Analysis of the thermoelastic rod collision with a heated rigid wall with due account for temperature and strain weak coupling*

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Abstract – The problem of the collision of a thermoelastic rod with a heated rigid wall is considered for the case of weak coupling between the strain and temperature fields when the thermoelastic behavior of the rod is described by the Green-Naghdy theory without energy dissipation. The lateral surfaces and free end of the rod are thermally insulated, and free thermal exchange is established within the contact domain with the wall. D'Alembert solution together with the perturbation technique are utilized as the method of solution. The proposed procedure allows one to construct an analytical solution enabling to study the influence of thermoelastic parameters on the contact duration, as well as to obtain the stress, velocity, temperature, and heat flow dependences of time and coordinate.

Keywords – Thermoelastic collision, thermoelastic rod, Green-Naghdy theory without energy dissipation, D'Alembert-type solution

I. Introduction

The generalized D'Alembert solution to the problem of coupled thermoelasticity was presented in pioneering work of Hetnarski in 1967, but this paper was unfairly forgotten. The solution was constructed for a parabolic-hyperbolic set of equations and involved both wave and diffusion terms. The coupling parameter was considered to be small, and the solution was constructed via the expansion in terms of a small parameter. However, the solution constructed in [1] was not applied for solving dynamic boundary-value problems.

It seems likely that D'Alembert expansions have not been used for solving boundary-value problems of dynamic theory of thermoelasticity till 1998 when the simplest problem for the Green-Naghdi hyperbolic theory of thermoelasticity without energy dissipation [2] has been considered [3], wherein the analytical solution of the Danilovskayas problem on the heat shock of a stress-free thermoelastic half-space has been derived. However, the solution suggested in [3] is applicable only for half-continuous media, since it does not allow one to consider reflected and refracted waves which occur in bodies of finite extent.

The solutions applicable for bodies of finite extent were proposed in [4, 5] for the collision of two thermoelastic rods and for the impact of a thermoelastic rod against a rigid heated barrier, respectively, using the Green-Naghdy theory but without coupling the temperature and strain fields. The procedure suggested has enabled for the first time to construct the longitudinal coordinate dependence of the desired functions at any fixed instant of the time beginning from the moment of the rods collision with the barrier up to the moment of its rebound, i.e., to obtain the analytical solution in the closed form for the main functions showing the distribution of the thermoelastic impact characteristics along the rod.

The procedure of the application of D'Alembert method in dynamic problems of uncoupled and coupled thermoelasticity is described in detail in [6].

In the present paper, the problem on impact of a thermoelastic rod against a heated barrier is considered with due account for the small coupling between the temperature and strain fields using the Green-Naghdy theory via the D'Alembert-type solution.

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II. Governing Equations and the D'Alembert-type Solution

The dynamic behavior of a thermoelastic rod based on the Green-Naghdy theory of thermoelasticity without energy dissipation [2] is described by the following set of equations:

$$\sigma_{,x} = \rho \dot{v}, \tag{1}$$

$$\dot{\sigma} = E v_{,x} - \gamma \dot{\theta}, \tag{2}$$

$$q_{,x} + c_\varepsilon \dot{\theta} + T_0 \gamma v_{,x} = 0, \tag{3}$$

$$\dot{q} = -æ \theta_{,x}, \tag{4}$$

where x is the coordinate, t is time, σ is the stress, v is the velocity, ρ is the density, $\theta = T - T_0$ is the relative temperature, T_0 is the initial temperature, E is the Young's modulus, $\gamma = E\alpha$, α is the coefficient of linear thermal expansion, c_ε is the specific heat at constant strain, q is the quantity of heat flowing through a rod cross-section area per a time unit, $æ = \lim\limits_{\tau\to\infty} (\lambda \tau^{-1})$, λ is the thermal conductivity of the material, τ is the thermal relaxation time, overdot denotes the time-derivative, and an index after a comma labels the derivative with respect to the coordinate.

Eliminating the stress σ from equation of motion (1) and Duhamel-Neumann law (2) once differentiated with respect to time, and the velocity of heat flow q from the law of conservation of energy (3) and the heat conduction law (4), we find

$$a^2 \theta_{,xx} - \ddot{\theta} = \frac{1}{\alpha}\, \varepsilon \dot{v}_{,x}, \tag{5}$$

$$c^2 v_{,xx} - \ddot{v} = c^2 \alpha \dot{\theta}_{,x}, \tag{6}$$

where $a = \sqrt{æ c_\varepsilon^{-1}}$ is the velocity of the pure thermal wave, $c = \sqrt{E\rho^{-1}}$ is the velocity of the pure elastic wave, $\varepsilon = T_0 \gamma^2 (c_\varepsilon E)^{-1}$ is a dimensionless parameter defining the strain and temperature fields coupling, which is a small value what is characteristic for such materials as different metals and alloys [7].

When $\varepsilon = 0$, i.e., for the uncoupled problem, the general solution of the D'Alembert type was derived by Rossikhin and Shitikova in the following form [5]:

$$\theta_0 = g_0(x - at) + k_0(x + at), \tag{7}$$

$$\begin{aligned} v_0 &= f_0(x - ct) + l_0(x + ct) \\ &+ A\left[g_0(x - at) - k_0(x + at)\right], \end{aligned} \tag{8}$$

$$\begin{aligned} \sigma_0 &= \rho c\left[-f_0(x - ct) + l_0(x + ct)\right] \\ &- \rho a A\left[g_0(x - at) + k_0(x + at)\right], \end{aligned} \tag{9}$$

$$q_0 = -æ a^{-1}\left[-g_0(x - at) + k_0(x + at)\right], \tag{10}$$

where $A = a c^2 \alpha \left(a^2 - c^2\right)^{-1}$ is the coefficient defining the influence of thermal characteristics on the velocity and stress fields, $f_0(x - ct)$, $l_0(x + ct)$, $g_0(x - at)$, and $k_0(x + at)$ are arbitrary functions.

In order to find the general solution of Eqs. (5) and (6) within an accuracy of $\varepsilon \neq 0$, let us substitute (8) into the right-hand side of Eq. (5). As a result we have [8]

$$\begin{aligned} a^2 \theta_{1,xx} - \ddot{\theta}_1 = \frac{1}{\alpha}\, \varepsilon &\left[-c f_{0,\xi\xi} + c l_{0,\eta\eta}\right. \\ &\left. - A a g_{0,\lambda\lambda} - A a k_{0,\mu\mu}\right], \end{aligned} \tag{11}$$

where indices after a comma denote the derivatives with respect to $\xi = x - ct$, $\eta = x + ct$, $\lambda = x - at$, and $\mu = x + at$.

The general solution of (11) has the form

$$\begin{aligned} \theta_1 &= g_1(x - at) + k_1(x + at) \\ &+ \frac{A\varepsilon}{c a \alpha^2}\left[-f_0(x - ct) + l_0(x + ct)\right] \\ &+ \frac{A\varepsilon}{2\alpha}\left[-g_{0,\lambda}(x - at) + k_{0,\mu}(x + at)\right] t, \end{aligned} \tag{12}$$

where $g_1(x - at)$, and $k_1(x + at)$ are arbitrary functions.

