Influence of a Bent or Battered Joint or Welding on the Dynamic Loads due to the Non-Suspended Masses of Railway Vehicles

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Abstract— Train circulation is a random dynamic phenomenon and, according to the different frequencies of the loads it imposes, there exists the corresponding response of track superstructure. Random dynamic phenomena are generally approached through the probability of occurrence (for stochastic processes see [1]). The railway track is modeled as a continuous beam on elastic support. At the moment when an axle of a railway vehicle passes from the location of a support point of a rail, that is a sleeper, a random dynamic load is applied on the sleeper. The theoretical approach for the estimation of the dynamic loading of a sleeper demands the analysis of the total load acting on the sleeper to individual component loads-actions, which, in general, can be divided into: the static component of the load, and the relevant to it reaction/action per support point of the rail (sleeper), the semi-static component of the load, and the relevant to it reaction/action per support point of the rail (sleeper) and the dynamic component of the load, and the relevant to it reaction/action per support point of the rail (sleeper). The motion of a railway vehicle on the rail running table –of the railway track– is described by formulas and it is illustrated through diagrams which have the form of a "signal". It is a random/ stochastic dynamic phenomenon. The general equation that describes the motion is the second order differential equation (of motion). In the present paper the dynamic component of the Load and the relevant Action/ Reaction on each support point of the rail are investigated through a sensitivity analysis by variating parameters of the second order differential equation of motion of the Non Suspended Masses of the Vehicle ([2], [3]) and specifically the transient response of the reaction/ action on each support point (sleeper) of the rail.

Keywords: *Second Order Differential Equation of Motion, Railway Track, Railway Vehicle, Non Suspended Masses, Suspended Masses, Actions, Reactions, Dynamic Loads, Stiffness, Joints, Weldings.*

I. Introduction - Background

Train circulation is a random dynamic phenomenon and, according to the different frequencies of the loads it imposes, there exists the corresponding response of track superstructure. The theoretical approach for the estimation of the dynamic loading of a sleeper demands the analysis of the total load acting on the sleeper to individual component loads-actions, which, in general, can be divided into: (a) the static component of the load, and the relevant to it reaction/action per support point of the rail (sleeper), (b) the semi-static component of the load, and the relevant to it reaction/action per support point of the rail (sleeper) and (c) the dynamic component of the load, and the relevant to it reaction/action per support point of the rail (sleeper).

The static component of the load on a sleeper, in the classical sense, is the load undertaken by the sleeper when a vehicle axle at standstill is situated exactly above the location of the sleeper. At low frequencies, however, the load is essentially static. The semi-static reaction/action is mainly owed to cant or superelevation deficiency in curves ([3], [4]). The dynamic component of the load of the track depends on the mechanical properties (stiffness, damping) of the system "vehicle-track", and on the excitation caused by the vehicle's motion on the track (Fig. 1).

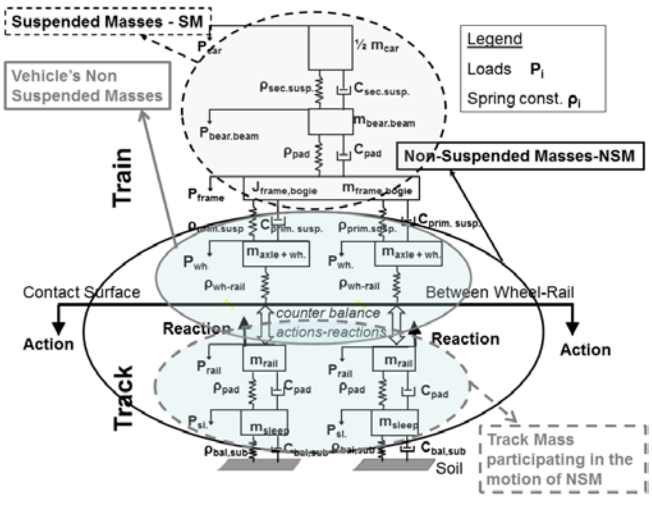

Figure 1 A three-axle bogie with the springs of the primary suspension inside the black elipse with continuous line and the springs of the secondary suspension in the black elipse with the dashed line.

The response of the track to the aforementioned excitation results in the increase of the static and semistatic loads on the superstructure. The dynamic load is primarily caused by the motion of the vehicle's Non-Suspended (Unsprung) Masses [2], which are excited by track geometry defects, and, to a smaller degree, by the effect of the Suspended (Sprung) Masses [3]. In the Non Suspended Masses a section of the Track Mass that participates in their motion is also included as depicted in Fig. 1 [5]. In order to formulate the theoretical equations for the calculation of the dynamic component of the load, the statistical probability of occurrence -in real conditions- should be considered. The general equation that describes the motion is the second order differential equation (of motion) [6].

II. Real Situation On A Railway Track

The railway vehicles consist of (a) the car-body, (b) the primary and the secondary suspension with the bogie in between the axles, and (c) the wheels. The heaviest vehicles are the locomotives which are "motive units" and have electric motors on the axles and/or the frame of the bogie. In Fig. 2 an electric-locomotive with two-axle bogie is depicted while the two axle bogie with the springs of the primary and secondary suspensions is depicted in Fig. 3.

Electric motors are either suspended totally from the frame of the bogie or they are suspended on the frame of the bogie at one end and supported on the axle at the other end. In the second case the electric motor is semi-suspended and a part of it is considered as Non Suspended Mass [2].

Figure 2 An Electric-Locomotive of the Greek railways with two-axle bogies, on a railway track. The two-axle bogie is marked with the grey elipses.

