An Investigation on Free/Forced Vibration of a Piezoelectric Circular Cylindrical Panel Located on an Elastic Foundation

A. Oveisi, M. Gudarzi and S. M. Hasheminejad

***Abstract*—**This paper presents an exact three-dimensional free and forced vibration analysis of an axially polarized transversely isotropic piezoelectric circular cylindrical panel on elastic foundation. Due to the wide use of piezoelectric materials as sensor/actuators, to the best knowledge of the authors, no one has studied the effect of imperfection in bonding of these piezo-layers to the host layer for cylindrical panels. Using separation of variables, three-dimensional exact solution is presented under generalized simply supported boundary conditions. In addition, the effect of elastic foundation on both structure natural frequency and steady state frequency response is investigated. For validation purposes the results are compared with those obtained from FEM and the results from previous works. Finally conclusions are made.

***Keywords*—**Piezoelectric, cylindrical panel, free vibration, frequency response, foundation.

I. Introduction

Piezoelectric materials have been widely used as transducers, sensors and actuators due to their fundamental direct and converse piezoelectric effects that take place between electric field and mechanical deformation. They are playing a key task as active components in many twigs of science and technology such as electronics, navigation, vibration control, etc.

Haskins and Walsh [1] investigated the free vibration of transversely isotropic piezoelectric cylindrical shell. The wall thickness of the cylinder was assumed to be negligible so two elastic constants are considered in their analysis. By employing the same assumption, Martin [2] studied the vibration of longitudinally polarized piezoelectric circular cylindrical shell. Drumheller and Kalnins [3] used a coupled theory for investigation of vibration of piezoceramic shells of revolution and analyzed the free vibration of a cylindrical shell. Burt [4] simplified the circular cylinder dynamic model to a two-dimensional one and studied the voltage response of radially polarized ceramic. Tzou and Zhong [5] presented a linear theory of piezoelectric shell vibration, which can be simplified to account for spheres, cylinders and plates. Ebenezer and Abraham [6] presented an Eigen function method to find the dynamic response of radially polarized piezoelectric circular cylindrical shells of finite length. Several works based on the methods of three-dimensional theory concentrated on the vibrations of cylinders. Paul [7] derived the frequency equation of a piezoelectric cylindrical shell without any numerical results. Ding *et al*. [8] exactly studied the free vibration of hollow and fluid-filled piezoelectric cylindrical shells on the basis of a decomposition formula for displacements. A more detailed description on related studies can be found in Saravanos and Heyliger [9]. Ding *et al*. [10] solved three dimension free vibration problem for transversely isotropic circular cylindrical panel. They used displacement function method to achieve the natural frequencies of the panel with different boundary conditions. Sharma and Pathania [11] investigated an exact analysis of the free vibration of a simply supported piezoelectric cylindrical panel using three displacement potential functions. They showed that a purely transverse mode is independent of piezoelectric effects and the rest of the motion. Yang *et al*. [12] studied the vibration characteristics of a circular cylindrical piezoelectric transducer using linear piezoelectricity theory. They solved the problem for both free and force vibration. Kapuria *et al*. [13] solved the free vibration and steady state response of cylindrical piezo electric panel by use of an exact two-dimensional piezo-elasticity solution. The piezoelectric layers assumed to be polarized along radial direction. Wang *et al*. [14] solved the exact vibration problem for magneto-electro-elastic circular cylinder with two simply supported ends. They used displacement function method. Kapuria and Kumari [15] considered a benchmark three-dimensional exact piezoelectricity solution in surface-bonded, embedded monolithic piezoelectric and piezoelectric fiber reinforced composite layers. The dynamic equations with variable coefficients are solved using the modified Frobenius method. Wang *et al*. [16] applied a dynamic model based on Love & Kirchhoff thin shell theory. They investigated the conversion of mechanical energy into electrical energy in a cylindrical piezoelectric panel with simply supported boundary conditions. Also, the effect of the curvature is studied. Bodaghi and Shakeri [17] investigated the free and forced vibration of simply supported circular cylindrical FGPM panel. The dynamic transient response of

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the system evaluated under blast pulses using Laplace inverse method. Recently, vibration controls of cylindrical panels by piezo-based actuator/sensors are studied widely. Lin *et al*. [18] considered the active acoustic pressure control in a cylindrical cavity with a flexible cylindrical panel using a pair of piezoelectric actuator/sensor. The dynamic governing equation solved by model expansion method and a LQG controller is designed and implemented. Karnaukhova *et al*. [19] studied the vibrations of elastic three-layer shells composed of metal layer and two piezoelectric layers. They presented analytical solution for different cases of electrodes placement. Kozlov *et al*. [20] used piezoelectric panels as sensor/actuator in order to inspect the active forced damping of viscoelastic shells. The vibration model is achieved by the Kirchhoff-Love hypotheses. Also, they consider the effect of the temperature which leads to a nonlinear model solved by FE method. Karnaukhov and Tkachenko [21] solved the active vibration damping in a circular cylindrical panel with clamped boundary conditions using piezo-actuator/sensors. Karnaukhov *et al*. [22] considered the problem of active vibration control of a viscoelastic cylindrical panel by using piezoelectric actuators. The dynamic response of the panel is achieved by using FE method. However, the mounted actuator/sensor in real applications is coupled to the host structure using special adhesive materials. This kind of attachment cannot be modeled as a perfect one.

The main subject of this study is to investigate the free vibration and harmonic force response of transversely isotropic piezoelectric cylindrical panel on elastic foundation. Based on the general solution of coupled equations for piezoelectric media presented by Ding *et al*. [23], three-dimensional exact solutions are obtained by using of the variable separation method. The obtained solutions not only satisfy the basic equations, but also satisfy any boundary conditions. A numerical example is finally presented with results compared to FEM results and good agreements are obtained.

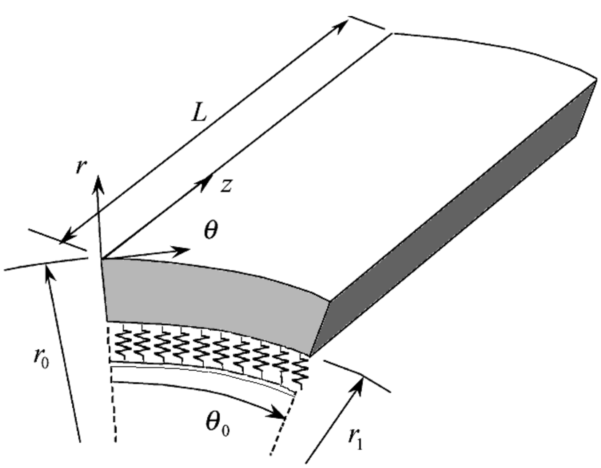

Fig. 1 A cylindrical panel on elastic foundation

II. Problem Formulation

A transversely isotropic piezoelectric cylindrical panel (length L, central angle α, inner radius r_1 and outer radius r_2, is considered (see Fig. 1). The (r,θ,z) cylindrical coordinate system is set at the bottom of central axes of the cylindrical panel, as shown in Fig.1. Because of considering 3D stress, all components of stress tensor are considered. Also, there are additionally three nonzero electric displacement components D_r, D_θ and D_z in the piezoelectric layer. In the following two subsections, the basic governing equations of the transversely isotropic piezoelectric circular cylindrical panel will be derived and suitable boundary condition will be set.