Substituting (12) into the right-hand side of Eq. (6) yields

$$\begin{aligned} c^2 v_{1,xx} - \ddot{v}_1 = c^2 \alpha &\left[a\left(-g_{1,\lambda\lambda} + k_{1,\mu\mu}\right)\right. \\ &+ \frac{A\varepsilon}{a\alpha^2}\left(f_{0,\xi\xi} + l_{0,\eta\eta}\right) + \frac{A\varepsilon}{2\alpha}\left(-g_{0,\lambda\lambda} + k_{0,\mu\mu}\right) \\ &\left. + \frac{A a \varepsilon}{2\alpha}\left(g_{0,\lambda\lambda\lambda} + k_{0,\mu\mu\mu}\right) t\right]. \end{aligned} \tag{13}$$

The general solution of (13) has the form

$$\begin{aligned} v_1 &= f_1(x - ct) + l_1(x + ct) \\ &+ A\left[g_1(x - at) - k_1(x + at)\right] \\ &+ \varepsilon\, \frac{c^2 A}{2}\left\{\frac{c^2 + a^2}{(c^2 - a^2)^2}\left[-g_0(x - at) + k_0(x + at)\right]\right. \\ &\left. + \frac{1}{c a \alpha}\left(f_{0,\xi} - l_{0,\eta}\right) t + \frac{a}{c^2 - a^2}\left(g_{0,\lambda} + k_{0,\mu}\right) t\right\}, \end{aligned} \tag{14}$$

where $f_1(x - ct)$, and $l_1(x + ct)$ are arbitrary functions.

Knowing the functions θ_1 and v_1, from relationships (2) and (4) we find

$$\begin{aligned}\sigma_1 = \rho c\left[-f_1(x-ct)+l_1(x+ct)\right]\\ -\rho a A\left[g_1(x-at)+k_1(x+at)\right]\\ +\varepsilon\rho c^2\left\{\frac{Aac^2}{(c^2-a^2)^2}\left[g_0(x-at)+k_0(x+at)\right]\right.\\ +\frac{A}{2ca\alpha}\left[f_0(x-ct)-l_0(x+ct)\right]\\ \left.-\frac{A}{2a\alpha}(f_{0,\xi}+l_{0,\eta})t+\frac{Aa^2}{2(c^2-a^2)}(-g_{0,\lambda}+k_{0,\mu})t\right\}, \quad (15)\end{aligned}$$

$$\begin{aligned}q_1 = -æ\left\{a^{-1}\left[-g_1(x-at)+k_1(x+at)\right]\right.\\ +\varepsilon\,\frac{A}{c^2a\alpha^2}\left[f_0(x-ct)+l_0(x+ct)\right]\\ -\varepsilon\,\frac{A}{2\alpha a^2}\left[-g_0(x-at)+k_0(x+at)\right]\\ \left.+\varepsilon\,\frac{A}{2\alpha a}(g_{0,\lambda}+k_{0,\mu})t\right\}. \quad (16)\end{aligned}$$

Thus, considering the zero-order (7)–(10) and first-order (12) and (14)–(16) approximations, the solution to be found could be written as

$$\theta = \theta_0 + \theta_1 + O(\varepsilon^2), \quad (17)$$

$$v = v_0 + v_1 + O(\varepsilon^2), \quad (18)$$

$$\sigma = \sigma_0 + \sigma_1 + O(\varepsilon^2), \quad (19)$$

$$q = q_0 + q_1 + O(\varepsilon^2). \quad (20)$$

Relationships (17)–(20) allow one to solve different boundary-value dynamic problems of coupled thermoelasticity.

III. Thermoelastic Rod Impact Against a Heated Barrier

Suppose that a thermoelastic rod of length L moves with the constant velocity V_0 along the x-axis towards a rigid heated wall with the temperature Θ_1. The temperature of the moving rod is equal to zero for simplicity. Impact occurs at $t = 0$ at the origin of coordinates $x = 0$ (Figure 1a). The rod's lateral surface and its free end are heat-insulated, while at the impact point free heat exchange between the striking rod and the barrier takes place.

In this problem, at the zero-order approximation the unknown functions f_0, l_0, g_0, and k_0 entering in Eqs. (7)–(10) are determined from the following initial and boundary conditions:

$$\begin{aligned}v_0(x,0) = -V_0, \quad \sigma_0(x,0) = 0,\\ \theta_0(x,0) = 0, \quad q_0(x,0) = 0 \quad (0 \le x \le L), \quad (21)\end{aligned}$$

$$v_0(0,t) = 0, \quad q_0(0,t) = -h(\theta_0 - \Theta_1), \quad (22)$$

$$\sigma_0(L,t) = 0, \quad q_0(L,t) = 0, \quad (23)$$

where h is the heat transfer coefficient.

Substituting (7)–(10) in (21)-(23) and assuming that $c < a < 2c$, we obtain

$$\begin{aligned}k_0(x) = g_0(x) = 0,\\ f_0(x) = l_0(x) = -\frac{V_0}{2} \quad (0 \le x < L), \quad (24)\end{aligned}$$

$$\begin{aligned}f_0(-ct) = \frac{2Aah}{æ+ah}\,k_0(at) - l_0(ct) - \frac{Aah}{æ+ah}\,\Theta_1,\\ g_0(-at) = \frac{æ-ah}{æ+ah}\,k_0(at) + \frac{ah}{æ+ah}\,\Theta_1, \quad (25)\end{aligned}$$

$$\begin{aligned}l_0(L+ct) = f_0(L-ct) + \frac{2Aa}{c}\,g_0(L-at),\\ k_0(L+at) = g_0(L-at). \quad (26)\end{aligned}$$

At the first-order approximation, to find the unknown functions f_1, l_1, g_1, and k_1 entering in Eqs. (12) and (14)–(16), it is necessary to utilize the zero initial conditions

$$\begin{aligned}v_1(x,0) = 0, \quad \sigma_1(x,0) = 0,\\ \theta_1(x,0) = 0, \quad q_1(x,0) = 0 \quad (0 \le x \le L), \quad (27)\end{aligned}$$

as well as the following boundary conditions

$$v_1(0,t) = 0, \quad q_1(0,t) = -h\theta_1, \quad (28)$$

$$\sigma_1(L,t) = 0, \quad q_1(L,t) = 0, \quad (29)$$

Substituting (12) and (14)–(16) in (27)-(29) with due accounf for (24)-(26), we obtain

$$\begin{aligned}-k_1(x) = g_1(x) = \frac{V_0}{2A}\,s,\\ f_1(x) = l_1(x) = -\frac{V_0}{2} \quad (0 \le x < L); \quad (30)\end{aligned}$$

$$f_1(-ct) = -\left(\frac{2b_1}{\delta}+1\right)l_1(ct) + \frac{2Ab_2}{\delta}\,k_1(at) + \frac{\Theta_1}{\delta},$$