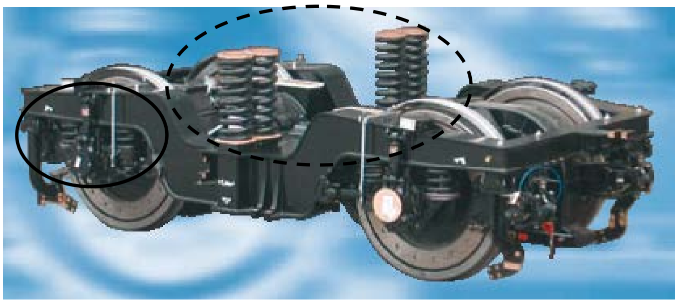

Figure 3 A two-axle bogie for the Electric-Locomotives with the springs of the primary suspension inside the black elipse with continuous line and the springs of the secondary suspension in the black elipse with the dashed line.

If we try to approach mathematically the motion of a vehicle on a railway track, we will end up with the model shown in Fig. 4, where both the vehicle and the railway track are composed of an ensemble of masses, springs and dashpots. As we can observe, the car body is supported by the secondary suspension that includes two sets of "springs-dashpots", seated on the frame of the bogie. The loads are transferred to the truss and the side frames of the bogie. Underneath the bogie there is the primary suspension, through which the bogie is seated onto the carrying axles and the wheels. Below the contact surface, between the wheel and the rail, the railway track also consists of a combination of masses-springs-dampers that simulates the rail, the sleepers, the elastic pad, the rail fastenings, the ballast and the ground.

The masses of the railway vehicle located under the primary suspension (axles, wheels and a percentage of the electric motor weight in the case of locomotives) are the Non Suspended Masses (N.S.M.) of the Vehicle, that act directly on the railway track without any damping at all. Furthermore a section of the track mass (m_{TRACK}) also participates in the motion of the vehicle's Non Suspended Masses, which also highly aggravates the stressing on the railway track (and on the vehicle too).

The remaining vehicle masses are called Suspended Masses (S.M.) or Sprung Masses: the car-body, the secondary suspension, the frame of the bogie, a part of the electric motor's weight and the primary suspension.

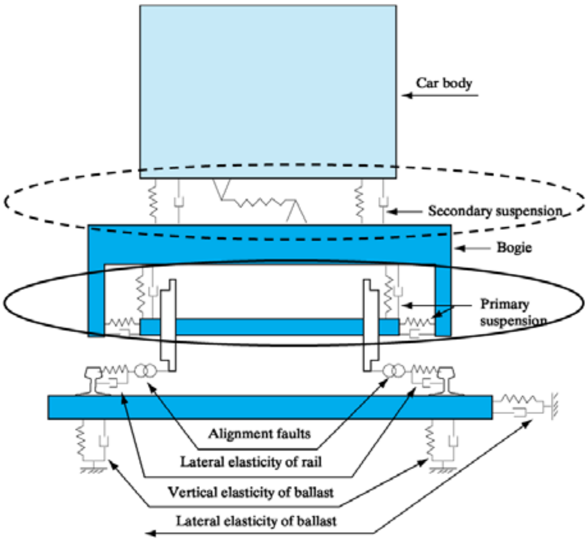

Figure 4 Model of the Railway system "Vehicle –Track", as an ensemble of springs and dashpots.

III. The Loads on a Railway Track: Static, Semi-Static and Dynamic

A. *The Loads on Track*

The system operates based on the classical principles of physics: Action-Reaction between the vehicle and the track. It is a dynamic stressing of random, vertical form.

The loading of the railway track from a moving vehicle consists of:

(a) the static load (static load of vehicle axle), as given by the rolling stock's producer.

(b) the semi-static load (cant/ superelevation deficiency at curves, which results in non-compensated lateral acceleration)

(c) the load from the Non-Suspended Masses of the vehicle (the masses that are not damped by any suspension, because they are under the primary suspension of the vehicle) and

(d) the load from the Suspended Masses of the vehicle, that is a damped force component of the total action on the railway track.

For High Speed Lines (V_{max} > 200 km/h), the component of the Load, due to the Non Suspended Masses, is of decisive importance for the Dynamic Load. In order to calculate the total Action/ Reaction on each support point of the rail (pair of fastenings on a sleeper) the static, the semistatic and the dynamic components should be added. The total dynamic component [(c)+(d)] is the square root of the second powers of the (c) and (d).

B. The Deflection of the Railway Track

The Load of the wheel of a railway vehicle is exerted on the railway track, at some point. Due to the elasticity (stiffness) of the track, the point subsides and the load is distributed along the track in many support points (sleepers). Thus the support point, under the acting load Q_{wheel}, undertakes a percentage of the total acting load, given by A□stat<1 [3], as depicted in Fig. 5.

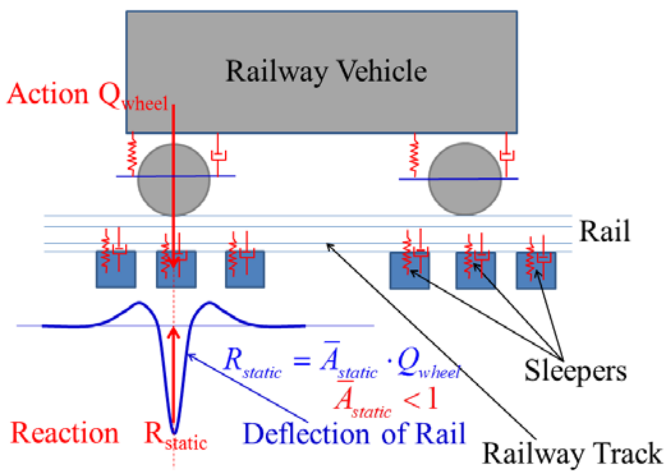

Figure 5 A railway vehicle on a railway track, which due to its elasticity (stiffness) subsides and distributes the load to the adjacent sleepers (support points of the rail). Consequantly the Action/ Reaction R < Qwheel, since A□stat<1.