A. Dynamic Modeling of Piezoelectric Cylindrical Panel

For dynamic modeling of the piezoelectric panel, two displacement functions Ψ and F are introduced. Ding *et al*. presented a general solution for the coupled linear dynamic equations of a transversely isotropic piezoelectric media [23]. In circular cylindrical coordinates (r,θ,z), if the media is axially polarized the general solution can be written as

$$u_r=\frac{1}{r}\frac{\partial\Psi}{\partial\theta}-\frac{\partial}{\partial r}\mathcal{A}_1F, u_\theta=-\frac{\partial\Psi}{\partial r}-\frac{1}{r}\frac{\partial}{\partial\theta}\mathcal{A}_1F, \quad W=\mathcal{A}_2F, \phi=\mathcal{A}_3F, \tag{1}$$

where u_r, u_θ and W are three displacement components, ϕ is the electric potential, and the differential operators $\mathcal{A}_1$, $\mathcal{A}_2$ and $\mathcal{A}_3$ are

$$\mathcal{A}_1=[(c_{13}+c_{44})\varepsilon_{33}+(e_{15}+e_{31})e_{33}]\frac{\partial^3}{\partial z^3}+[(c_{13}+c_{44})\varepsilon_{11}+(e_{15}+e_{31})e_{15}]\Lambda\frac{\partial}{\partial z}, \tag{2}$$

$$\mathcal{A}_2=c_{44}\varepsilon_{33}\frac{\partial^4}{\partial z^4}+\left\{\left[c_{11}\varepsilon_{33}+c_{44}\varepsilon_{11}+(e_{15}+e_{31})^2\Lambda-\rho\varepsilon_{33}\frac{\partial^2}{\partial t^2}\right]\right\}\frac{\partial^2}{\partial z^2}+c_{11}\varepsilon_{11}\Lambda^2-\rho\varepsilon_{11}\Lambda\frac{\partial^2}{\partial t^2}, \tag{3}$$

$$\mathcal{A}_3=c_{44}e_{33}\frac{\partial^4}{\partial z^4}+\left\{[c_{11}e_{33}+c_{44}e_{15}-(c_{13}+c_{44})(e_{15}+e_{31})]\Lambda-\rho e_{33}\frac{\partial^2}{\partial t^2}\right\}\frac{\partial^2}{\partial z^2}+c_{11}e_{15}\Lambda^2-\rho e_{15}\Lambda\frac{\partial^2}{\partial t^2}, \tag{4}$$

where $\Lambda=\partial^2/\partial r^2+(1/r)\partial/\partial r+(1/r^2)\partial^2/\partial\theta^2$ is the two-dimensional Laplacian operator. The displacement functions Ψ and F must satisfy the following two equations

$$\left(c_{66}\Lambda+c_{44}\frac{\partial^2}{\partial z^2}-\rho\frac{\partial^2}{\partial t^2}\right)\Psi=0, L_0F=0, \tag{5}$$

where

$$
\begin{aligned}
L_0 = a_4 \Lambda\Lambda\Lambda + \left(a_3 \frac{\partial^2}{\partial z^2} + a_6 \frac{\partial^2}{\partial t^2}\right) \Lambda\Lambda \\
+ \left(a_2 \frac{\partial^4}{\partial z^4} + a_5 \frac{\partial^4}{\partial t^4} + a_7 \frac{\partial^4}{\partial z^2 \partial t^2}\right) \Lambda \\
+ a_1 \frac{\partial^6}{\partial z^6} + a_8 \frac{\partial^6}{\partial z^4 \partial t^2} + a_9 \frac{\partial^6}{\partial z^2 \partial t^4},
\end{aligned} \tag{6}
$$

Here a_n $(n = 1,2,\dots,9)$ can be expressed in terms of elastic constants c_{ij}, dielectric constants ε_{ij} and piezoelectric coefficients e_{ij} as follows

$$
\begin{aligned}
a_1 &= c_{44}(e_{33}^2 + c_{33}\varepsilon_{33}), \\
a_2 &= c_{33}[c_{44}\varepsilon_{11} + (e_{15} + e_{31})^2] \\
&\quad + \varepsilon_{33}[c_{11}c_{33} + c_{44}^2 - (c_{11} + c_{44})^2] \\
&\quad + e_{33}[2c_{44}e_{15} + c_{11}e_{33} \\
&\quad - 2(c_{13} + c_{44})(e_{15} + e_{31})], \\
a_3 &= c_{44}[c_{11}\varepsilon_{33} + (e_{15} + e_{31})^2] \\
&\quad + \varepsilon_{11}[c_{11}c_{33} + c_{44}^2 - (c_{13} + c_{44})^2] \\
&\quad + e_{15}[2c_{11}e_{33} + c_{44}e_{15} \\
&\quad - 2(c_{13} + c_{44})(e_{15} + e_{31})], \\
a_4 &= c_{11}(e_{15}^2 + c_{44}\varepsilon_{11}), \\
a_5 &= \rho^2 \varepsilon_{11}, \\
a_6 &= -\rho[e_{15}^2 + (c_{11} + c_{44})\varepsilon_{11}], \\
a_7 &= -\rho^2[2e_{15}e_{33} + (c_{44} + c_{33})\varepsilon_{11} + (c_{11} + c_{44})\varepsilon_{33} \\
&\quad + (e_{15} + e_{31})^2], \\
a_8 &= -\rho[e_{33}^2 + (c_{44} + c_{33})\varepsilon_{33}], \\
a_9 &= \rho^2 \varepsilon_{33},
\end{aligned} \tag{7}
$$

Consider a circular cylindrical panel with outer radius r_0, inner radius r_1, circular center angle α and length L, as one is shown in Fig. 1. If the panel vibrate with a resonant frequency ω, the displacement functions can be assumed as

$$
\begin{aligned}
F &= \frac{r_0^5}{c_{11}\varepsilon_{33}} P(\xi)\Theta(\mu\theta)Z(\beta\zeta)e^{\mathrm{i}\omega t}, \\
\Psi &= r_0^2 P_4(\xi)\Theta'(\mu\theta)Z'(\beta\zeta)e^{\mathrm{i}\omega t},
\end{aligned} \tag{8}
$$

where $\xi = r/R$, $\zeta = z/L$ are the dimensionless coordinates (R is mean radius of the panel) in r and z directions, and $\Theta'(\mu\theta)$ and $Z'(\beta\zeta)$ denote the derivation of , $\Theta(\mu\theta)$ with respect to $\mu\theta$ and the derivation of $Z(\beta\zeta)$ with respect to $\beta\zeta$, respectively. In addition,

$$
\begin{aligned}
\Theta(\mu\theta) &= C_1 \cos(\mu\theta) + C_2 \sin(\mu\theta), \\
Z(\beta\zeta) &= C_3 \sin(\beta\zeta) + C_4 \cos(\beta\zeta),
\end{aligned} \tag{9}
$$

where C_m $(m = 1,2,3,4)$ are constants. Substitution of (8) into (5) yields

$$
(\Delta + k_4^2)P_4(\xi) = 0, \tag{10}
$$

$$
(\Delta + (k_1)^2)(\Delta + (k_2)^2)(\Delta + (k_3)^2)P(\xi) = 0, \tag{11}
$$

where $\Delta = \partial^2/\partial(\xi)^2 + (1/\xi)\partial/\partial(\xi) - \mu^2/\xi^2$ and