$$Ag_1(-at) = \frac{2b_1}{\delta}\, l_1(ct) - \left(\frac{2b_2}{\delta} - 1\right) Ak_1(t) - \frac{\Theta_1}{\delta}, \tag{31}$$

$$l_1(L+ct) = (1+sg)f_1(L-ct) + 2gA\, g_1(L-at),$$

$$Ak_1(L+at) = 2s\, f_1(L-ct) + (1+sg)A\, g_1(L-at), \tag{32}$$

where $\delta = b_2(1+d_2) - b_1(1+d_1)$,

$$d_1 = \frac{æ}{ch},\;\; d_2 = \frac{æ}{ah},\;\; g = \frac{a}{c},\;\; b_1 = \varepsilon\, \frac{c}{\alpha(a\alpha^2 - c^2)},$$

$$b_2 = -\frac{1}{\alpha ac^2}\left[a^2 - c^2 + \varepsilon\, \frac{c^2(a^2+c^2)}{2(a^2-c^2)}\right],$$

and the parameter s involving the small coupling parameter ε takes the form

$$s = -\varepsilon \frac{a^2c^2}{(a^2-c^2)^2}. \tag{33}$$

Formulas (24)-(26) and (30)-(32) allow one to find all desired functions of the problem under consideration (17)-(20) for any magnitude of the coordinate $0 \le x \le L$ and for arbitrary instant of time $0 \le t \le t_{\text{cont}}$, where t_{cont} is the duration of contact. Really, substituting (24)-(26) in (7)-(10) and (30)-(32) in (12) and (14)–(16), putting $t_2/t_1 = 3/2$, where $t_1 = 2La^{-1}$ and $t_2 = 2Lc^{-1}$, we could find the desired functions at the characteristic instants of time:

I: at $t = \frac{1}{2}t_1$ the faster wave, i.e., thermal wave Σ_1 traveling with the velocity a, reaches the free end of the rod:

I a) $0 < x < \frac{2}{3}L$

$$\begin{aligned}
v &= 0,\\
\sigma &= -\rho c V_0 - \frac{\gamma\Theta_1}{(1+g^{-1})(1+d_2)}\\
&+ s\rho a V_0\left(1 - g^{-1}\right)\frac{(d_2+g^{-1})}{(1+d_2)},\\
\theta &= \frac{\Theta_1}{1+d_2}\\
&- \frac{s}{1+d_2}\left[\Theta_1 g^{-1} + V_0 d_2\left(1-g^{-1}\right)\frac{a^2-c^2}{\alpha ac^2}\right],\\
q &= \frac{\Theta_1 d_2 h}{1+d_2}\\
&- s\frac{d_2 h}{1+d_2}\left[\Theta_1 - V_0\left(1-g^{-1}\right)\frac{a^2-c^2}{\alpha ac^2}\right];
\end{aligned} \tag{34}$$

I b) $\frac{2}{3}L < x < L$

$$\begin{aligned}
v &= -V_0 + \frac{A\Theta_1}{1+d_2} - sV_0\frac{d_2+g^{-1}}{1+d_2},\\
\sigma &= -\frac{\gamma\Theta_1}{(1-g^{-2})(1+d_2)} + s\rho a V_0\frac{d_2+g^{-1}}{1+d_2},\\
\theta &= \frac{\Theta_1}{1+d_2} - sV_0\frac{(d_2+g^{-1})(a^2-c^2)}{(1+d_2)\alpha ac^2},\\
q &= \frac{d_2\Theta_1 h}{1+d_2}\\
&+ sV_0 h d_2\frac{(d_2+g^{-1})(a^2-c^2)}{(1+d_2)\alpha ac^2};
\end{aligned} \tag{35}$$

II: at $t = \frac{1}{2}t_2$ the slower wave, i.e., elastic wave Σ_2 traveling with the velocity c, reaches the free end of the rod:

II a) $0 < x < \frac{1}{2}L$

Within this segment, the solution coincides with that in the case Ia, i.e., it has the form of (34).

II b) $\frac{1}{2}L < x < \frac{2}{3}L$

$$\begin{aligned}
v &= -\frac{A\Theta_1}{1+d_2} + s\left[-\frac{2A\Theta_1 g}{1+d_2} + V_0\frac{d_2+g^{-1}}{1+d_2}\right],\\
\sigma &= -\rho c V_0 - \frac{\gamma(2-g^{-1})\Theta_1}{(1-g^{-2})(1+d_2)}\\
&- s\rho c\left[\frac{2A\Theta_1 g^2}{1+d_2} - V_0\frac{(d_2+g^{-1})(2g^{-1}-1)}{1+d_2}\right],\\
\theta &= \frac{2\Theta_1}{1+d_2} + s\left[\frac{\Theta_1(2g^2-1)g^{-1}}{1+d_2}\right.\\
&- \left.V_0\frac{a^2-c^2}{\alpha ac^2}\left(\frac{2(d_2+g^{-1})}{1+d_2} - g^{-1}\right)\right],\\
q &= -shd_2\left[V_0\frac{a^2-c^2}{\alpha ac^2} + \frac{\Theta_1(1+2g)}{1+d_2}\right];
\end{aligned} \tag{36}$$

II c) $\frac{2}{3}L < x < L$

$$\begin{aligned}
v &= \frac{A(2g-1)\Theta_1}{1+d_2} + s\left[\frac{2A\Theta_1 g(g-1)}{1+d_2}\right.\\
&- \left.V_0\frac{(d_2+g^{-1})(2g-1)}{1+d_2}\right],\\
\sigma &= -\rho c V_0 + \frac{g^{-1}\gamma\Theta_1}{(1-g^{-2})(1+d_2)}\\
&- s\rho c V_0\frac{d_2+g^{-1}}{1+d_2},
\end{aligned}$$

$$\begin{aligned}\theta &= \frac{2\Theta_1}{1+d_2} + s\left[\frac{\theta_1 g^{-1}}{1+d_2}\left(2g^2-2g-1\right)\right.\\ &- \left. V_0\frac{a^2-c^2}{\alpha a c^2}\left(\frac{2(d_2+g^{-1})}{1+d_2}-g^{-1}\right)\right],\\ q &= shd_2\left[V_0\frac{a^2-c^2}{\alpha a c^2}-\frac{\Theta_1}{1+d_2}\right]; \qquad (37)\end{aligned}$$

III: at $t=\frac{7}{8}t_1$ the reflected thermal wave Σ_1^1, which is generated when the incident thermal wave Σ_1 is reflected from the rod's free end, is approaching the place of contact:

III a) $0<x<\frac{1}{4}L$

Within this segment, the solution coincides with those in the cases Ia and IIa, i.e., it has the form of (34).

III b) $\frac{1}{4}L<x<\frac{1}{2}L$

Within this segment, the solution coincides with that in the case IIb, i.e., it has the form of (36).

III c) $\frac{1}{2}L<x<\frac{3}{4}L$

Within this segment, the solution coincides with that in the case IIc, i.e., it has the form of (37).

