathbf{y}\,\hat{\mathbf{V}}^\dagger \tag{50}$$

Even in this case, the approximation error decreases as p increases.

Note that the observer Markov parameters thus identified may not necessarily appear to be asymptotically decaynig during the first $p-1$ steps. To solve for $\hat{\bar{\mathbf{Y}}}$ uniquely, all rows of $\hat{\mathbf{V}}$ must be linearly independent. Furthermore, to minimize any numerical error due to the computation of the pseudo-inverse, the rows of $\hat{\mathbf{V}}$ should be chosen as independently as possible. As a result, the maximum number of p is the number that maximizes the number $(r+m)\,p+r$ of the independent rows of $\hat{\mathbf{V}}$. The maximum p means the upper bound of the order of the deadbeat observer. It is possible to proof that the lower bound for the order of the observer is the minimum p such that $m\,p \geq n$, where m is the number of the outputs and n is the order of the system.

The Markov Parameters include the system Markov parameters $\mathbf{Y}_k$ and the observer gain Markov parameters $\mathbf{Y}_k^0$. The system Markov parameters are used to compute the system matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{D}$ whereas the observer gain Markov parameters are used to determine the observer gain matrix $\mathbf{G}$.

It can be showed that to recover the system Markov parameters $\mathbf{Y}_k$ from the observer Markov parameters $\bar{\mathbf{Y}}_k$ the following relations hold:

$$\begin{cases} \mathbf{D} = \mathbf{Y}_0 = \bar{\mathbf{Y}}_0 \\ \mathbf{Y}_k = \bar{\mathbf{Y}}_k^{(1)} - \sum\limits_{i=1}^{k} \bar{\mathbf{Y}}_i^{(2)}\,\mathbf{Y}_{k-i} \quad , \quad k = 1, 2, \dots, p \\ \mathbf{Y}_k = -\sum\limits_{i=1}^{p} \bar{\mathbf{Y}}_i^{(2)}\,\mathbf{Y}_{k-i} \quad , \quad k = p, p+1, \dots \end{cases} \tag{51}$$

These equations show that for $k > p$ the system Markov parameters are a linear combination of their p past system Markov parameters or, in other words, it is possible to identify only p independent system Markov parameters starting from the observer Markov parameters.

F. Observer Gain Markov Parameters

The observer gain Markov parameters $\mathbf{Y}_k^0$ are defined as:

$$\mathbf{Y}_1^0 = \mathbf{C}\,\mathbf{G}\,,\ \mathbf{Y}_2^0 = \mathbf{C}\,\mathbf{A}\,\mathbf{G}\,,\ \ldots\,,\ \mathbf{Y}_k^0 = \mathbf{C}\,\mathbf{A}^{k-1}\,\mathbf{G} \tag{52}$$

It can be showed that to recover the observer gain Markov parameters $\mathbf{Y}_k^0$ from the observer Markov parameters $\bar{\mathbf{Y}}_k$ the following relations hold:

$$\left\{ \begin{array}{c} \mathbf{Y}_1^0 = \mathbf{C}\,\mathbf{G} = \bar{\mathbf{Y}}_1^{(2)} \\ \mathbf{Y}_k^0 = \bar{\mathbf{Y}}_k^{(2)} - \sum_{i=1}^{k-1} \bar{\mathbf{Y}}_i^{(2)}\,\mathbf{Y}_{k-i}^0 \quad , \quad k = 2\,,\ldots,p \\ \mathbf{Y}_k^0 = -\sum_{i=1}^{p} \bar{\mathbf{Y}}_i^{(2)}\,\mathbf{Y}_{k-i}^0 \quad , \quad k = p+1\,,\ldots \end{array} \right. \tag{53}$$

Conventional time-domain system identification methods use only the system Markov parameters $\mathbf{Y}_k$ to determine $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{D}$. Consider the combined system and observer gain Markov parameters:

$$\mathbf{\Gamma}_k = \left[\begin{array}{cc} \mathbf{Y}_k & \mathbf{Y}_k^0 \end{array} \right] \tag{54}$$