$$
k_4^2 = \frac{\Omega^2 c_{11}}{c_{66}} - \frac{\gamma^2 c_{44}}{c_{66}}, \tag{12}
$$

$$
\Omega^2 = \frac{\rho\omega^2 r_0^2}{c_{11}}, \qquad \gamma = \beta t_1, \qquad t_1 = \frac{r_0}{h_0}, \tag{13}
$$

where $h_0 = r_0 - r_1$ and $k_m^2, (m = 1,2,3)$ (assuming $Re[k_m] \geq 0$) are the eigenvalues of the following equation

$$
\begin{aligned}
\bar{a}_4 k^6 + (\bar{a}_6\Omega^2 + \bar{a}_3\gamma^2)k^4 \\
+ (\bar{a}_2\gamma^4 + \bar{a}_7\gamma^2\Omega^2 + \bar{a}_5\Omega^4)k^2 \\
+ (\bar{a}_1\gamma^6 + \bar{a}_8\gamma^4\Omega^2 + \gamma^2\Omega^4) = 0,
\end{aligned} \tag{14}
$$

In which

$$
\begin{aligned}
\bar{a}_n &= a_n/(c_{11}^2\varepsilon_{33}), (n = 1,2,3,4), \\
\bar{a}_n &= a_n/(\rho c_{11}\varepsilon_{33}), (n = 6,7,8), \\
\bar{a}_5 &= a_5/(\rho^2\varepsilon_{33}),
\end{aligned} \tag{15}
$$

The solution of Eq. (11) can be assumed as

$$
P(\xi) = P_1(\xi) + P_2(\xi) + P_3(\xi), \tag{16}
$$

where $P_m(\xi), (m = 1,2,3,4)$ is obtained as

$$
P_m(\xi) = \begin{cases} A_m J_\mu(k_m\xi) + B_m Y_\mu(k_m\xi), k_m^2 > 0, \\ A_m \xi^\mu + B_m \xi^{-\mu}, k_m^2 = 0, \\ A_m I_\mu(k_m'\xi) + B_m K_\mu(k_m'\xi), k_m^2 = -(k_m')^2 < 0, \end{cases} \tag{17}
$$

Substituting (8) into (1) gives the mechanical displacements and electric potential as bellow

$$
u_r = -r_0\left[\frac{\mu}{\xi}P_4(\xi) + \sum_{m=1}^{3} \alpha_{1m}P_m'(\xi)\right]\Theta(\mu\theta)Z'(\beta\zeta)e^{\mathrm{i}\omega t}, \tag{18}
$$

$$
u_\theta = -r_0\left[P_4'(\xi) + \frac{\mu}{\xi}\sum_{m=1}^{3} \alpha_{1m}P_m(\xi)\right]\Theta'(\mu\theta)Z'(\beta\zeta)e^{\mathrm{i}\omega t}, \tag{19}
$$

$$
W = r_0\left[\sum_{m=1}^{3} \alpha_{2m}P_m(\xi)\right]\Theta(\mu\theta)Z(\beta\zeta)e^{\mathrm{i}\omega t}, \tag{20}
$$

$$
\phi = r_0\sqrt{\frac{c_{11}}{\varepsilon_{33}}}\left[\sum_{m=1}^{3} \alpha_{3m}P_m(\xi)\right]\Theta(\mu\theta)Z(\beta\zeta)e^{\mathrm{i}\omega t}, \tag{21}
$$

where

$$\begin{aligned}\alpha_{1m} &= -[(c_{13}+c_{44})(\varepsilon_{11}k_m^2+\varepsilon_{33}\gamma^2) + (e_{15}+e_{31})(e_{15}k_m^2+e_{33}\gamma^2)]\gamma/(c_{11}\varepsilon_{33}),\\ \alpha_{2m} &= \left[\left(c_{11}k_m^2+c_{44}\gamma^2-c_{11}\Omega^2\right)(\varepsilon_{11}k_m^2+\varepsilon_{33}\gamma^2) + (e_{15}+e_{31})^2k_m^2\gamma^2\right]/(c_{11}\varepsilon_{33}),\\ \alpha_{3m} &= \left[\left(c_{11}k_m^2+c_{44}\gamma^2-c_{11}\Omega^2\right)(e_{15}k_m^2+e_{33}\gamma^2) - (c_{13}+c_{44})(e_{15}+e_{31})^2k_m^2\gamma^2\right]/\left(c_{11}\sqrt{c_{11}\varepsilon_{33}}\right), (m=1,2,3),\end{aligned} \tag{22}$$

Utilizing the constitutive relations of piezoelectricity together with Eqs. (18)–(22), one can derive the stress components and electric displacement components as

$$\begin{aligned}\sigma_r &= \Bigg\{(c_{12}-c_{11})\left[\frac{\mu}{\xi}P_4'(\xi)-\frac{\mu}{\xi^2}P_4(\xi)\right] + (c_{12}-c_{11})\sum_{m=1}^{3}\alpha_{1m}P_m''(\xi)\\ &+ \sum_{m=1}^{3}\Big(c_{12}\alpha_{1m}k_m^2+c_{13}\gamma\alpha_{2m} + e_{31}\sqrt{c_{11}/\varepsilon_{33}}\,\gamma\alpha_{3m}\Big)P_m(\xi)\Bigg\}\Theta(\mu\theta)Z'(\beta\zeta)e^{i\omega t}\end{aligned} \tag{23}$$

$$\begin{aligned}\sigma_\theta &= \Bigg\{(c_{11}-c_{12})\left[\frac{\mu}{\xi}P_4'(\xi)-\frac{\mu}{\xi^2}P_4(\xi)\right] + (c_{11}-c_{12})\sum_{m=1}^{3}\alpha_{1m}P_m''(\xi)\\ &+ \sum_{m=1}^{3}\Big(c_{11}\alpha_{1m}(k_m^{Pi})^2+c_{13}\gamma\alpha_{2m} + e_{31}\sqrt{c_{11}/\varepsilon_{33}}\,\gamma\alpha_{3m}\Big)P_m(\xi)\Bigg\}\Theta(\mu\theta)Z'(\beta\zeta)e^{i\omega t}\end{aligned} \tag{24}$$

$$\sigma_z = \sum_{m=1}^{3}\left[\left(c_{13}\alpha_{1m}k_m^2+c_{33}\gamma\alpha_{2m} + e_{33}\sqrt{c_{11}/\varepsilon_{33}}\,\gamma\alpha_{3m}\right)P_m(\xi)\right]\Theta(\mu\theta)Z'(\beta\zeta)e^{i\omega t} \tag{25}$$

$$\tau_{\theta z} = \Bigg\{c_{44}\gamma P_4'(\xi) + \frac{\mu}{\xi}\sum_{m=1}^{3}\Big[\Big(c_{44}\gamma\alpha_{1m}+c_{44}\alpha_{2m} + e_{15}\sqrt{c_{11}/\varepsilon_{33}}\,\alpha_{3m}\Big)P_m(\xi)\Big]\Bigg\}\Theta'(\mu\theta)Z(\beta\zeta)e^{i\omega t} \tag{26}$$