III d) $\frac{3}{4}L<x<\frac{5}{6}L$

$$\begin{aligned}v &= \frac{A(2g-1)\Theta_1}{1+d_2} + s\left[\frac{2A\Theta_1}{1+d_2}\left(1-g+g^2\right)\right.\\ &- \left. V_0\left(\frac{(d_2+g^{-1})(2g-1)}{1+d_2}+2\right)\right],\\ \sigma &= -\rho c V_0 + \frac{ca^{-1}\gamma\Theta_1}{(1-g^{-2})(1+d_2)}\\ &+ s\rho c\left[\frac{2A\Theta_1 g}{1+d_2}+V_0\left(\frac{d_2+g^{-1}}{1+d_2}-2g\right)\right],\\ \theta &= \frac{2\Theta_1}{1+d_2} - s\left\{\frac{\Theta_1 g^{-1}}{1+d_2}\left(1+4g-2g^2\right)\right.\\ &+ \left. V_0\frac{a^2-c^2}{\alpha a c^2}\left[\frac{2(d_2+g^{-1})}{1+d_2}-\left(2+g^{-1}\right)\right]\right\},\\ q &= shd_2\left[\frac{\Theta_1}{1+d_2}\right.\\ &+ \left. V_0\frac{a^2-c^2}{\alpha a c^2}\left(1+\frac{d_2+g^{-1}}{1+d_2}\right)\right]; \qquad (38)\end{aligned}$$

III e) $\frac{5}{6}L<x<L$

$$\begin{aligned}v &= V_0 + \frac{2A(g-1)\Theta_1}{1+d_2}\\ &+ \frac{2s}{1+d_2}(g-1)^2\left[A\Theta_1+V_0 g^{-1}\right],\\ \sigma &= 0,\\ \theta &= \frac{2\Theta_1}{1+d_2}+\frac{2s}{1+d_2}\left[\Theta_1(g-2)\right.\\ &+ \left. V_0\frac{(a^2-c^2)(1-g^{-1})}{\alpha a c^2}\right],\\ q &= 0; \qquad (39)\end{aligned}$$

IV: at $t=\frac{9}{8}t_1$ the reflected thermal wave Σ_2^1, which is generated when the incident elastic wave Σ_2 is reflected from the rod's free end, and elastic wave Σ_1^2, which is generated when the incident thermal wave Σ_1 is reflected from the rod's free end, are approaching the place of contact:

IV a) $0<x<\frac{1}{6}L$

$$\begin{aligned}v &= 0,\\ \sigma &= -\rho c V_0 - \frac{\left[1+g^{-1}+d_2(3-g^{-1})\right]\gamma\Theta_1}{(1-g^{-2})(1+d_2)^2}\\ &- s\rho c\frac{4A\Theta_1 g^2(d_2+g^{-1})}{(1+d_2)^2}\\ &+ s\rho c V_0\frac{g(d_2+g^{-1})\left[1+g^{-1}+d_2(3-g^{-1})\right]}{(1+d_2)^2},\\ \theta &= \frac{\Theta_1(1+3d_2)}{(1+d_2)^2}\\ &- sV_0\frac{a^2-c^2}{\alpha a c^2}\left[\frac{(d_2+g^{-1})(1+3d_2)}{(1+d_2)^2}-g^{-1}\right],\\ &+ s\frac{\Theta_1}{1+d_2}\left[\frac{2d_2}{1+d_2}(2g-1)-g^{-1}\right]\\ q &= \frac{\Theta_1 h d_2(d_2-1)}{(1+d_2)^2}\\ &- shd_2V_0\frac{a^2-c^2}{\alpha a c^2}\left[\frac{(d_2+g^{-1})(d_2-1)}{(1+d_2)^2}-1\right]\\ &- shd_2\frac{\Theta_1(d_2-1+4g)}{(1+d_2)^2}; \qquad (40)\end{aligned}$$

IV b) $\frac{1}{6}L<x<\frac{1}{4}L$

$$\begin{aligned}v &= \frac{2A(g-1+d_2g)\Theta_1}{(1+d_2)^2}+s\frac{2Ag(g-2+d_2g)\Theta_1}{(1+d_2)^2}\\ &- sV_0\left[\frac{2(d_2+g^{-1})(g-1+d_2g)}{(1+d_2)^2}-\frac{1-g^{-1}}{1+d_2}\right],\\ \sigma &= -\rho c V_0 + \frac{\left[1+g^{-1}-d_2(1-g^{-1})\right]\gamma\Theta_1}{(1-g^{-2})(1+d_2)^2}\\ &+ s\rho c\frac{2Ag^2(1-d_2)\Theta_1}{(1+d_2)^2}\end{aligned}$$

$$- \quad s\rho c V_0 \frac{g(d_2+g^{-1})\left[1+g^{-1}-d_2(1-g^{-1})\right]}{(1+d_2)^2},$$

$$\theta = \frac{\Theta_1(1+3d_2)}{(1+d_2)^2} + s\frac{\Theta_1}{1+d_2}\left[\frac{4gd_2}{1+d_2} - g^{-1} - 2\right]$$
$$- \quad sV_0\,\frac{a^2-c^2}{\alpha a c^2}\left[\frac{(d_2+g^{-1})(1+3d_3)}{(1+d_2)^2} - g^{-1}\right],$$

$$q = \frac{\Theta_1 h d_2(d_2-1)}{(1+d_2)^2} + s\,\frac{\Theta_1 h d_2}{1+d_2}\left[\frac{2g(d_2-1)}{1+d_2} - 1\right]$$
$$- \quad s h d_2 V_0\,\frac{a^2-c^2}{\alpha a c^2}\left[\frac{(d_2+g^{-1})(d_2-1)}{(1+d_2)^2} - 1\right]; \quad (41)$$

IV c) $\frac{1}{4}L < x < \frac{1}{2}L$

Within this segment, the solution coincides with that in the case IIId, i.e., it has the form of (38).

IV d) $\frac{1}{2}L < x < L$

Within this segment, the solution coincides with that in the case IIIe, i.e., it has the form of (39).