In this paper the the combined system and observer gain Markov parameters $\mathbf{\Gamma}_k$ are used to identify the matrices $\mathbf{A}$, $\left[\begin{array}{cc} \mathbf{B} & \mathbf{G} \end{array} \right]$, $\mathbf{C}$ and $\mathbf{D}$. There are several advantages for this approach. First, the observer gain $\mathbf{G}$ is obtained directly and it is possible to proof that it is related to the steady-state Kalman filter gain K in this way:

$$\mathrm{K} = -\mathbf{G} \tag{55}$$

Second, the number of independent Markov parameters has been compressed by using the observer. This allows to drastically reduce the computational effort in the identification algorithm. Third, one can identify the number of independent system Markov parameters from a single set of data for lightly damped systems with multiple inputs and multiple outputs.

G. ERA/OKID Identification Method

ERA stands for Eigensystem Realization Algorithm and it is a time-domain state-space realization method originally developed by Ho and Kalman. OKID means Observer/Kalman Filter Identification and it is a numerical procedure based on ERA developed by Juang and Pahn.

The ERA/OKID identification method is a time-domain identification method which compute a minimum realization of system and the observer gain matrix starting from the combined system and observer gain Markov parameters $\mathbf{\Gamma}_k$. A realization is a triplet of matrices $\{\mathbf{A}\,,\mathbf{B}\,,\mathbf{C}\}$ that satisfies the discrete-time state-space equations (11) and (12). Obviously, the same system has an infinite set of realizations which will predict the identical response for any particular input. Minimum realization means a model the smallest state space dimensions among all the realizable systems that have the same input-output relations. All minimum realizations have the same set of eigenvalues and eigenvectors, which are the modal parameters of the system itself. Assume that the state matrix $\mathbf{A}$ has a complete set of linearly independent eigenvectors $\{\phi_1\,,\phi_2\,,\ldots\,,\phi_n\}$ with corresponding eigenvalues $\{\lambda_1\,,\lambda_2\,,\ldots\,,\lambda_n\}$:

$$\mathbf{A}\,\mathbf{\Phi} = \mathbf{\Phi}\,\mathbf{\Lambda} \tag{56}$$

where $\mathbf{\Lambda}$ is the diagonal matrix of the eigenvalues and $\mathbf{\Phi}$ is the matrix of the eigenvectors. The realization $\{\mathbf{A}\,,\mathbf{B}\,,\mathbf{C}\}$ can be transformed in the realization $\left\{\mathbf{\Lambda}\,,\mathbf{\Phi}^{-1}\,\mathbf{B}\,,\mathbf{C}\,\mathbf{\Phi}\right\}$ by using the eigenvalues and eigenvectors matrices. The diagonal matrix $\mathbf{\Lambda}$ contains the informations of modal damping rates and damped natural frequencies. The matrix $\mathbf{\Phi}^{-1}\,\mathbf{B}$ defines the initial modal amplitudes and the matrix $\mathbf{C}\,\mathbf{\Phi}$ the mode shapes at the sensor points. All the modal parameters of a dynamic system can thus be identified by the triplet $\left\{\mathbf{\Lambda}\,,\mathbf{\Phi}^{-1}\,\mathbf{B}\,,\mathbf{C}\,\mathbf{\Phi}\right\}$.