$$\tau_{rz} = \Bigg\{c_{44}\gamma\frac{\mu}{\xi}Q(\xi) + \Bigg[\sum_{m=1}^{3}\Big(c_{44}\gamma\alpha_{1m}+c_{44}\alpha_{2m} + e_{15}\sqrt{c_{11}/\varepsilon_{33}}\,\alpha_{3m}\Big)P_m'(\xi)\Bigg]\Bigg\}\Theta(\mu\theta)Z(\beta\zeta)e^{i\omega t} \tag{27}$$

$$\tau_{r\theta} = c_{66}\Bigg[-k_4^2P_4(\xi)-2P_4''(\xi) + \frac{2\mu}{\xi^2}\sum_{m=1}^{3}\alpha_{1m}P_m(\xi) - \frac{2\mu}{\xi}\sum_{m=1}^{3}\alpha_{1m}P_m'(\xi)\Bigg]\Theta'(\mu\theta)Z'(\beta\zeta)e^{i\omega t} \tag{28}$$

$$D_r = \Bigg\{e_{15}\gamma\frac{\mu}{\xi}P_4(\xi) + \Bigg[\sum_{m=1}^{3}\Big(e_{15}\gamma\alpha_{1m}+e_{15}\alpha_{2m} + \varepsilon_{11}\sqrt{c_{11}/\varepsilon_{33}}\,\alpha_{3m}\Big)P_m'(\xi)\Bigg]\Bigg\}\Theta(\mu\theta)Z(\beta\zeta)e^{i\omega t} \tag{29}$$

$$D_\theta = \Bigg\{e_{15}\gamma P_4'(\xi) + \frac{\mu}{\xi}\Bigg[\sum_{m=1}^{3}\Big(e_{15}\gamma\alpha_{1m}+e_{15}\alpha_{2m} + \varepsilon_{11}\sqrt{c_{11}/\varepsilon_{33}}\,\alpha_{3m}\Big)P_m(\xi)\Bigg]\Bigg\}\Theta'(\mu\theta)Z(\beta\zeta)e^{i\omega t} \tag{30}$$

$$D_z = \Bigg[\sum_{m=1}^{3}\Big(e_{31}\alpha_{1m}k_m^2+e_{33}\gamma\alpha_{2m} - \sqrt{c_{11}\varepsilon_{33}}\,\gamma\alpha_{3m}\Big)P_m(\xi)\Bigg]\Theta(\mu\theta)Z'(\beta\zeta)e^{i\omega t} \tag{31}$$

B. Boundary Conditions

The piezoelectric panel has eight boundary conditions consist of six mechanical and two electrical. By considering generalized simply support boundary conditions at $\theta = 0$ and $\theta = \alpha$ we will have

$$w = u_r = 0, \sigma_\theta = 0, \phi = 0\,, \tag{32}$$

One can take

$$C_1 = 0, C_2 = 1, \mu = \frac{(2m+1)\pi}{2\alpha}, m = 0,1,2,\ldots \tag{33}$$

And by considering generalized simply support boundary conditions at $\zeta = 0$ and $\zeta = 1$

$$u_r = u_\theta = 0, \sigma_z = 0\, , D_z = 0, \tag{34}$$

One can take

$$C_3 = 0, C_4 = 1, \beta = n\pi, n = 0,1,2, \ldots \tag{35}$$

By considering the free vibration of the piezoelectric panel on an elastic foundation, no external force acts on the structure. Now, the coupled free vibration problem is considered.

Because of the effect of the foundation, the boundary conditions at inner surface r_1 become

$$\sigma_r = \Sigma, \tau_{r\theta} = \tau_{rz} = 0, \phi = 0 \; at \; r = r_1, \tag{36}$$

where Σ is the reactive force of the foundation, which satisfies the following equation for Kerr model [24]

$$\left(1 + \frac{\kappa}{\vartheta}\right)\Sigma - \frac{\nu}{\vartheta}\Delta\Sigma = \kappa u_r - \mu \Delta u_r, \tag{37}$$

where $\Delta = {\partial^2}/{\partial z^2} + {\partial^2}/{r^2 \partial\theta^2}$ κ and ϑ are the spring constants of the spring layers and ν is the shear constant of the shear layer. From (22)-(38), we get the coupled free vibration frequency equation as follows

$$\left|T_{ij}\right| = 0, (i, j = 1,2, \ldots, 8) \tag{38}$$

where

$$T_{1(2m-1)} = (c_{12} - c_{11})\alpha_{1m} k_m^2 J''_\mu(k_m s) + \left[c_{12}\alpha_{1m}k_m^2 + c_{13}\gamma\alpha_{2m} + e_{31}\sqrt{c_{11}/\varepsilon_{33}}\,\gamma\alpha_{3m}\right] J_\mu(k_m s) \tag{39}$$

$$T_{2(2m-1)} = \left(c_{44}\gamma\alpha_{1m} + c_{44}\alpha_{2m} + e_{15}\sqrt{c_{11}/\varepsilon_{33}}\,\alpha_{3m}\right) k_m J'_\mu(k_m s) \tag{40}$$

$$T_{3(2m-1)} = 2c_{66}\mu\alpha_{1m}\left[-\frac{k_m}{s}J'_\mu(k_m s) + \frac{1}{s^2}J_\mu(k_m s)\right], (m = 1,2,3) \tag{41}$$

$$T_{17} = (c_{12} - c_{11})\mu\left[\frac{k_4}{s}J'_\mu(k_4 s) - \frac{1}{s^2}J_\mu(k_4 s)\right] \tag{42}$$

$$T_{27} = c_{44}\gamma\frac{\mu}{s}J_\mu(k_4 s)$$
$$T_{37} = -c_{66}k_4^2 J_\mu(k_4 s) - 2\, c_{66}k_4^2 J''_\mu(k_4 s) \tag{43}$$

where prime denotes derivation with respect to s. According to the electrical condition, which is open circuit at both inner and outer surface, one can obtain

$$T_{4(2m-1)} = \left(e_{15}\gamma\alpha_{1m} + e_{15}\alpha_{2m} - \varepsilon_{11}\sqrt{c_{11}/\varepsilon_{33}}\,\alpha_{3m}\right) k_m J'_\mu(k_m s) \tag{44}$$

$$T_{47} = e_{15}\gamma\frac{\mu}{s}J_\mu(k_4 s), (m = 1,2,3) \tag{45}$$

In which $s = r_0/r_1$ is the ratio of the inner radius to the outer radius in the piezoelectric panel. Also, for closed circuit electrical condition at both inner and outer surface of piezoelectric layer

$$T_{4(2m-1)} = \alpha_{3m}J_\mu(k_m s), T_{47} = 0\, , (m = 1,2,3) \tag{46}$$

In (39)-(46) only elements of $(2m - 1)th, (m = 1, \ldots, 4)$ columns are listed, however, the elements of $2m\, th, (m = 1, \ldots, 4)$ can easily obtained by replacing J_μ by Y_μ and zero elements remain zero. And the elements of $(n + 4)th$ order, if not mentioned, can obtained by replacing s by 1 in $n\, th$ order form except for

$$T_{51} = T_{11}|_{s=1} - p\mu J_\mu(k_1), T_{52} = T_{12}|_{s=1} - p\mu Y_\mu(k_1) \tag{47}$$
$$T_{53} = T_{13}|_{s=1} - pI'_\mu(k_2), T_{54} = T_{14}|_{s=1} - pK'_\mu(k_2) \tag{48}$$
$$T_{55} = T_{15}|_{s=1} - pI'_\mu(k_3), T_{56} = T_{16}|_{s=1} - pK'_\mu(k_3) \tag{49}$$

where

$$p = [p_1 + p_2(t_L^2 + \beta^2/t_2^2)]/[1 + p_3 + p_2 p_3\, (t_L^2 + \beta^2/t_2^2)/p_1],$$

And, $p_1 = \kappa R/c_{44}, p_2 = \nu/(c_{44}R)$ and $p_3 = \kappa/\vartheta$ are the three non-dimensional foundation parameters. It can be seen that if we take $p_3 = 0$ in (47)-(49), then the effect of a Kerr foundation on the frequencies will be identical with that of a Pasternak foundation, in which only two foundation parameters are involved [25]. Moreover, if we take $p_3 = p_2 = 0$, then frequency equation degenerates to the one of a panel on a Winkler foundation [26]. Note that if $p = 0$, frequency equation will be the same as the uncoupled one.