C. The Static Component of the Load

According to the equation –referred to as the Zimmermann theory or formula, based on Winkler– that examines the track as a continuous beam on an elastic support [7]:

$$\frac{d^4y}{dx^4} = -\frac{1}{E \cdot J} \cdot \frac{d^2M}{dx^2} \tag{1}$$

where y is the deflection of the rail, M is the moment that stresses the beam, J is the moment of inertia of the rail, and E is the modulus of elasticity of the rail. The general differential equation, according to Winkler is approached in [8], [9].

Solving the differential equation of the formula (1) it is derived that the action/ reaction R_{static}, on each support point of the rail (sleeper), is:

$$R_{stat} = \frac{Q_{wheel}}{2\sqrt{2}} \cdot \sqrt[4]{\frac{\ell^3 \cdot \rho}{E \cdot J}} \Rightarrow \frac{R_{stat}}{Q_{wheel}} = \bar{A} = \bar{A}_{stat} = \frac{1}{2\sqrt{2}} \cdot \sqrt[4]{\frac{\ell^3 \cdot \rho}{E \cdot J}} \tag{2}$$

where Q_{wheel} the static wheel load, ℓ the distance among the support points of the rail (sleepers), E and J the modulus of elasticity and the moment of inertia of the rail, R_{stat} the static action/reaction on the sleeper, ρ reaction coefficient of the sleeper which is defined as: ρ=R/y, and is a quasi-coefficient of track elasticity (stiffness) or a spring constant of the track, according to the Hooke's Law, A□=A□$_{stat}$ < 1 (fluctuating between 0,30 and 0,70 normally) equals to R_{stat}/Q_{wheel}, that is the percentage of the acting (static) load of the wheel that each support point of the rail undertakes as (static) reaction.

In reality, the track consists of a sequence of materials –in the vertical axis, up to down– (rail, elastic pad of the fastening, sleeper, ballast, substructure), that are characterized by their individual coefficients of elasticity (static stiffness coefficients) ρ_i (Fig. 6).

Hence, for each material:

$$\rho_i = \frac{R}{y_i} \Rightarrow$$

$$y_{total} = \sum_{i=1}^{v} y_i = \sum_{i=1}^{v} \frac{R}{\rho_i} = R \cdot \sum_{i=1}^{v} \frac{1}{\rho_i} \Rightarrow \frac{1}{\rho_{total}} = \sum_{i=1}^{v} \frac{1}{\rho_i} \tag{3}$$

where ν is the number of various layers of materials that exist: rail, elastic pad of the fastening, sleeper, ballast, substructure.

The ρ_{pad} and the ρsubstructure are of crucial importance and influence for the magnitude of the total static stiffness coefficient of the track ρ_{total}, since they contribute over 85% to the total static stiffness coefficient of the track and the final values of actions/reactions [10].

D. The Semi-Static Component of the Load

This load is produced by the centrifugal acceleration exerted on the wheels of a vehicle that is running in a curve with cant (superelevation) deficiency. Cant deficiency or unbalanced superelevation ([11], p. 604) is defined as the difference (deficit or excess in mm) of the designed superelevation in a curve, from the theoretic one that is needed to fully counterbalance the centrifugal/ centripetal acceleration in the cross section of a track on a curve. In a curve there is always need to achieve a superelevation α (Fig. 7), in order to (Fig. 7):

$$\frac{m \cdot V^2}{R} \cdot \cos\varphi = m \cdot g \cdot \sin\varphi$$

Since in real conditions both passenger high speed trains and freight trains, much slower in speed, are running, consequently there is a superelevation (cant) deficiency or excess d_α. The following equation:

$$Q_\alpha = \frac{2 \cdot d_\alpha \cdot h_{CG}}{e^2} \cdot Q_{wheel} \qquad (4)$$

provides the increase Q_α of the vertical static load Q_{wheel} of the wheel, at curves with cant deficiency/ excess. In the above equation d_α is the cant deficiency, h_{CG} the height of the center of gravity of the vehicle from the rail head, e the track gauge.

It is not, however, a dynamic load in the sense of the load referred to in the next paragraph. Therefore, it is often considered to be a semi-static load. The following equation [3, 12, 13]:

The semi-static reaction of the sleeper is:

$$R_{semi-stat} = \overline{A} \cdot Q_\alpha$$

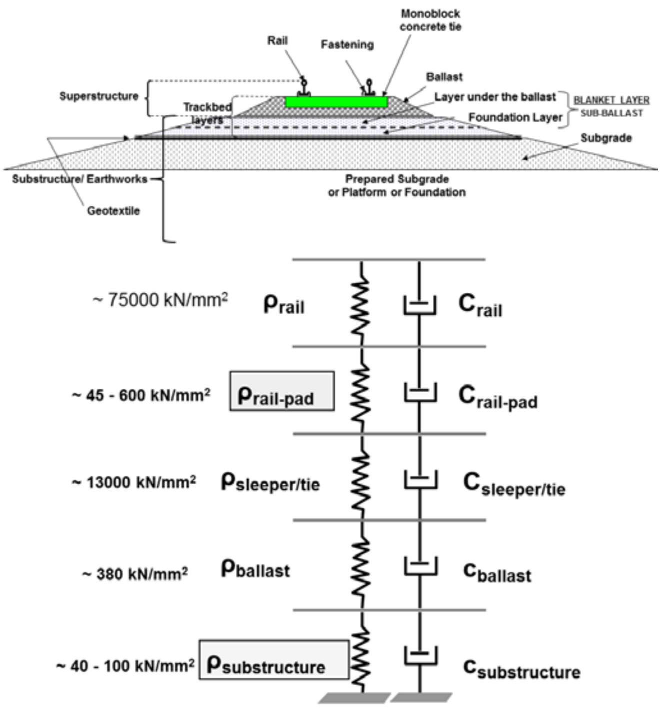

Figure 6 (2) The cross-section of a railway track with monoblock sleepers with terminology according to U.I.C. (upper illustration) and its simulation as an ensemble of springs and dashpots (lower illustration). The characteristic values of the static stiffness coefficients of the layers ρi are depicted.