Once having identified the combined system and observer gain Markov parameters, the next step consist in forming the generalized Hankel matrix $\bar{\mathbf{H}}(k-1)$:

$$\bar{\mathbf{H}}(k-1) = \left[\begin{array}{cccc} \mathbf{\Gamma}_k & \mathbf{\Gamma}_{k+1} & \ldots & \mathbf{\Gamma}_{k+\beta-1} \\ \mathbf{\Gamma}_{k+1} & \mathbf{\Gamma}_{k+2} & \ldots & \mathbf{\Gamma}_{k+\beta} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{\Gamma}_{k+\alpha-1} & \mathbf{\Gamma}_{k+\alpha} & \ldots & \mathbf{\Gamma}_{k+\alpha+\beta-2} \end{array} \right] \tag{57}$$

for the case $k = 1$ one get:

$$\bar{\mathbf{H}}(0) = \left[\begin{array}{cccc} \mathbf{\Gamma}_1 & \mathbf{\Gamma}_2 & \ldots & \mathbf{\Gamma}_\beta \\ \mathbf{\Gamma}_2 & \mathbf{\Gamma}_3 & \ldots & \mathbf{\Gamma}_{\beta+1} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{\Gamma}_\alpha & \mathbf{\Gamma}_{\alpha+1} & \ldots & \mathbf{\Gamma}_{\alpha+\beta-1} \end{array} \right] \tag{58}$$

Decomposing the matrix $\bar{\mathbf{H}}(0)$ using singular value decomposition leads to:

$$\bar{\mathbf{H}}(0) = \bar{\mathbf{R}}_n\,\bar{\mathbf{\Sigma}}_n\,\bar{\mathbf{S}}_n^T \tag{59}$$

Now examining the singular value $\bar{\mathbf{\Sigma}}_n$ of the Hankel matrix $\bar{\mathbf{H}}(0)$ it is possible to determine the order of the system. In order to compute a minimum order realization of the system $\left\{\hat{\mathbf{A}}\,,\hat{\mathbf{B}}\,,\hat{\mathbf{C}}\right\}$, it is necessary to construct a shifted Hankel matrix $\bar{\mathbf{H}}(1)$:

$$\bar{\mathbf{H}}(1) = \left[\begin{array}{cccc} \mathbf{\Gamma}_2 & \mathbf{\Gamma}_3 & \ldots & \mathbf{\Gamma}_{\beta+1} \\ \mathbf{\Gamma}_3 & \mathbf{\Gamma}_4 & \ldots & \mathbf{\Gamma}_{\beta+2} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{\Gamma}_{\alpha+1} & \mathbf{\Gamma}_{\alpha+2} & \ldots & \mathbf{\Gamma}_{\alpha+\beta} \end{array} \right] \tag{60}$$

Finally, the basic formulation of the minim order realization for the ERA/OKID is:

$$\left\{ \begin{array}{c} \hat{\mathbf{C}} = \mathbf{E}_m^T\,\bar{\mathbf{R}}_n\,\bar{\mathbf{\Sigma}}_n^{-1/2} \\ \hat{\mathbf{A}} = \bar{\mathbf{\Sigma}}_n^{-1/2}\,\bar{\mathbf{R}}_n^T\,\bar{\mathbf{H}}(1)\,\bar{\mathbf{S}}_n\,\bar{\mathbf{\Sigma}}_n^{-1/2} \\ \left[\begin{array}{cc} \hat{\mathbf{B}} & \hat{\mathbf{G}} \end{array} \right] = \bar{\mathbf{\Sigma}}_n^{-1/2}\,\bar{\mathbf{S}}_n^T\,\mathbf{E}_{r+m} \end{array} \right. \tag{61}$$

where the matrix $\mathbf{E}_{r+m}$ is defined as follows:

$$\mathbf{E}_{r+m} = \left[\begin{array}{c} \mathbf{I}_{r+m} \\ \mathbf{O}_{r+m} \\ \vdots \\ \mathbf{O}_{r+m} \end{array} \right] \tag{62}$$

Now one can easily find the eigensolution of the realized state matrix $\hat{\mathbf{A}}$:

$$\hat{\mathbf{A}}\,\hat{\mathbf{\Psi}} = \hat{\mathbf{\Psi}}\,\hat{\mathbf{\Lambda}} \tag{63}$$