For obtaining the frequency response function of the piezoelectric structure on the aforementioned foundation, one can suppose that an external force act on the upper surface of the panel. So,

$$\sigma_r = F_{\text{ex}}, \tau_{r\theta} = \tau_{rz} = 0, \phi = \Phi_{\text{ex}} \; at \; r = r_0 \tag{50}$$

where F_{ex} and Φ_{ex} are external force and electric potential, respectively. For obtaining frequency response of cylindrical panel under a harmonic external excitation, the following matrix equation must solved

$$\mathbf{T}\,\mathrm{X}_m = \mathrm{F}_m, \quad \mathbf{T}\,\mathrm{X}_e = \mathrm{F}_e, \tag{51}$$

where $\mathbf{T}$ is the coefficient matrix and given in (40)-(50), and

$$\mathrm{X}_m = [A_1^m, B_1^m, A_2^m, B_2^m, A_3^m, B_3^m, A_4^m, B_4^m]^{\mathrm{T}}$$
$$\mathrm{X}_e = [A_1^e, B_1^e, A_2^e, B_2^e, A_3^e, B_3^e, A_4^e, B_4^e]^{\mathrm{T}}$$

Also,

$$\begin{aligned} \mathrm{F}_e &= [0,0,0,0,0,0,\iota_{nm},0]^{\mathrm{T}} \\ \mathrm{F}_m &= [p_{nm},0,0,0,0,0,0,0]^{\mathrm{T}} \end{aligned} \tag{52}$$

In addition,

$$\begin{aligned} &F_{\mathrm{ex}}(r_0,\theta,z,\omega) \\ &= C_{11}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} p_{nm}(r_0,\omega)\sin(\mu\theta)\sin(\eta\zeta)\,e^{\mathrm{i}\omega t}, \\ &\Phi_{ex}(r_0,\theta,z,\omega) \\ &= L\sqrt{C_{11}/\varepsilon_{11}}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \iota_{nm}(r_0,\omega)\sin(\mu\theta)\cos(\eta\zeta)\,e^{\mathrm{i}\omega t}, \end{aligned} \tag{53}$$

where

$$\begin{aligned} &p_{nm}(r_0,\omega) \\ &= \frac{4}{\alpha L C_{11}}\iint_A F_{\mathrm{ex}}(r_0,\theta,z,\omega)\sin(\mu\theta)\sin(\eta\zeta)\,\mathrm{d}\zeta\mathrm{d}\theta, \\ &\iota_{nm}(r_0,\omega) \\ &= \frac{4}{\alpha L^2\sqrt{C_{11}/\varepsilon_{11}}}\iint_A \Phi_{\mathrm{ex}}(r_0,\theta,z,\omega)\sin(\mu\theta)\cos(\eta\zeta)\,\mathrm{d}\zeta\mathrm{d}\theta, \end{aligned} \tag{54}$$

In which A is the upper surface of the panel. After finding these unknown constants that are functions of m,n, by replacing them in the displacement and stress and electric displacement of corresponding equations all of the system variables can easily determined. However, the voltage obtained from the piezoelectric layer is calculated as

$$q = \int_{Area} \overline{D}.\,\mathrm{d}A_{Area}, \tag{55}$$

In which, $\overline{D} = D_r\hat{r} + D_\theta\hat{\theta} + D_z\hat{z}$ is the electric displacement vector in the principle cylindrical coordinates. $Area$ in the integration stands for the place that the layer is active and voltage is measured and $\mathrm{d}A_{Area} = (\mathrm{d}z \times \mathrm{d}\theta)\hat{r}$ which simplifies the above equation as

$$q_r = \int_{\theta_1}^{\theta_2}\int_{z_1}^{z_2} D_r\,\mathrm{d}z\mathrm{d}\theta, \tag{56}$$

where θ_1, θ_2, z_1 and z_2 shows the integration bounds (electroded area). And by considering the piezoelectric layer as an electric capacitance $V = q/c_P$ one can obtain the voltage as

$$V = \frac{1}{c_P}\int_{\theta_1^s}^{\theta_2^s}\int_{z_1^s}^{z_2^s} D_r\,dzd\theta \tag{57}$$

where c_P is the capacitance of the piezoelectric layer.

III. Numerical Examples and Discussion

A piezoelectric panel with different relative radius ratios, and fabricated from $\mathrm{Ba_2NaNb_5O_{15}}$/ PZT4 with the mechanical and electrical material properties as given in Table. I is considered.

Computations were performed on a network of personal computer with a maximum truncation constant of $\mathrm{n_{max}} = 10$ in Fourier expansions in all stress, displacement and electric voltage equations to assure convergence in the high frequency range. Before presenting the main results, the overall validity of the formulation should be demonstrated. To do this, the panel natural frequencies are computed for the same geometry used in [10]. The outcome, as shown in Table. II, shows good agreements with those calculated using commercial finite element software and [10].

TABLE I
Material properties of the panel

		Piezoelectric Layer $\rho = 7500$ (Kg/m^3)
c_{ij} (N/m^2)	c_{11}	13.9×10^{10}
	c_{12}	10.4×10^{10}
	c_{13}	7.43×10^{10}
	c_{22}	13.9×10^{10}
	c_{23}	7.43×10^{10}
	c_{33}	11.5×10^{10}
	c_{44}	2.56×10^{10}
	c_{55}	2.56×10^{10}
	c_{66}	3.06×10^{10}
e_{ij} (C/m^2)	e_{15}	12.7
	e_{24}	12.7
	e_{31}	-5.2
	e_{32}	-5.2
	e_{33}	15.1
ε_{ij} (F/m)	ε_{11}	650×10^{-11}
	ε_{22}	650×10^{-11}
	ε_{33}	560×10^{-11}

TABLE II
Comparison of first three natural frequencies ($\mu = 1.8, s = 0.5$)

Model	1^{st}N. F.	2^{nd}N. F.	3^{rd}N. F.
Reference [10]	0.5408	1.6544	2.4273
FEM	0.5409	1.6561	2.4278
Present	0.5404	1.6556	2.4270

In the validation procedure, FEM parameters are selected as shown in Table. III. In addition, mesh size sensitivity analysis was carried out for numerical convergence checking.

The first five natural frequencies of the piezoelectric panel computed without any foundation, are listed in the Table. IV. As one can see the natural frequency decreases as the thickness ratio increases.