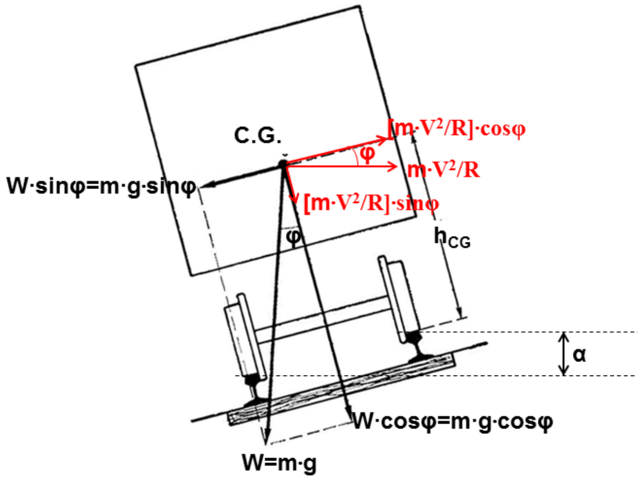

Figure 7 A railway vehicle on a curve. The weight W and the centrifugal/ centripetal force are depicted.

E. Differential Equation for Railway Track

The railway track is an infinite beam on elastic foundation, and the elastic foundation can be simulated by a large number of closely spaced translational springs [16]:

$$EI \frac{\partial^4 z}{\partial x^4} + \rho_1 \cdot \frac{\partial^2 z}{\partial x^2} + k_1 \cdot z = 0$$

when there is no external force, or [13]:

$$EI \frac{\partial^4 z}{\partial x^4} + \rho_1 \cdot \frac{\partial^2 z}{\partial x^2} + k_1 \cdot z = -Q \cdot \delta(x)$$

In these equations z is the deflection of the beam, ρ_1 is the mass of the track participating in the motion, k_1 the viscous damping of the track, E, I, the elasticity modulus and the moment of inertia of the rail and Q the force/ load from the wheel (when the force is present).

The solution of these equations becomes challenging if we want to take into account all the parameters [3]. However, if we make some simplifying hypotheses we will be able to approximate the influence of certain parameters provided that we will verify the theoretical results with experimental measurements.

In real conditions the increase of the dynamic component of the load due to the Non Suspended Masses of the Vehicle plus the track mass participating in their motion is given by the equation (6) above.

F. The Dynamic Component of the Load

The Suspended (Sprung) Masses of the vehicle –masses situated above the primary suspension (Fig. 4)– create forces with very small influence on the wheel's trajectory and on the system's excitation. This enables the simulation of the track as an elastic media with damping as shown in Fig. 8, depicting the rolling wheel on the rail running table [14]. Forced oscillation is caused by the irregularities of the rail running table (like an input random signal) –which are represented by n–, in a gravitational field with acceleration g. There are two suspensions on the vehicle for passenger comfort purposes: primary and secondary suspension. Moreover, a section of the mass of the railway track participates in the motion of the Non-Suspended (Unsprung) Masses of the vehicle. These Masses are situated under the primary suspension of the vehicle.

If the random excitation (track irregularities) is given, it is difficult to derive the response, unless the system is linear and invariable. In this case the input signal can be defined by its spectral density and from this we can calculate the spectral density of the response (see relevantly [5], [2]). The theoretical results confirm and explain the experimental verifications ([13], p. 39, 71).

The equation for the interaction between the vehicle's axle and the track becomes ([7], [6]):

$$\left(m_{NSM}+m_{TRACK}\right)\cdot\frac{d^2y}{dt^2}+\Gamma\cdot\frac{dy}{dt}+h_{TRACK}\cdot y=$$
$$=-m_{NSM}\cdot\frac{d^2n}{dt^2}+\left(m_{NSM}+m_{SM}\right)\cdot g \qquad (6)$$

where: m_{NSM} the Non-Suspended Masses of the vehicle, m_{TRACK} the mass of the track that participates in the motion, m_{SM} the Suspended Masses of the vehicle that are cited above the primary suspension of the vehicle, Γ damping constant of the track, h_{TRACK} the total dynamic stiffness coefficient of the track (for its calculation see [6]), n the fault ordinate of the rail running table and y the total deflection of the track.

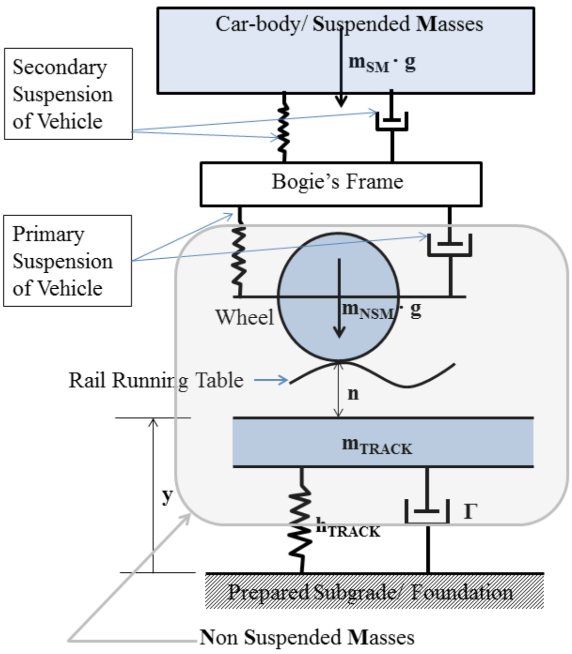

Figure 8 (3) Model of a rolling wheel on the rail running table. The acting Loads due to the Suspended Masses and the Non Suspended Masses of the Railway Vehicle are depicted as well as the Track Mass participating in the motion of NSM.