V: at $t = \frac{11}{8}t_1$ the reflected elastic wave Σ_2^2, which is generated when the incident elastic wave Σ_2 reached the rod's free end, is approaching the place of contact:

V a) $0 < x < \frac{1}{6}L$

$$v = 0,$$

$$\sigma = -\rho c V_0 + \frac{\gamma[3-g^{-1}+d_2(1+g^{-1})]\Theta_1}{(1-g^{-2})(1+d_2)^2}$$
$$+ \quad s\rho c\frac{2A\Theta_1}{(1+d_2)^2}\left[2g(1+d_2)+(g-1)(g-4)\right]$$
$$- \quad s\rho c V_0 \frac{g(d_2+g^{-1})}{(1+d_2)^2}\left[3-g^{-1}+d_2(1+g^{-1})\right]$$
$$+ \quad 4s\rho c V_0\frac{g-1}{1+d_2},$$

$$\theta = \frac{\Theta_1(1+3d_2)}{(1+d_2)^2}$$
$$- \quad s\frac{\Theta_1}{1+d_2}\left[\frac{2d_2}{1+d_2}\left(5-2g^{-1}\right)+\frac{1}{g}\right]$$
$$- \quad sV_0\,\frac{a^2-c^2}{\alpha a c^2}\left[\frac{(d_2+g^{-1})(1+3d_2)}{(1+d_2)^2}\right.$$
$$\left. - \quad g^{-1}\left(1+\frac{4d_2 g}{1+d_2}\right)\right],$$

$$q = \frac{\Theta_1 h d_2(d_2-1)}{(1+d_2)^2}$$
$$- \quad s\,\frac{\Theta_1 h d_2\left[d_2-1+4(g-2)\right]}{(1+d_2)^2}$$
$$- \quad s h d_2 V_0\,\frac{a^2-c^2}{\alpha a c^2}\left[\frac{3-d_2}{1+d_2}\right.$$
$$\left. + \quad \frac{(d_2+g^{-1})(d_2-1)}{(1+d_2)^2}\right]; \quad (42)$$

V b) $\frac{1}{6}L < x < \frac{1}{4}L$

$$v = V_0 + \frac{A(2g-1)\Theta_1}{1+d_2}$$
$$+ \quad s\frac{2A\Theta_1}{(1+d_2)^2}\left[g(g-1)(1+d_2)+4\right]$$
$$- \quad sV_0\left[\frac{(d_2+g^{-1})(2g-1)}{1+d_2}+\frac{4+d_2}{1+d_2}-2g\right],$$

$$\sigma = \frac{\gamma\left(1-2g^{-1}-d_2\right)\Theta_1}{(1-g^{-2})(1+d_2)^2}$$
$$- \quad s\rho a\frac{2A\Theta_1}{(1+d_2)^2}\left[(g-1)(1+d_2)-2(g-1)+4\right]$$
$$- \quad s\rho a V_0\left[\frac{(1-2g^{-1}-d_2)(d_2+g^{-1})}{(1+d_2)^2}-\frac{1-d_2}{1+d_2}\right],$$

$$\theta = \frac{\Theta_1(1+3d_2)}{(1+d_2)^2}$$
$$+ \quad s\frac{2\Theta_1}{1+d_2}\left[\frac{4-2g+g^{-1}}{1+d_2}+2g-3\right]$$
$$- \quad sV_0\,\frac{a^2-c^2}{\alpha a c^2}\left[\frac{(d_2+g^{-1})(1+3d_2)}{(1+d_2)^2}-\frac{4d_2}{1+d_2}\right],$$

$$q = \frac{\Theta_1 h d_2(d_2-1)}{(1+d_2)^2}$$
$$+ \quad s\frac{2\Theta_1 h d_2\left[(d_2+1)(g-1)+5-2g\right]}{(1+d_2)^2}$$
$$- \quad s h d_2 V_0\,\frac{a^2-c^2}{\alpha a c^2}\left[\frac{(d_2+g^{-1})}{1+d_2}-2\right]\frac{d_2-1}{1+d_2}; \quad (43)$$

V c) $\frac{1}{4}L < x < \frac{1}{2}L$

$$v = V_0 + \frac{A(2g-1)\Theta_1}{1+d_2} - s\frac{2A\Theta_1}{1+d_2}\left(g^2+g-1\right)$$
$$- \quad sV_0\left[\frac{(d_2+g^{-1})(2g-1)}{1+d_2}-2(g-1)\right],$$

$$\sigma = \frac{\gamma\left(1-2g^{-1}-d\right)\Theta_1}{(1-g^{-2})(1+d_2)^2}\left(1+2sg\right)$$
$$- \quad s\rho a V_0\left[\frac{(1-2g^{-1}-d_2)(d_2+g^{-1})}{(1+d_2)^2}+1\right],$$

$$\theta = \frac{\Theta_1(1+3d_2)}{(1+d_2)^2}+s\frac{2\Theta_1}{1+d}\left[2(g-1)-\frac{2g-g^{-1}}{1+d}\right]$$
$$- \quad sV_0\,\frac{a^2-c^2}{\alpha a c^2}\left[\frac{(d_2+g^{-1})(1+3d_2)}{(1+d_2)^2}-3\right],$$

$$\begin{aligned} q &= \frac{\Theta_1 h d_2(d_2-1)}{(1+d-2)^2} + s\frac{2\Theta_1 h d_2(d_2-1+2g)}{(1+d_2)^2} \\ &- shd_2V_0\,\frac{a^2-c^2}{\alpha a c^2}\left[\frac{(d_2+g^{-1})(d_2-1)}{(1+d_2)^2}-2\right]; \end{aligned} \tag{44}$$

V d) $\frac{1}{2}L < x < \frac{3}{4}L$

$$\begin{aligned} v &= V_0 + \frac{A[2g-3+d_2(2g-1)]\Theta_1}{(1+d_2)^2} \\ &- s\frac{2A\Theta_1}{1+d_2}\left(g^2-g+1+\frac{2g}{1+d_2}\right) \\ &+ sV_0\,[2(g-1) \\ &- \left.\frac{(d_2+g^{-1})[2g-3+d_2(2g^{-1}-1)]}{(1+d_2)^2}\right], \end{aligned}$$

$$\begin{aligned} \sigma &= \frac{\gamma\,(1-d_2)\,\Theta_1}{(1-g^{-2})(1+d_2)^2}(1+2sg) \\ &- s\rho c V_0\left[\frac{g\,(1-d_2)\,(d_2+g^{-1})}{(1+d_2)^2}\right. \\ &+ \left.\frac{2\,(1-g^{-1})}{1+d_2}\right], \end{aligned}$$

$$\begin{aligned} \theta &= \frac{\Theta_1(1+3d_2)}{(1+d_2)^2} \\ &+ s\,\frac{4\Theta_1 g\,[d_2(1-g^{-1})-g^{-1}]}{(1+d_2)^2} \\ &- sV_0\,\frac{a^2-c^2}{\alpha a c^2}\left[\frac{(d_2+g^{-1})(1+3d_2)}{(1+d_2)^2}-2\right], \end{aligned}$$

$$\begin{aligned} q &= \frac{\Theta_1 h d_2(d_2-1)}{(1+d_2)^2}(1+2sg) \\ &+ shd_2V_0\,\frac{a^2-c^2}{\alpha a c^2}\,[2(1+g) \\ &- \left.\frac{(d_2+g^{-1})(d_2-1)}{(1+d_2)^2}\right]; \end{aligned} \tag{45}$$

V e) $\frac{3}{4}L < x < L$

Within this segment, the solution coincides with that in the case IIIe, i.e., it has the form of (39).

VI: at $t = \frac{3}{2}t_1 = t_2$, the reflected quasi-elastic wave Σ_2^2 arrives at the place of contact.