An inspection of the system eigensolution shows that while the identified eigenvalues $\hat{\mathbf{\Lambda}}$ are equal to the true ones $\mathbf{\Lambda}$, the identified eigenvectors $\hat{\mathbf{\Psi}}$ are different from the true one $\mathbf{\Phi}$ because they are not expressed terms of a displacement-velocity basis. It is easy to transform the identified eigenvectors $\hat{\mathbf{\Psi}}$ into another equivalent eigenvectors matrix expressed in terms of a displacement-velocity basis using the following formula:

$$\hat{\mathbf{\Phi}} = \begin{bmatrix} \hat{\mathbf{C}}\,\hat{\mathbf{\Psi}}\,\hat{\mathbf{\Lambda}}^{-c} \\ \hat{\mathbf{C}}\,\hat{\mathbf{\Psi}}\,\hat{\mathbf{\Lambda}}^{-c+1} \end{bmatrix} \tag{64}$$

where $c = 0$ for displacement sensing, $c = 1$ for velocity sensing and $c = 2$ for acceleration sensing.

III. Case-Study

Consider the following mechanical system:
Assume the following data:

$$m_1 = 2.00\ [kg]\,, m_2 = 3.00\ [kg] \tag{65}$$

$$\begin{aligned} &r_1 = 2.00\ [N\,s/m]\,, r_2 = 4.00\ [N\,s/m]\,, \\ &r_3 = 3.00\ [N\,s/m] \end{aligned} \tag{66}$$

$$\begin{aligned} &k_1 = 20.00\ [N/m]\,, k_2 = 40.00\ [N/m]\,, \\ &k_3 = 30.00\ [N/m] \end{aligned} \tag{67}$$

This system has $n_2 = 2$ degrees of freedom. Suppose that one is able to measure the $m = 2$ accelerations of the two masses and the $r = 1$ force acting on the first mass. Consider the following generalized displacement vector:

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \tag{68}$$

The system equations of motion are:

$$\mathbf{M}\,\ddot{\mathbf{x}}(t) + \mathbf{R}\,\dot{\mathbf{x}}(t) + \mathbf{K}\,\mathbf{x}(t) = \mathbf{B_2}\,u(t) \tag{69}$$

where $\mathbf{M}$, $\mathbf{R}$ and $\mathbf{K}$ are the mass, damping and stiffness matrices, respectively. For this system, these matrices are:

$$\mathbf{M} = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} = \begin{bmatrix} 2.00 & 0 \\ 0 & 3.00 \end{bmatrix} \tag{70}$$

$$\mathbf{R} = \begin{bmatrix} r_1 + r_2 & -r_2 \\ -r_2 & r_2 + r_3 \end{bmatrix} = \begin{bmatrix} 6.00 & -4.00 \\ -4.00 & 7.00 \end{bmatrix} \tag{71}$$

$$\mathbf{K} = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix} = \begin{bmatrix} 60.00 & -40.00 \\ -40.00 & 70.00 \end{bmatrix} \tag{72}$$

and the matrix $\mathbf{B}_2$ is defined as follows:

$$\mathbf{B}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{73}$$

The system output equations are:

$$\mathbf{y}(t) = \mathbf{C_a}\,\ddot{\mathbf{x}}(t) \tag{74}$$

where the output influence matrix for acceleration $\mathbf{C_a}$ is an identity matrix:

$$\mathbf{C_a} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{75}$$

If $\mathbf{z}(t)$ is the state vector of the system:

$$\mathbf{z}(t) = \begin{bmatrix} \mathbf{x}(t) \\ \dot{\mathbf{x}}(t) \end{bmatrix} = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} \tag{76}$$