The phenomena of the wheel-rail contact and of the wheel hunting, particularly the equivalent conicity of the wheel and the forces of pseudo-glide, are non-linear. In any case the use of the linear system's approach is valid for speeds lower than the Vcritical≈500 km/h. The integration for the non-linear model (wheel-rail contact, wheel-hunting and pseudoglide forces) is performed through the Runge Kutta method ([13], p.94-95, 80, see also [15], p. 171, 351).

The solution of this second order differential equation of motion (forced damped vibration) gives the increase of the R_{stat}+ $R_{semi\text{-}stat}$ of the equations (2) and (5), by the dynamic component of the Load due to the Non Suspended and the Suspended Masses of the Vehicle, mainly based on the steady-state solution. The solution for the dynamic component due to the Non Suspended Masses and its verification through measurements is cited in [2] and [5].

The solution for the Suspended Masses is cited in [5] and [17]. In the next paragraphs, a sensitivity analysis by variating parameters of the transient component of the general solution of the equation (6), is attempted.

In high frequencies, as in the case of High Speeds, the response of the superstructure is negligible due to its low eigenfrequency, therefore, it has been proposed by the author and has been verified on track, that dynamic loads (semi-statics due to cant deficiency are also included) are not distributed to the adjacent sleepers, in contrast to static loads. Thus the Action/ Reaction $R_{dynamic}$ due to the Dynamic Component of the Load ($Q_{dynamic}$) is equal to ([7], [3]):

$$R_{dynamic}=Q_{dynamic}$$

and not reduced as the Actions/ Reactions due to the Static and the Semi-Static Components (see more analytically in [3]). The standard deviation of the dynamic component of the Load is given by ([7], [3]):

$$\sigma\left(\Delta Q_{dynamic}\right)=\sqrt{\sigma^2\left(\Delta Q_{NSM}\right)+\sigma^2\left(\Delta Q_{SM}\right)}$$

IV. The Second Order Differential Equation of Motion and the Transient Term of Its Solution

A. *Defects of Trigonometric Form*

The theoretical analysis for the additional –to the static and semi-static component– dynamic component of the load due to the Non Suspended Masses and the Suspended Masses of the vehicle, lead to the examination of the influence of the Non Suspended Masses only, since the frequency of oscillation of the Suspended Masses is much smaller than the frequency of the Non Suspended Masses. If m_{NSM} represents the Non Suspended Mass, m_{SM} the Suspended Mass and m_{TRACK} the Track Mass participating in the motion of the Non Suspended Masses of the vehicle, the differential equation is:

$$m_{NSM}\cdot\frac{d^2z}{dt^2}+h_{TRACK}\cdot z=m_{NSM}\cdot g \qquad (7a)$$

Where: g the acceleration of gravity and the dynamic track stiffness coefficient h_{TRACK}:

$$h_{TRACK} = 2\sqrt{2} \cdot \sqrt[4]{\frac{EJ\rho_{total}^3}{\ell^3}} \quad (8)$$

ρ_{total} the total static stiffness coefficient of the track, ℓ the distance among the sleepers, E, J the modulus of elasticity and the moment of inertia of the rail.

The theoretic calculation of m_{TRACK} gives as result ([2], [16]):

$$m_{TRACK} = 2\sqrt{2} \cdot m_0 \cdot \sqrt[4]{\frac{EJ\ell}{\rho_{total}}} \quad (9)$$

The equation (7a) is transformed:

$$\left(m_{NSM} + m_{TRACK}\right) \cdot \frac{d^2 z}{dt^2} + h_{TRACK} \cdot z = m_{NSM} \cdot g \quad (7b)$$

For a comparison of the theoretical track mass to measurements' results see [2] and [16]. The particular solution of the differential equation (8b) corresponds to the static action of the weight of the wheel:

$$z = \frac{m_{TRACK} \cdot g}{h_{TRACK}} \quad (10)$$

Let's suppose that the rolling wheel run over an isolated sinusoidal defect of length λ of the form:

$$n = \frac{a}{2} \cdot \left(1 - \cos\frac{2\pi x}{\lambda}\right) = \frac{a}{2} \cdot \left(1 - \cos\frac{2\pi Vt}{\lambda}\right) \quad (11)$$

Where n is the ordinate of the defect, consequently the ordinate of the trajectory of the center of inertia of the wheel is n+z. If we name τ_1 the time needed for the overpassing of the defect by the wheel rolling at a speed V:

$$\tau_1 = \frac{\lambda}{V} \quad (12)$$

The differential equation of the motion of the wheel is:

$$m_{NSM} \cdot \frac{d^2}{dt^2}(z+n) + m_{TRACK} \cdot \frac{d^2 z}{dt^2} + h_{TRACK} \cdot z = 0 \Rightarrow$$
$$\left(m_{NSM} + m_{TRACK}\right) \cdot \frac{d^2 z}{dt^2} + h_{TRACK} \cdot z = -m_{NSM} \cdot \frac{d^2 n}{dt^2} =$$
$$= -m_{NSM} \cdot \frac{2a\pi^2}{\tau_1^2} \cdot \cos\frac{2\pi t}{\tau_1} \quad (13)$$

B. Battered Rail-Joints or a Bent or Deformed Welding

The rail-joints during their Life-Cycle are battered and consequently the rail edges present deformations and bends. In Fig. 9, a wheel passes a deformed, bent joint between two consecutive rails jointed through fishplates and bolts.