VI a) $0 < x < \frac{1}{3}L$

$$\begin{aligned} v &= V_0 - \frac{A\Theta_1}{1+d_2}(1+2sg) \\ &+ sV_0\left[\frac{d_2+g^{-1}}{1+d_2}+2g\right], \end{aligned}$$

$$\begin{aligned} \sigma &= \frac{\gamma\,(3-2g^{-1}+d_2)\,\Theta_1}{(1-g^{-2})(1+d_2)^2} \\ &+ s\rho a\frac{2A\Theta_1}{(1+d_2)^2}\left[1+d_2+2(1-g^{-1})(g-2)\right] \\ &- s\rho a V_0\left[\frac{(3-2g^{-1}+d_2)(d_2+g^{-1})}{(1+d_2)^2}\right. \\ &- \left.\frac{4(1-g^{-1})}{1+d_2}+2\right], \end{aligned}$$

$$\begin{aligned} \theta &= \frac{\Theta_1(1+3d_2)}{(1+d_2)^2} \\ &+ s\,\frac{2\Theta_1}{1+d_2}\left[2g-3+\frac{g^{-1}-2(g-2)}{1+d_2}\right] \\ &- sV_0\,\frac{a^2-c^2}{\alpha a c^2}\left[\frac{(d_2+g^{-1})(1+3d_2)}{(1+d_2)^2}\right. \\ &- \left.\frac{4d_2}{1+d_2}\right], \end{aligned}$$

$$\begin{aligned} q &= \frac{\Theta_1 h d_2(d_2-1)}{(1+d_2)^2} \\ &- shd_2V_0\,\frac{a^2-c^2}{\alpha a c^2}\left[\frac{(d_2+g^{-1})(d_2-1)}{(1+d_2)^2}-1\right] \\ &- s\,\frac{2\Theta_1 h d_2(d_2-4+g)}{(1+d_2)^2}; \end{aligned} \tag{46}$$

VI b) $\frac{1}{3}L < x < \frac{1}{2}L$

Within this segment, the solution coincides with that in the case Vb, i.e., it has the form of (43).

VI c) $\frac{1}{2}L < x < \frac{2}{3}L$

Within this segment, the solution coincides with that in the case Vc, i.e., it has the form of (44).

VI d) $\frac{2}{3}L < x < L$

Within this segment, the solution coincides with that in the case Vd, i.e., it has the form of (45).

To find the duration of contact t_{cont}, let us investigate the time-dependence of the contact stress $\sigma(0,t)$ putting $x=0$ in (19), i.e., considering (9) and (15) at $x=0$,

$$\begin{aligned} \sigma(0,t) &= \rho[-cf_0(-ct)+cl_0(ct) \\ &- Aag_0(-at)-Aak_0(at)] \\ &+ \rho[-cf_1(-ct)+cl_1(ct) \\ &- Aag_1(-at)-Aak_1(at)] \\ &+ \varepsilon\rho c^2\left\{\frac{Aac^2}{(c^2-a^2)^2}\,[g_0(-at)+k_0(at)]\right. \\ &+ \frac{A}{2ca\alpha}[f_0(-ct)-l_0(ct)] \\ &- \frac{A}{2a\alpha}(f_{0,\xi}+l_{0,\eta})t \\ &+ \left.\frac{Aa^2}{2(c^2-a^2)}(-g_{0,\lambda}+k_{0,\mu})t\right\}. \end{aligned} \tag{47}$$

Considering (24)-(26) and (30)-(46), in the case of small coupling between the strain and temperature fields we find from (47)

$$\begin{aligned}
\sigma &= -\rho c V_0 - \frac{\gamma\Theta_1}{(1+g^{-1})(1+d_2)} \\
&+ s\rho a V_0 (1-g^{-1})\frac{(d_2+g^{-1})}{(1+d_2)} < 0 \quad (0<t<t_1),
\end{aligned}$$

$$\begin{aligned}
\sigma &= -\rho c V_0 - \frac{[1+g^{-1}+d_2(3-g^{-1})]\gamma\Theta_1}{(1-g^{-2})(1+d_2)^2} \\
&+ s\rho a \frac{(d_2+g^{-1})}{(1+d_2)^2}\left[V_0\left(1+g^{-1}+d_2(3-g^{-1})\right)\right. \\
&- \left.4A\Theta_1 g\right] < 0 \quad \left(t_1 < t < \frac{1}{2}t_1 + \frac{1}{2}t_2\right),
\end{aligned}$$

$$\begin{aligned}
\sigma &= -\rho c V_0 + \frac{\gamma\left[3-g^{-1}+d_2(1+g^{-1})\right]\Theta_1}{(1-g^{-2})(1+d_2)^2} \\
&+ s\rho c\frac{2A\Theta_1}{(1+d_2)^2}\left[2g(1+d_2)+(g-1)(g-4)\right] \\
&- s\rho a V_0\left[\frac{(d_2+g^{-1})}{(1+d_2)^2}\left[3-g^{-1}+d_2(1+g^{-1})\right]\right. \\
&- \left.\frac{4(1-g^{-1})}{1+d}\right] \quad \left(\frac{1}{2}t_1+\frac{1}{2}t_2 < t < t_2\right),
\end{aligned}$$

$$\begin{aligned}
\sigma &= \frac{\gamma(3-2g^{-1}+d_2)\Theta_1}{(1-g^{-2})(1+d_2)^2} \\
&+ s\rho a\frac{2A\Theta_1}{(1+d_2)^2}\left[1+d_2+2(1-g^{-1})(g-2)\right] \\
&- s\rho a V_0\left[\frac{(d_2+g^{-1})(3-2g^{-1}+d_2)}{(1+d_2)^2}\right. \\
&- \left.\frac{4(1-g^{-1})}{1+d_2}+2\right] > 0 \quad (t=t_2). \qquad (48)
\end{aligned}$$

Reference to (48) shows that the rod's rebound from the rigid wall can occur in two cases: when $t=\frac{1}{2}t_1+\frac{1}{2}t_2$, if the thermoelastic parameters entering into the relationship for the contact stress vanish it, or when $t=t_2$. At the moment $t=\frac{1}{2}t_1+\frac{1}{2}t_2$, as distinct from the uncoupled case [5], two reflected waves return at a time to the impact point, namely: the quasi-elastic waveΣ_1^2 and quasi-thermal wave Σ_2^1, which are generated as a result of the action at the rod's free end of the incident quasi-thermal Σ_1 and quasi-elastic $\Sigma_2$waves, respectively. If the rebound does not occur at this moment of time, then the collision will terminate necessarily at time of $t=t_2$, i.e., when the slowest reflected quasi-elastic wave Σ_2^2 reaches the place of contact.

If we neglect coupling, i.e., put $s=0$, then formulas (48) go over into formulas (41) and (43) of [5].

IV. Numerical Example

For illustrating the above analysis of the motion of incident and reflected waves, the schemes of the wave fronts' location are presented in Figures 1a-g at the instants of time when the surfaces of discontinuities during their propagation reach the free end of the colliding rod or come in close proximity to the place of contact. Solid and dashed lines in all diagrams refer to the cases of uncoupled and coupled thermoelasticity, respectively.