The first five mode shapes of thickness ratio $s = 0.1, 0.3, 0.6, 0.9$ are shown in Figs 2, 3 and 4. Figs 2, 3 and 4 show the displacements due to each mode shape in the x-direction, y-direction and z-direction, respectively.

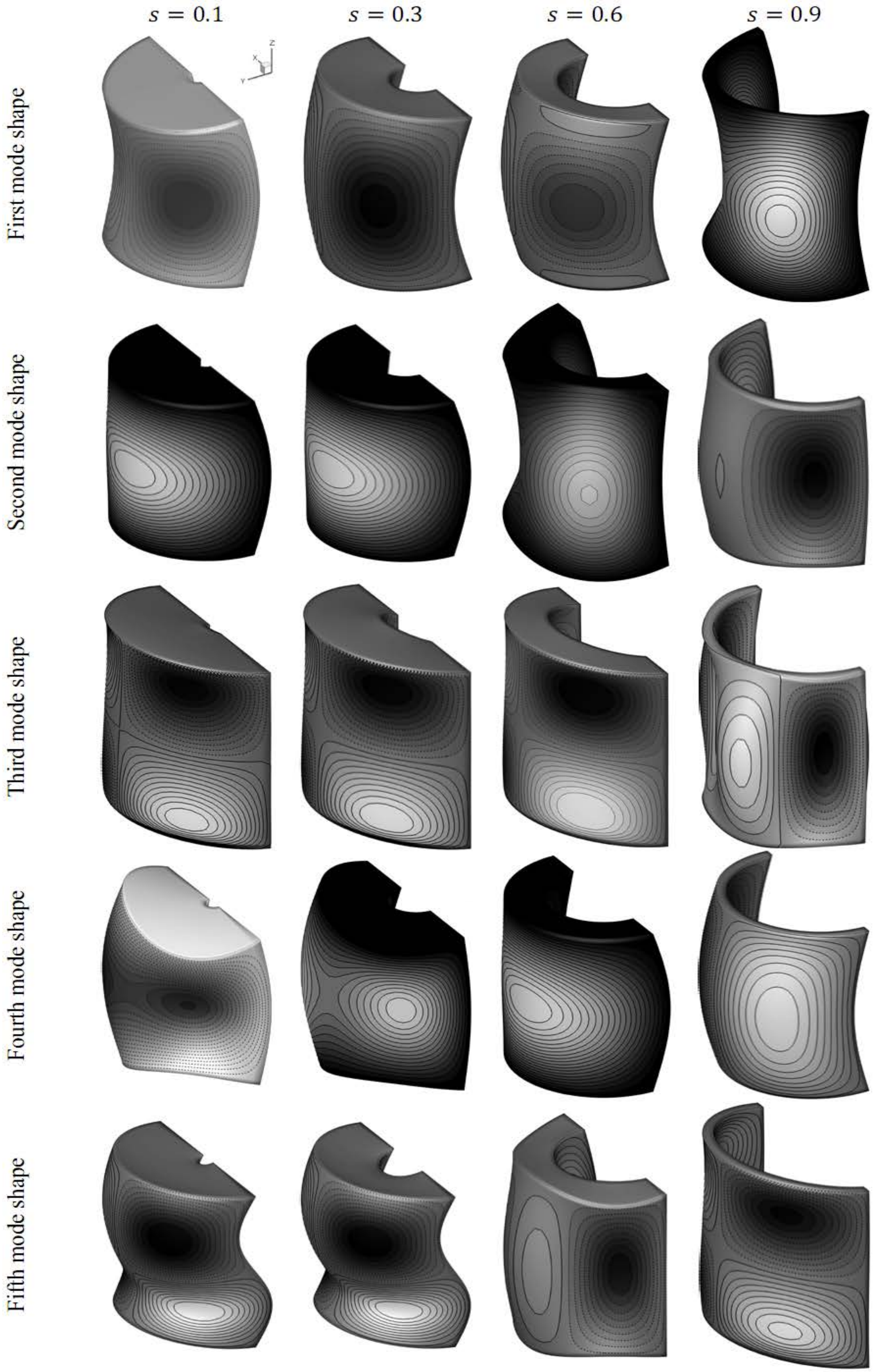

Fig. 2 Displacement of the first five mode shapes in x direction

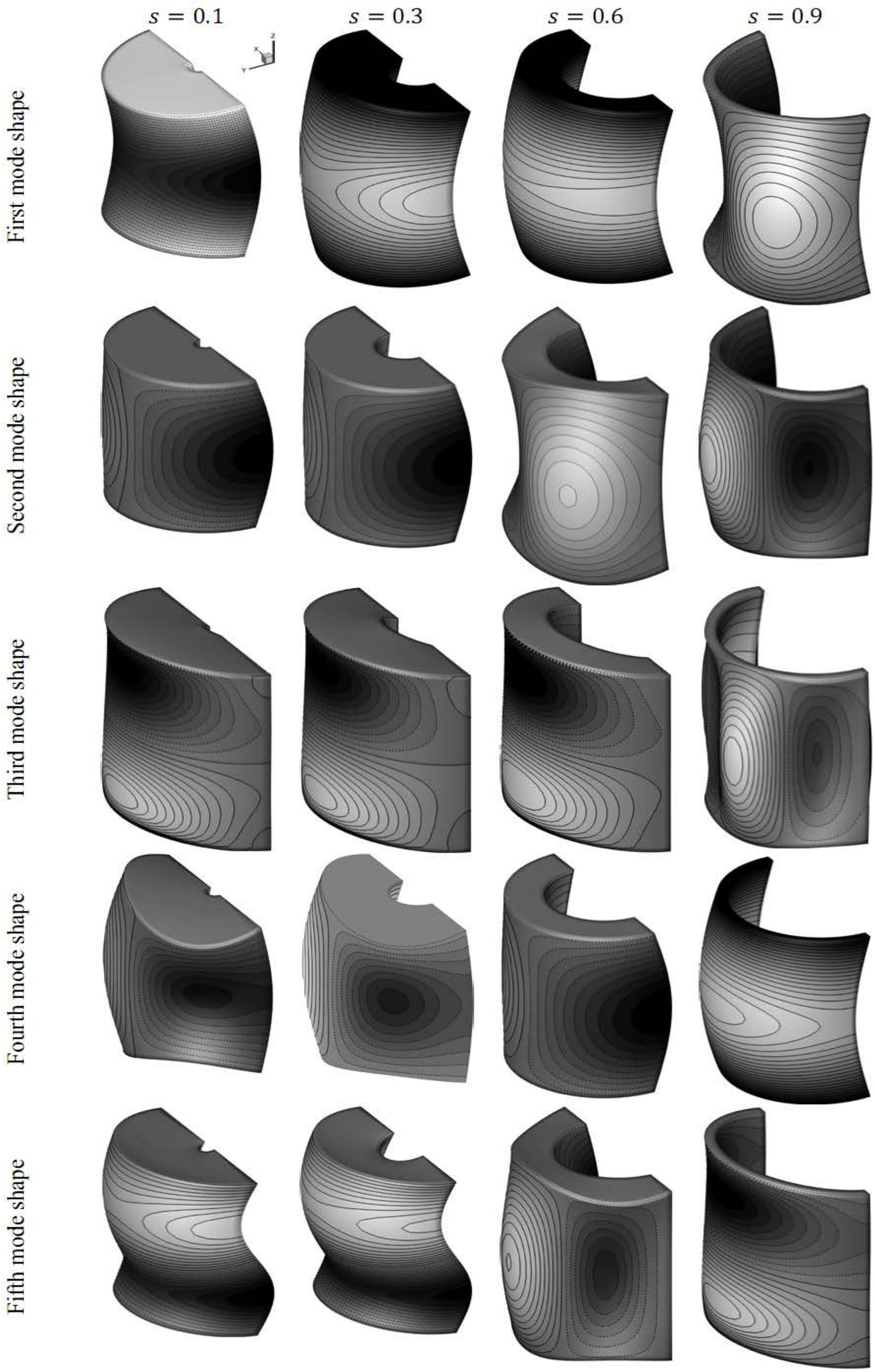

Fig. 3 Displacement of the first five mode shapes in y direction

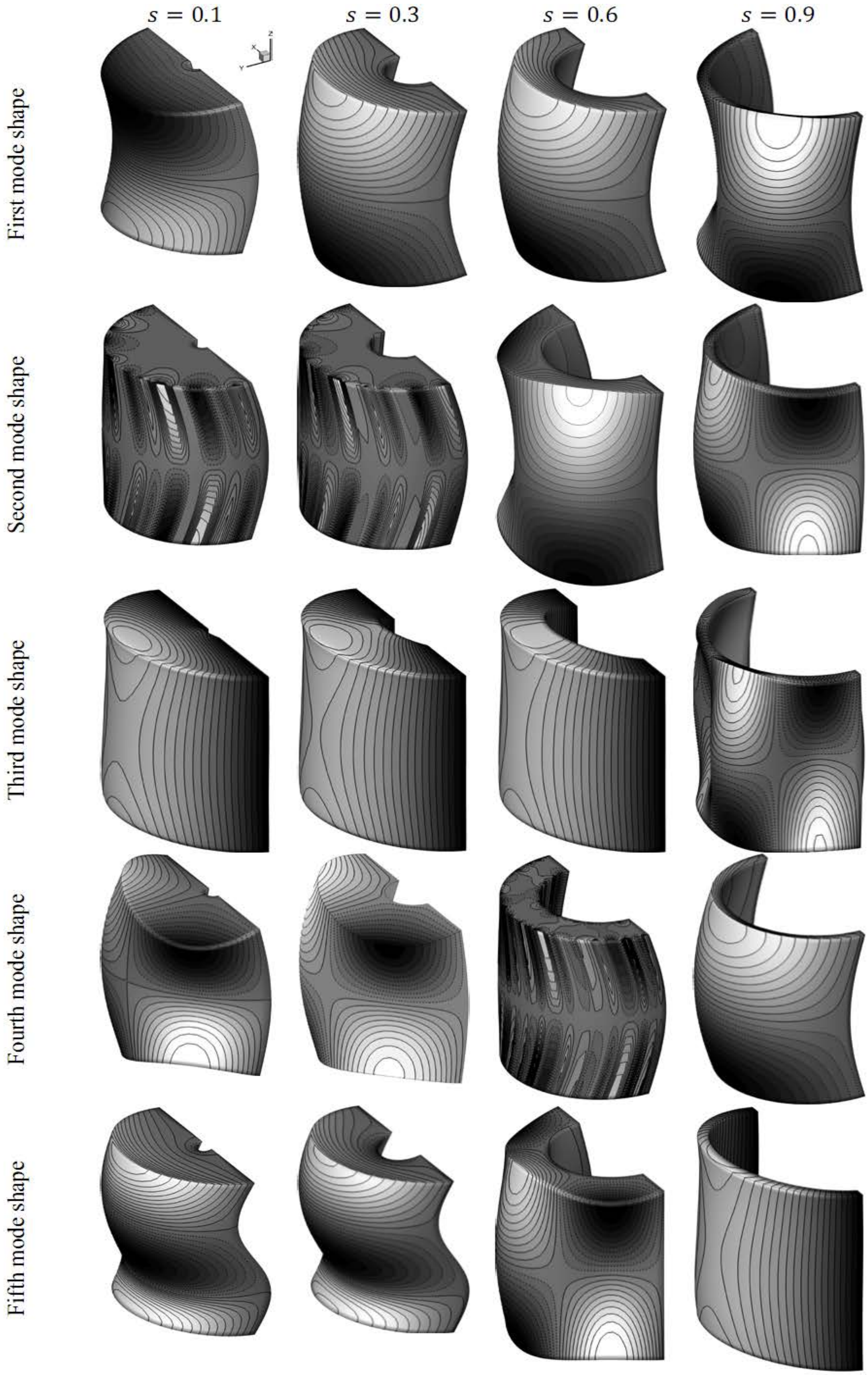

Fig. 4 Displacement of the first five mode shapes in z direction

Also, the first five natural frequencies of the circular cylindrical piezoelectric located on a foundation with $\mathrm{K} = p_1 = 50 \times 10^6$ for different thickness ratios are listed as Table. V. As one can see, adding the Winkler foundation increases the panel natural frequencies.

TABLE III
CONSIDERED FEM MODEL PARAMETERS

ABAQUS elastic coefficients: Engineering Constants	
Parameters	Magnitude
$\mathbf{E_1}$	8.12394×10^{10}
$\mathbf{E_2}$	8.12394×10^{10}
$\mathbf{E_3}$	6.4073×10^{10}
$\mathbf{Nu_{12}}$	0.327441
$\mathbf{Nu_{13}}$	0.434532
$\mathbf{Nu_{23}}$	0.434532
$\mathbf{G_{12}}$	306×10^8