The continuous-time state-space model of the system is:

$$\dot{\mathbf{z}}(t) = \mathbf{A_c}\,\mathbf{z}(t) + \mathbf{B_c}\,u(t) \tag{77}$$

$$\mathbf{y}(t) = \mathbf{C}\,\mathbf{z}(t) + \mathbf{D}\,u(t) \tag{78}$$

where the matrices $\mathbf{A_c}$, $\mathbf{B_c}$, $\mathbf{C}$ and $\mathbf{C}$ can be obtained using equations (6), (7), (8) and (9):

$$\mathbf{A}_c = \begin{bmatrix} 0 & 0 & 1.00 & 0 \\ 0 & 0 & 0 & 1.00 \\ -30.00 & 20.00 & -3.00 & 2.00 \\ 13.33 & -23.33 & 1.33 & -2.33 \end{bmatrix} \tag{79}$$

$$\mathbf{B}_c = \begin{bmatrix} 0 \\ 0 \\ 0.50 \\ 0 \end{bmatrix} \tag{80}$$

$$\mathbf{C} = \begin{bmatrix} -30.00 & 20.00 & -3.00 & 2.00 \\ 13.33 & -23.33 & 1.33 & -2.33 \end{bmatrix} \tag{81}$$

$$\mathbf{D} = \begin{bmatrix} 0.50 \\ 0 \end{bmatrix} \tag{82}$$

The discrete-time state-space model of the system is:

$$\mathbf{z}(k+1) = \mathbf{A}\,\mathbf{z}(k) + \mathbf{B}\,u(k) \tag{83}$$

$$\mathbf{y}(k) = \mathbf{C}\,\mathbf{z}(k) + \mathbf{D}\,u(k) \tag{84}$$

where the matrices $\mathbf{A}$ and $\mathbf{B}$ can be obtained using the equations (13) and (14) with $\Delta t = 10^{-2}$:

$$\mathbf{A} = \begin{bmatrix} 1.00 & 0.00 & 0.01 & 0.00 \\ 0.00 & 1.00 & 0.00 & 0.01 \\ -0.29 & 0.19 & 0.97 & 0.02 \\ 0.13 & -0.23 & 0.01 & 0.98 \end{bmatrix} \tag{85}$$

$$\mathbf{B} = 10^{-2} \begin{bmatrix} 0.00 \\ 0.00 \\ 0.49 \\ 0.00 \end{bmatrix} \tag{86}$$

The eigenvalues and eigenvectors of the discrete-time state matrix $\mathbf{A}$ are:

$$\lambda = \begin{bmatrix} 0.98 + \mathbf{i}\,0.061 \\ 0.98 - \mathbf{i}\,0.061 \\ 0.99 + \mathbf{i}\,0.031 \\ 0.99 - \mathbf{i}\,0.031 \end{bmatrix} \tag{87}$$

$$\mathbf{\Lambda} = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{bmatrix} \tag{88}$$

$$\begin{aligned} \phi_1 &= \begin{bmatrix} 0.041 + \mathbf{i}\,0.12 \\ -0.027 - \mathbf{i}\,0.079 \\ -0.82 \\ 0.55 \end{bmatrix}, \phi_2 = \begin{bmatrix} 0.041 - \mathbf{i}\,0.12 \\ -0.027 + \mathbf{i}\,0.079 \\ -0.82 \\ 0.55 \end{bmatrix}, \\ \phi_3 &= \begin{bmatrix} -0.037 - \mathbf{i}\,0.21 \\ -0.037 - \mathbf{i}\,0.21 \\ 0.67 \\ 0.67 \end{bmatrix}, \phi_4 = \begin{bmatrix} -0.037 + \mathbf{i}\,0.21 \\ -0.037 + \mathbf{i}\,0.21 \\ 0.67 \\ 0.67 \end{bmatrix} \end{aligned} \tag{89}$$

$$\mathbf{\Phi} = \left[\begin{array}{cccc} \phi_1 & \phi_2 & \phi_3 & \phi_4 \end{array}\right] \tag{90}$$

Consider a time span $T = 10\,[s]$. Using the definition (18) it is possible to compute exactly the system Markov parameters. This sequence is showed in figure (1). Assume that the input force is a white noise as showed in figure (2). By a numerical simulation it is possible to compute the displacements, velocities and accelerations of the two masses. Assume that the measured outputs are the accelerations as showed in figure (3). The application of equations (50) and (51) to the two output measurements and to the input measurement allow us to identify the sequence of Markov parameters as showed in figure (4). Now it is possible to compute the observer gain Markov parameters using equation (53) and to construct the generalized Hankel matrices $\bar{\mathbf{H}}(0)$ and $\bar{\mathbf{H}}(1)$ by using the equation (58) and (60). The singular values of the Hankel matrix $\bar{\mathbf{H}}(0)$ are showed in figure (4).