For the numerical analysis, it is convenient to rewrite the longitudinal coordinate dependence of the velocity, stress, temperature, and heat flowing (34)-(46) in the dimensionless form introducing the following dimensionless values:

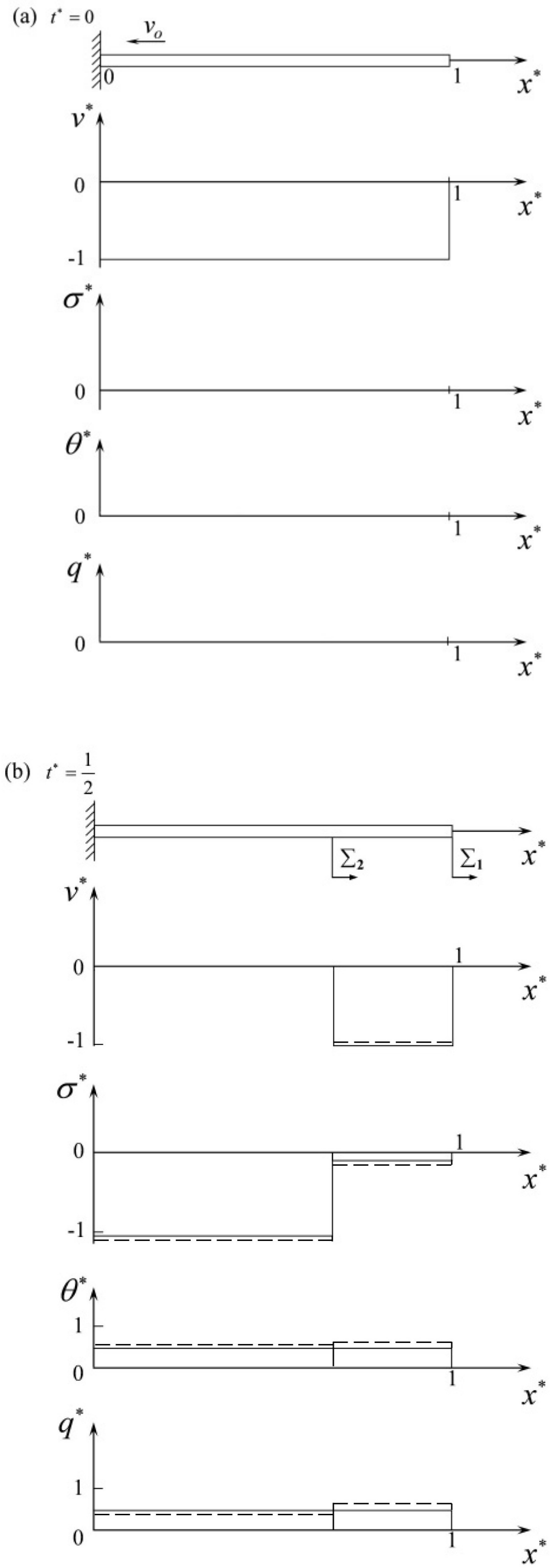
$$x* = \frac{x}{L}, \quad t* = \frac{t}{t_1} \quad v* = \frac{v}{V_0}$$

$$\sigma* = \frac{\sigma}{\rho c V_0}, \quad \theta* = \frac{\theta}{\Theta_1}, \quad q* = \frac{q}{\Theta_1 h}$$

Calculations according to relationships (34)-(48) written in the dimensionless form have been carried out at the following magnitudes of the dimensionless values: $\alpha\Theta_1 c/V_0 = 0.1$ and $d_2 = 1$. Then the wave surface Σ_1 will be the first to reach the free end $x=L$ at the moment $t=\frac{1}{2}t_1$. As this takes place, two reflected waves Σ_1^1 (thermal from thermal) and Σ_1^2 (elastic from thermal) are generated at a time.

Reference to numerical investigation shows that the contact stress remains its sign at $t=t_1$, i.e., at the moment of arrival at the contact point of the reflected thermal wave Σ_1^1, which is generated when the incident thermal wave Σ_1 arrives at the free rod's end. At time of $t=\frac{1}{2}t_1+\frac{1}{2}t_2$, the reflected elastic wave Σ_1^2, which is generated at the free rod's end from the incident thermal wave Σ_1, returns to the impact point. At this instant the sign of the contact stress is determined by the magnitude of $\delta = \rho c V_0(\gamma\Theta_1)^{-1}$: if $\delta \le \nu$, where $\nu = (1-c^2a^{-2})^{-1}(1+d_2)^{-2}[3-ca^{-1}+d_2(1+ca^{-1})]$, then the contact stress vanishes to zero or becomes positive, resulting in the rebound of the rod from the rigid heated wall, but if $\delta > \nu$, then the collision terminates at time of $t=t_2$ (the instant of arrival at the contact point of the elastic wave Σ_2^2 reflected from the free end when the incident elastic wave Σ_2 reaches it).

Reference to Figure 1d shows that in the case of coupling thermoelasticity, a new wave of small amplitude generates when the elastic wave Σ_2 reaches the free end of the rod, namely: the reflected thermal wave Σ_2^1, which occurrence is connected by variation in the rod's temperature during its deformation in the case of weak coupling between the strain and temperature fields. This wave disappears if coupling is