$\mathbf{G_{13}}$	256×10^8
$\mathbf{G_{23}}$	256×10^8
ABAQUS Mesh Element Type for Piezoelectric	
C3D20RE: A20 − node quadratic piezoelectric brick, reuced integration	

TABLE IV
THE FIRST FIVE NATURAL FREQUENCIES OF THE PANEL ($\mathrm{K} = 0$)

s	1^{st}N. F.	2^{nd}N. F.	3^{rd}N. F.	4^{th}N. F.	5^{th}N. F.
0.1	334.45	458.20	611.02	738.25	806.13
0.2	331.49	458.25	573.76	667.07	794.78
0.3	327.22	458.22	527.22	583.12	777.45
0.4	322.01	458.35	481.94	502.13	753.29
0.5	316.15	425.76	441.95	458.34	720.53
0.6	309.97	352.65	407.67	458.42	666.99
0.7	283.08	303.94	378.39	458.45	509.97
0.8	221.25	298.67	344.72	353.18	458.48
0.9	177.21	182.08	279.82	294.82	331.23

TABLE V
The first five natural frequencies of the panel ($\mathrm{K} = 50 \times 10^6$)

s	1^{st}N. F.	2^{nd}N. F.	3^{rd}N. F.	4^{th}N. F.	5^{th}N. F.
0.1	334.48	458.20	611.02	738.25	806.14
0.2	331.53	458.25	573.76	667.09	794.80
0.3	327.30	458.30	527.22	583.17	777.49
0.4	322.13	458.35	481.94	502.23	753.35
0.5	316.33	425.93	441.95	458.39	720.63
0.6	310.23	352.96	407.67	458.42	667.16
0.7	283.67	304.35	378.39	458.45	510.32
0.8	222.52	299.37	345.61	353.19	458.48
0.9	180.66	185.82	282.35	296.38	331.25