The weldings, in the Continuously Welded Rails (CWR), due to non-correct execution of the welding procedure (mainly poor alignment) or "softer" material in the area of welding, could present also the same situation. In Fig. 10 a wheel passes a deformed, bent welding.

We can approach the matter beginning with a discontinuity of the rail running table –a change in the inclination of the rail running table along the track– in the form of one angle (as in Fig. 11-upper illustration), instead of two parabolic arcs (as in Fig. 10). We use the "mass-spring-damper" model as depicted in Fig. 8.

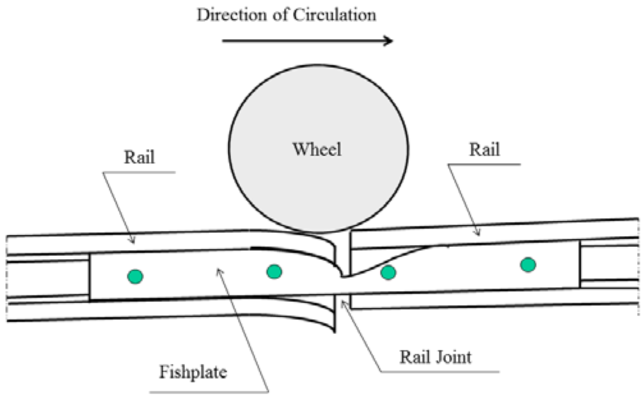

Figure 9 (4) Battered rail joint between two consecutive rails jointed through fishplates and bolts.

The equation of the form of the defect is:

$$n = -\alpha \cdot x = -\alpha \cdot V \cdot t \quad (14)$$

where α is the angle in rad and V the speed, for x>0 or t>0.

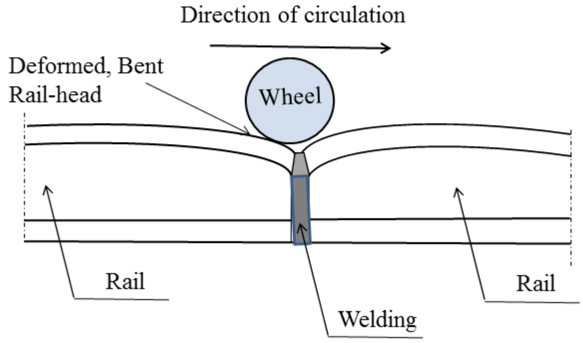

Figure 10 (5) Deformed, bent rail welding between two successive welded rails.

At this point we have to remember the delta (or Dirac) function δ(x), and the unit step (Heaviside's) function H(t). The delta function is usually defined as follows ([18, p. 270] and [19, p. 74]):

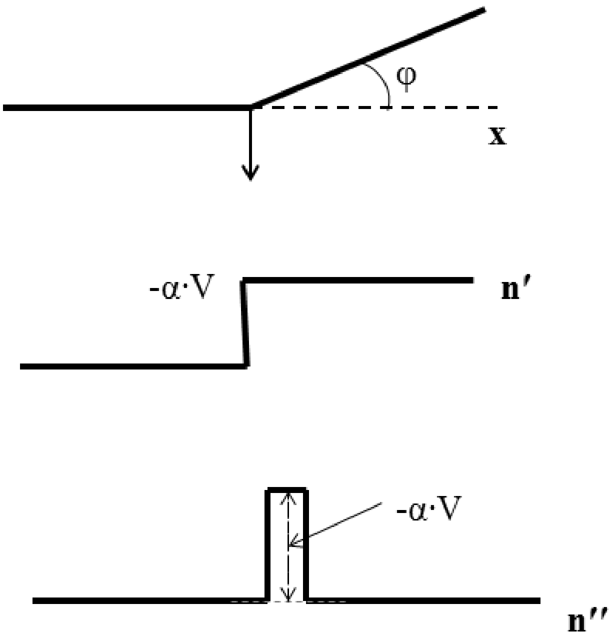

Figure 11 (6) Battered rail joint or welding between two adjacent rails. The rail running tables are considered linear forming an angle π-φ between them (upper illustration) with the mathematical formula given by the equation (14). The first derivative of the equation (14) in the middle and the second derivative in the lower illustration.

$$\delta(t)=0, \quad for \quad t\neq 0, \quad and \quad \int_{-\infty}^{+\infty}\delta(t)\cdot dt=1 \tag{15}$$

The unit step function ([18], p. 38] and [19, p. 61]) is defined:

$$H(t)=\frac{1}{2}+\frac{1}{2}\cdot\operatorname{sgn} t=\begin{cases}0 & if \quad t<0\\ \left(\frac{1}{2} \quad if \quad t=0\right)\\ 1 & if \quad t>0\end{cases} \tag{16}$$

Where the sign function is defined ([19, p. 65]):

$$\operatorname{sgn} t=\begin{cases}-1 & for \quad t<0\\ +1 & for \quad t>0\end{cases} \tag{17}$$

The delta function or Dirac Impulse is depicted in Fig. 12 and comparing Fig. 12 to Figure 11-lower illustration, we conclude that they have similar form.

The unit step function of Heaviside is depicted in Fig. 13 and comparing Fig. 13 to Figure 11-middle illustration, we conclude that they have similar form.