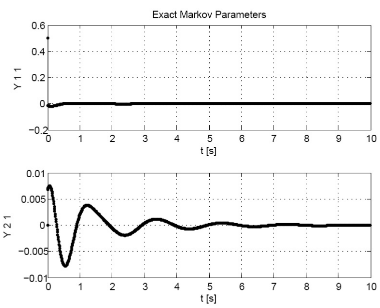

Fig. 1. Exact Markov Parameters

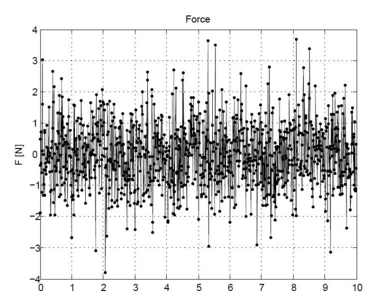

Fig. 2. Force

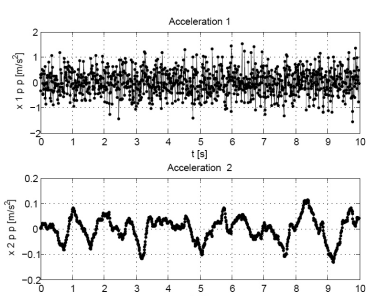

Fig. 3. Acceleration of Masses 1 and 2

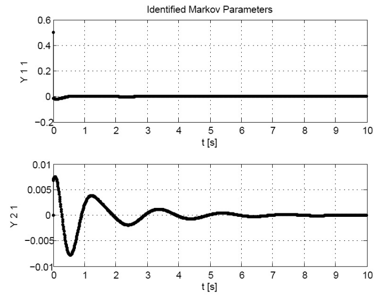

Fig. 4. Identified Markov Parameters

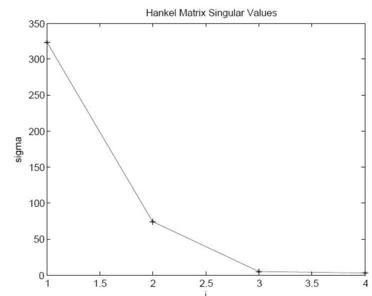

Fig. 5. Singular Values

Now it is possible to compute a minimum realization of the system $\left\{\hat{\mathbf{A}}\,,\hat{\mathbf{B}}\,,\hat{\mathbf{C}}\right\}$ by using the ERA/OKID equations (61):

$$\hat{\mathbf{A}} = \left[\begin{array}{cccc} 1.00 & 0.01 & 0.24 & 0.01 \\ 0.00 & 1.00 & -0.02 & -0.34 \\ 0.00 & 0.00 & 0.99 & 0.01 \\ 0.00 & 0.01 & 0.00 & 0.96 \end{array}\right] \tag{91}$$

$$\hat{\mathbf{B}} = 10^{-2}\left[\begin{array}{c} 0.05 \\ -0.26 \\ 0.02 \\ 0.04 \end{array}\right] \tag{92}$$

$$\hat{\mathbf{C}} = \left[\begin{array}{cccc} -8.97 & 4.34 & 1.14 & 0.73 \\ -9.02 & -4.27 & 1.09 & -0.77 \end{array}\right] \tag{93}$$