The effect of increasing K on the panel natural frequencies is investigated in Table. VI. These results are obtained for the thickness ratio $s = 0.8$

TABLE VI
The first five natural frequencies of the panel for different K

K	1^{st}N. F.	2^{nd}N. F.	3^{rd}N. F.	4^{th}N. F.	5^{th}N. F.
0	221.25	298.67	344.72	353.18	458.48
$\mathbf{25 \times 10^6}$	221.89	299.02	345.16	353.19	458.48
$\mathbf{50 \times 10^6}$	222.52	299.37	345.61	353.19	458.48
$\mathbf{75 \times 10^6}$	223.16	299.72	346.05	353.19	458.48
$\mathbf{100 \times 10^6}$	223.79	300.07	346.50	353.19	458.48
$\mathbf{125 \times 10^6}$	224.42	300.42	346.94	353.20	458.48
$\mathbf{150 \times 10^6}$	225.05	300.77	347.38	353.20	458.48
$\mathbf{175 \times 10^6}$	225.68	301.12	347.82	353.20	458.48
$\mathbf{200 \times 10^6}$	226.30	301.47	348.27	353.20	458.48

Obtained results show that, the natural frequencies increase by increasing the foundation constant ($\mathrm{K} = p_1$). Note that this effect decreases for higher order mode shapes, and in the fifth mode shape, the natural frequency does not change due to adding the foundation. This shows that structure vibration in the high frequencies does not depend on the foundation.

The effect of large increasing of K is studied in Table. VII. This table shows that by increasing the Winkler foundation constant, the natural frequencies increase and tend to a constant bound, finally. This constant bound is the structure natural frequencies with clamped boundary condition at lower surface.

In order to investigate the panel frequency response, it is excited by a harmonic pressure force distributed on the upper surface. Fig. 5 compares the electric voltage induced due to this excitation for the case $\mathrm{K} = 0$ and $\mathrm{K} = 100 \times 10^6$. This figure shows that the response peaks slip to higher frequency by adding the Winkler foundation, which is in agreement with the calculated natural frequencies. In high frequencies this effect is insignificant. Also, as shown in Fig. 5 and Fig. 6 the vibration amplitude increases by adding the foundation.

TABLE VII
The first five natural frequencies of the panel for different large K

K	1^{st}N. F.	2^{nd}N. F.	3^{rd}N. F.	4^{th}N. F.	5^{th}N. F.
$\mathbf{100 \times 10^5}$	221.50	298.81	344.89	353.18	458.48
$\mathbf{100 \times 10^6}$	223.79	300.07	346.50	353.19	458.48
$\mathbf{100 \times 10^7}$	245.41	312.27	353.27	362.09	458.48
$\mathbf{100 \times 10^8}$	353.75	396.73	401.21	458.51	489.56
$\mathbf{100 \times 10^9}$	354.41	458.81	549.50	772.82	786.24
$\mathbf{100 \times 10^{10}}$	354.66	461.76	573.81	773.35	847.84
Clamped boundary condition					
0	354.79	458.48	574.29	773.76	851.46

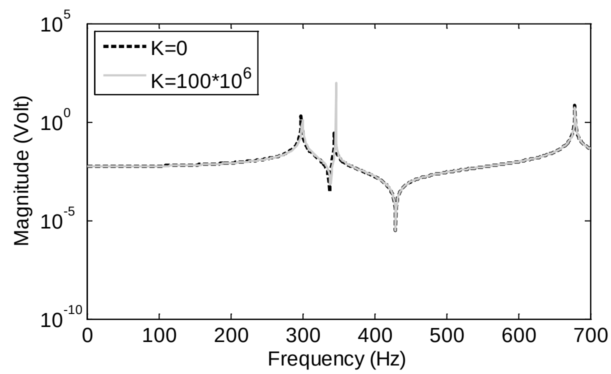

Fig. 5 Comparison magnitude of electric potential

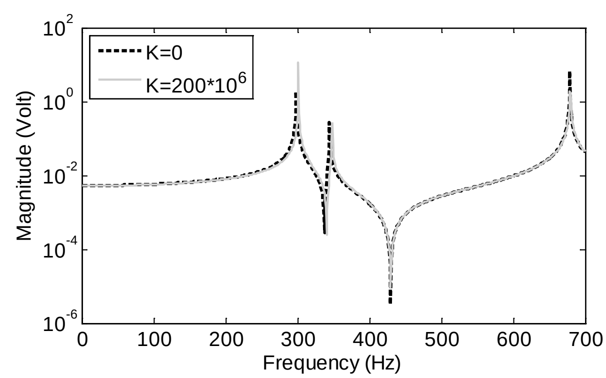

Fig. 6 Comparison magnitude of electric potential

By comparing Fig. 5 and Fig. 6, it can be easily seen that by increasing the elastic coefficient the vibration amplitude decreases.

Fig. 7 and Fig. 8 show the vibration amplitude in R-direction for two cases of no foundation and elastic foundation.

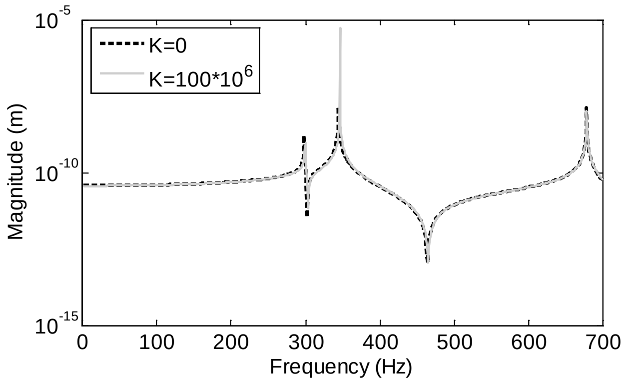

Fig. 7 Comparison magnitude of electric potential

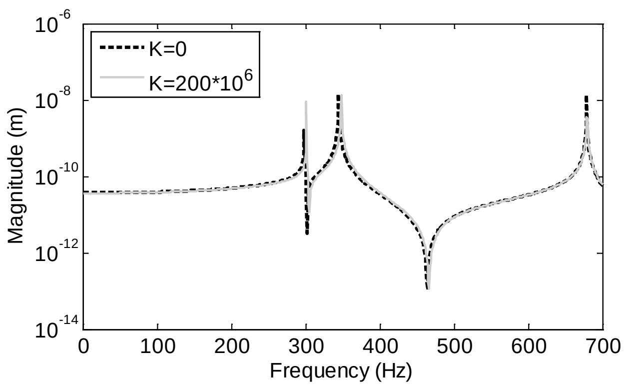

Fig. 8 Comparison magnitude of electric potential

As one can see, the effect of increasing K on vibration amplitude in lower frequencies is more than higher frequencies.

IV. Conclusions

The free/forced vibration of a transversely isotropic piezoelectric circular cylindrical panel supported on an elastic Winkler foundation was investigated in this paper. An exact, three-dimensional frequency response is presented. The effect of foundation constant on the natural frequencies of piezoelectric panel is numerically investigated for transversely isotropic materials with a generalized simply support boundary condition. Also, the frequency response of the panel compared for different foundations. As it is shown in the results, the imperfections in the placement of the piezoelectric layers as sensor/actuator have important effect on the structure vibration response. In addition, for validation purposes, calculated results compared with the results from FEM and results from a previous work.

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