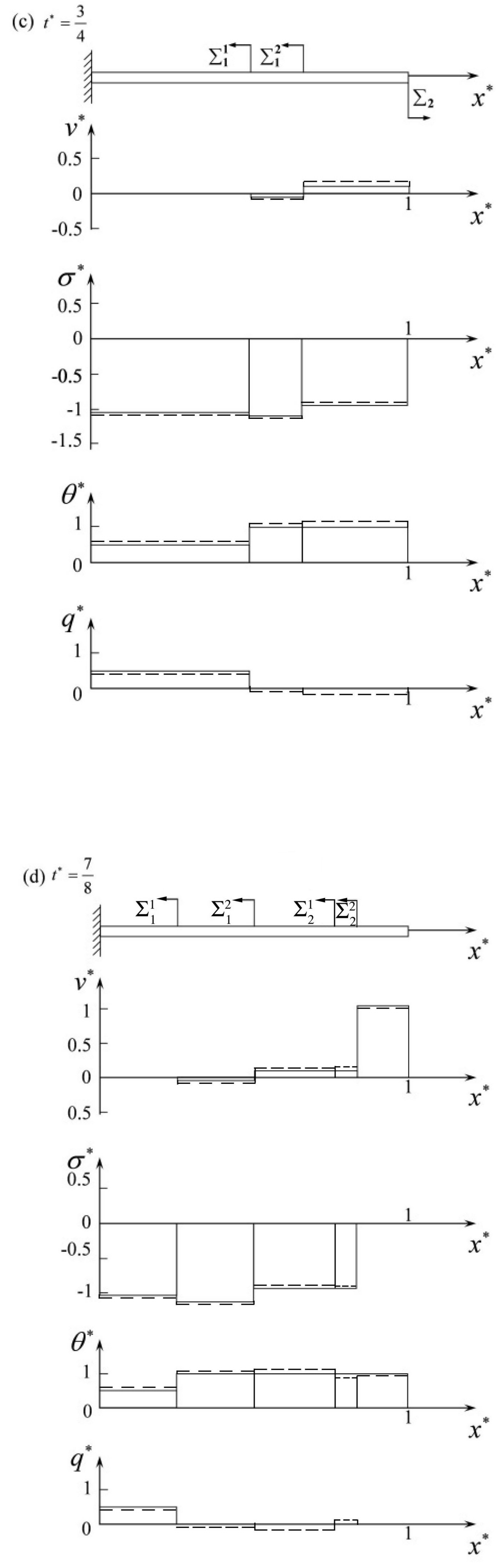
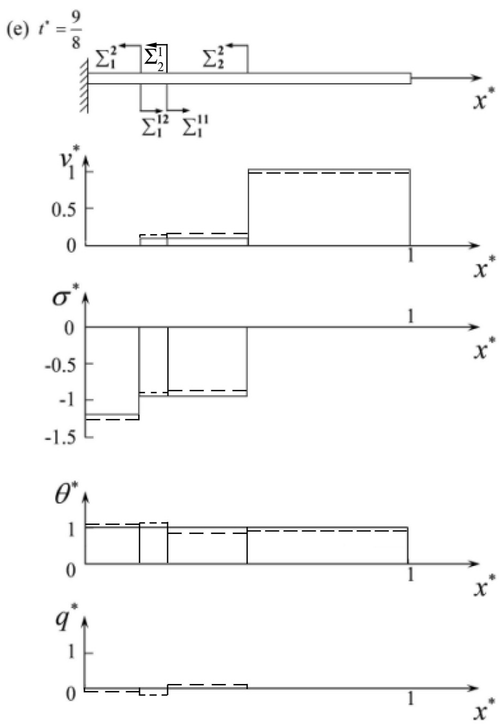
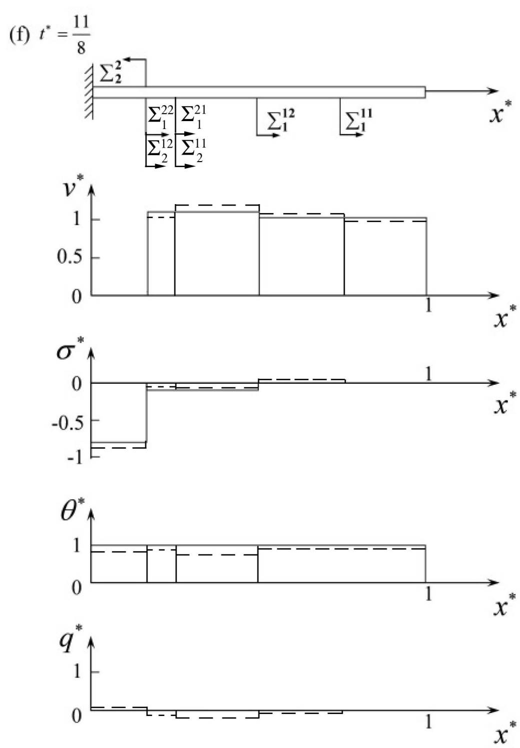


Fig. 1 Schemes of the wave fronts' location and the longitudinal coordinate dependence of the velocity, stress, temperature, and heat flowing at the characteristic instants of time





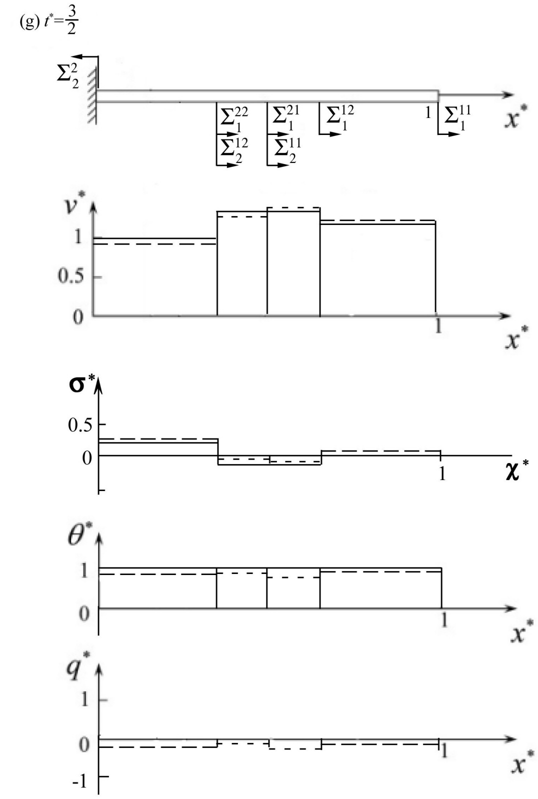


Fig. 1 Continued

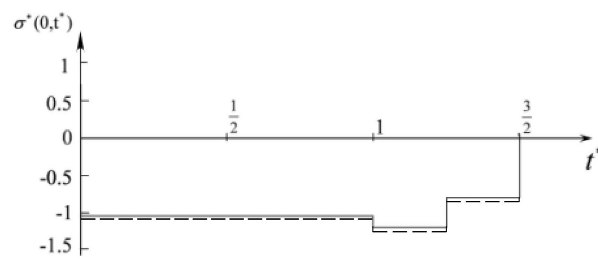


Fig.2 The dimensionless time dependence of the dimensionless contact stress

neglected. The wave Σ_2^1 arriving at the place of contact generates, in its turn, two another wave surfaces of small discontinuity of the order of ε, which reflect from the rigid wall: the thermal wave Σ_2^{11} and elastic wave Σ_2^{12}.

From Figures 1b-1g it is evident that all other incident and reflected waves are the same as in the uncoupled case, while coupling weakly influences the magnitudes of discontinuities of the values under consideration on these wave fronts either slightly increasing or decreasing their amounts as compared with the uncoupled case.

From Figure 2 illustrating the time dependence of the contact stress, it is seen that in the given example the rod's rebound occurs at $t_{\text{cont}} = t_2$.

V. Conclusion

Thus, it has been shown how the D'Alembert method could be used for solving dynamic boundary-value problems of coupled generalized thermoelasticity with due account for weak coupling between the temperature and strain fields. The solution of D'Alembert's type involving four arbitrary functions is found for the set of equations describing the dynamic behavior of a thermoelastic rod using Green-Naghdy theory. This solution is used for solving the problem of impact of a thermoelastic rod against a heated rigid wall, but it could be generalized also for the case of the collision of two thermoelastic rods.

The procedure proposed enables one to construct the longitudinal coordinate dependence of the desired functions at any fixed instant of the time beginning from the moment of the rod's collision with the wall up to the moment of its rebound, i.e., to obtain the analytical solution in the closed form for the main functions showing the distribution of the thermoelastic impact characteristics along the rod.

Based on the detailed analytical treatment, it has been shown that small coupling between the strain and temperature fields results in the generation of a new shock wave of small amplitude Σ_2^1, namely, the thermal wave reflected from the incident elastic wave at the free rod's end.

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