Consider the identified discrete-time state matrix $\hat{\mathbf{A}}$. The identified eigenvalues and eigenvector are:

$$\hat{\lambda} = \left[\begin{array}{c} 0.98 + \mathbf{i}\,0.061 \\ 0.98 - \mathbf{i}\,0.061 \\ 0.99 + \mathbf{i}\,0.031 \\ 0.99 - \mathbf{i}\,0.031 \end{array}\right] \tag{94}$$

$$\hat{\mathbf{\Lambda}} = \left[\begin{array}{cccc} \hat{\lambda}_1 & 0 & 0 & 0 \\ 0 & \hat{\lambda}_2 & 0 & 0 \\ 0 & 0 & \hat{\lambda}_3 & 0 \\ 0 & 0 & 0 & \hat{\lambda}_4 \end{array}\right] \tag{95}$$

$$\hat{\psi}_1 = \begin{bmatrix} -0.99 \\ 0.06 \\ 0.02 - \mathbf{i}\,0.12 \\ \mathbf{i}\,0.01 \end{bmatrix}, \hat{\psi}_2 = \begin{bmatrix} -0.99 \\ 0.06 \\ 0.02 + \mathbf{i}\,0.12 \\ -\mathbf{i}\,0.01 \end{bmatrix},$$
$$\hat{\psi}_3 = \begin{bmatrix} -0.09 \\ 0.98 \\ -0.02 - \mathbf{i}\,0.03 \\ 0.67 - \mathbf{i}\,0.17 \end{bmatrix}, \hat{\psi}_4 = \begin{bmatrix} -0.09 \\ 0.98 \\ -0.02 + \mathbf{i}\,0.03 \\ 0.67 + \mathbf{i}\,0.17 \end{bmatrix} \tag{96}$$

$$\hat{\mathbf{\Psi}} = \begin{bmatrix} \hat{\psi}_1 & \hat{\psi}_2 & \hat{\psi}_3 & \hat{\psi}_4 \end{bmatrix} \tag{97}$$

Finally one can recover the eigenvectors of the system in terms of a displacement-velocity basis by using equations (64) with $c = 2$:

$$\hat{\phi}_1 = \begin{bmatrix} 0.041 + \mathbf{i}\,0.12 \\ -0.027 - \mathbf{i}\,0.079 \\ -0.82 \\ 0.55 \end{bmatrix}, \hat{\phi}_2 = \begin{bmatrix} 0.041 - \mathbf{i}\,0.12 \\ -0.027 + \mathbf{i}\,0.079 \\ -0.82 \\ 0.55 \end{bmatrix},$$
$$\hat{\phi}_3 = \begin{bmatrix} -0.037 - \mathbf{i}\,0.21 \\ -0.037 - \mathbf{i}\,0.21 \\ 0.67 \\ 0.67 \end{bmatrix}, \hat{\phi}_4 = \begin{bmatrix} -0.037 + \mathbf{i}\,0.21 \\ -0.037 + \mathbf{i}\,0.21 \\ 0.67 \\ 0.67 \end{bmatrix} \tag{98}$$

$$\hat{\mathbf{\Phi}} = \begin{bmatrix} \hat{\phi}_1 & \hat{\phi}_2 & \hat{\phi}_3 & \hat{\phi}_4 \end{bmatrix} \tag{99}$$

Note that even if the identified matrix $\hat{\mathbf{A}}$, $\hat{\mathbf{B}}$ and $\hat{\mathbf{C}}$ are different from the real ones $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$, they have the same eigenvalues, eigenvectors and sequence of Markov parameters as showed in figure (5).

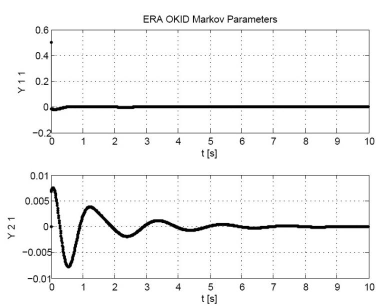

Fig. 6. Singular Values

IV. Conclusion

A method for identifying parameters of mechanical linear systems has been proposed. This procedure has been tested on a light-damped mechanical linear apparatus. Numerical results show a good agreement with real system parameters. Authors think that this method can be used to describe mechanical systems in order to obtain a model for performing parameters identification of nonlinear force fields acting on the system itself.

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