Differentiating in relation to time t the equations (7a, 7b) we can derive:

$$n'=-\alpha\cdot V=-\alpha\cdot V\cdot H(t) \tag{18}$$

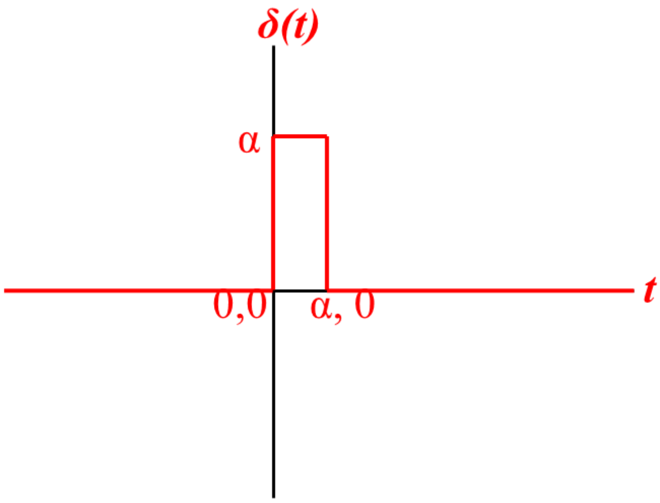

Figure 12 The Dirac Impulse or Dirac function .

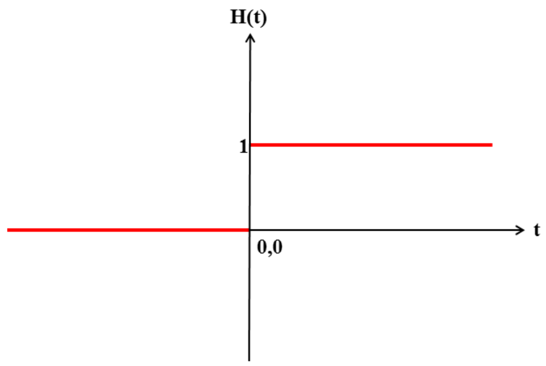

Figure 13 (7) The unit step function of Heaviside.

In Figure 11-middle illustration the first derivative n′ is depicted. From the properties of the delta function and the unit step function we know that ([19], p. 98, and [20], p. 42), the first derivative of the unit step function H′(t), is the Dirac's delta function δ(t), consequently:

$$n''=-\alpha\cdot V\cdot H'(t)=-\alpha\cdot V\cdot\delta(t) \tag{19}$$

In Figure 11-lower illustration the second derivative n″ is depicted. From the equations (7a, 7b), replacing the second term of the forcing external load due to the angle on the rail running table, and adding the term for damping, we derive, from equation (6):

$$\left(m_{NSM}+m_{TRACK}\right)\cdot\frac{d^2z}{dt^2}+\Gamma\cdot\frac{dy}{dt}+h_{TRACK}\cdot z=$$
$$=-m_{NSM}\cdot\frac{d^2n}{dt^2}=-m_{NSM}\cdot\left(-\alpha\cdot V\cdot\delta(t)\right)\Rightarrow$$

$$\Rightarrow \frac{d^2z}{dt^2}+\frac{\Gamma}{(m_{NSM}+m_{TRACK})}\cdot\frac{dy}{dt}+\frac{h_{TRACK}}{(m_{NSM}+m_{TRACK})}\cdot z=$$

$$=\underbrace{\frac{m_{NSM}}{(m_{NSM}+m_{TRACK})}}_{\approx 1}\cdot\alpha\cdot V\cdot\delta(t)\Rightarrow$$

$$\Rightarrow \frac{d^2z}{dt^2}+2\cdot\zeta\cdot\omega_n\cdot\frac{dy}{dt}+\omega_n^2\cdot z\approx\alpha\cdot V\cdot\delta(t)$$

V. The Solution of the Second Order Differential Equation of Motion

For the free oscillation (without external force) the equation is:

$$\mathrm{m}\cdot\ddot{\mathrm{z}}+\mathrm{k}\cdot\mathrm{z}=0\Rightarrow$$

$$\Rightarrow\ddot{\mathrm{z}}+\frac{\mathrm{k}}{\mathrm{m}}\cdot\mathrm{z}=0\Rightarrow\ddot{\mathrm{z}}+\omega_n^2\cdot\mathrm{z}=0 \qquad (21)$$

The general solution is [4]:

$$\mathrm{z}(\mathrm{t})=\mathrm{A}\cdot\cos(\omega_n\mathrm{t})+\mathrm{B}\cdot\sin(\omega_n\mathrm{t})=$$

$$=\mathrm{z}(0)\cdot\cos(\omega_n\mathrm{t})+\frac{\dot{\mathrm{z}}(0)}{\omega_n}\cdot\sin(\omega_n\mathrm{t}) \qquad (22)$$

Where:

$$\mathrm{A}=\mathrm{z}(0),\quad \mathrm{B}=\frac{\dot{\mathrm{z}}(0)}{\omega_n} \qquad (23)$$

If we pass to the damped harmonic oscillation of the form:

$$\mathrm{m}\cdot\ddot{\mathrm{z}}+\mathrm{c}\cdot\dot{\mathrm{z}}+\mathrm{k}\cdot\mathrm{z}=\mathrm{p}_0\cdot\cos(\omega\mathrm{t})\Rightarrow$$

$$\Rightarrow\ddot{\mathrm{z}}+\frac{\mathrm{c}}{\mathrm{m}}\cdot\dot{\mathrm{z}}+\omega_n^2\cdot\mathrm{z}=\frac{\mathrm{p}_0}{\mathrm{m}}\cdot\cos(\omega\mathrm{t})\Rightarrow$$

$$\Rightarrow\ddot{\mathrm{z}}+2\cdot\zeta\cdot\omega_n\cdot\dot{\mathrm{z}}+\omega_n^2\cdot\mathrm{z}=\omega_n^2\cdot\frac{\mathrm{p}_0}{\mathrm{k}}\cdot\cos(\omega\mathrm{t}) \qquad (24)$$

where:

$$\omega_n^2=\frac{\mathrm{k}}{\mathrm{m}}\Rightarrow\mathrm{m}=\frac{\mathrm{k}}{\omega_n^2} \qquad (25)$$

The particular solution of the linear second order differential equation (24) is of the form:

$$\mathrm{z}_p(\mathrm{t})=\mathrm{C}\cdot\sin(\omega\mathrm{t})+\mathrm{D}\cdot\cos(\omega\mathrm{t})\Rightarrow$$

$$\Rightarrow\dot{\mathrm{z}}_p(\mathrm{t})=\omega\cdot\mathrm{C}\cdot\cos(\omega\mathrm{t})-\omega\cdot\mathrm{D}\cdot\sin(\omega\mathrm{t})\Rightarrow$$

$$\Rightarrow\ddot{\mathrm{z}}_p(\mathrm{t})=-\omega^2\cdot\mathrm{C}\cdot\sin(\omega\mathrm{t})-\omega^2\cdot\mathrm{D}\cdot\sin(\omega\mathrm{t}) \qquad (26)$$

Substituting eq. (26) to eq. (24) and after the mathematical procedure we derive ([7], p.110, [21]):

$$\left[\left(\omega_n^2-\omega^2\right)C-2\zeta\omega_n\omega D\right]\cdot\sin(\omega t)+$$

$$+\left[2\zeta\omega_n\omega C-\left(\omega_n^2-\omega^2\right)D\right]\cdot\cos(\omega t)=$$

$$=\omega_n^2\frac{p_0}{k}\sin(\omega t) \qquad (27)$$

For the equation (27) to be valid for every t, the coefficients of the sine and cosine terms of the equation must be equal and finally solving a two equations system, we derive:

$$C=\frac{p_0}{k}\cdot\frac{1-\left(\frac{\omega}{\omega_n}\right)^2}{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2+\left[2\zeta\left(\frac{\omega}{\omega_n}\right)\right]^2} \qquad (22) \qquad (28a)$$

$$D=\frac{p_0}{k}\cdot\frac{-2\zeta\left(\frac{\omega}{\omega_n}\right)^2}{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2+\left[2\zeta\left(\frac{\omega}{\omega_n}\right)\right]^2} \qquad (28b)$$

The complete solution, for the equation (24), is the addition of the solution (22) and of the solution of the equation (24) combined with the equations (28a) and (28b):

$$\mathrm{z}(\mathrm{t})=\underbrace{\mathrm{e}^{-\zeta\omega_n\mathrm{t}}\cdot\left(\mathrm{A}\cdot\cos(\omega_D\mathrm{t})+\mathrm{B}\cdot\sin(\omega_D\mathrm{t})\right)}_{\text{transient-term}}+$$

$$+\underbrace{\mathrm{C}\cdot\sin(\omega\mathrm{t})+\mathrm{D}\cdot\cos(\omega\mathrm{t})}_{\text{steady-state-term}} \qquad (29a)$$

$$\text{where}\quad\omega_D=\omega_n\sqrt{1-\zeta^2} \qquad (29b)$$

In the case of equation (20), we have a constant external force and *ω=0*, consequently *sin(ωt)=0* and *D=0*. There is no steady state term in the solution, but only transient term. The equation (29a) is transformed to:

$$z(t) = \underbrace{e^{-\zeta\omega_n t} \cdot \left(A \cdot \cos(\omega_D t) + B \cdot \sin(\omega_D t)\right)}_{\text{transient-term}} \quad (30)$$

which is the transient term of the solution of the second order differential equation of motion.

VI. Sensitivity Analysis for The Transient Term of the Solution of the Second Order Differential Equation of Motion

Equation (30) can be written (choosing appropriately the sine form function and not the cosine, since for *t=0* the value of *z=0*) also in the form of polar coordinates ([22], p. 28 and [23], p. 22, 24):

$$z(t) = \left[(0)^2 + \left(\frac{\alpha V + (0)\zeta\omega_n}{\omega_n \sqrt{1-\zeta^2}} \right)^2 \right]^{\frac{1}{2}} \cdot \cos\left(\omega_n \sqrt{1-\zeta^2} \cdot t\right) \cdot e^{-\zeta\omega_n t} \Rightarrow$$

$$\Rightarrow z(t) = \frac{\alpha V}{\omega_n \sqrt{1-\zeta^2}} \sqrt{e^{-\zeta\omega_n t}} \cdot \cos\left(\omega_n \sqrt{1-\zeta^2} \cdot t\right) \quad (31)$$

Where $p_0=\alpha \cdot V$, $\theta=0$ since there is no phase difference between the external force and the eigenfrequency. We have for *t=0*, then *z(0)=0* as depicted in Fig. 11:

$$z(t) = r \cdot \cos(\omega_D t + \theta) \cdot e^{-\zeta\omega_n t}$$

$$r = \left[z(0)^2 + \left(\frac{\dot{z}(0) + z(0)\zeta\omega_n}{\omega_D} \right)^2 \right]^{\frac{1}{2}} \quad (32)$$

$$\theta = -\tan^{-1}\left(\frac{\dot{z}(0) + z(0)\zeta\omega_n}{\omega_D z(0)} \right)$$

Since the action and the deflection take simultaneously their maximum values at the support point of the rail (sleeper), then the maximum increase of the total action/ reaction, due to the dynamic component owed to the defect, is observed for:

$$\omega_n \sqrt{1-\zeta^2} \cdot t = \pi \Rightarrow t = \frac{\pi}{\omega_n \sqrt{1-\zeta^2}} \quad (33)$$

at a remote point -from the defect's peak- and so more remote as the ω_n is small this means that in cases of very soft prepared subgrade (or platform).

The dynamic increase of the load, due to a deformed, bent joint or welding is equal to:

$$Q_{dynamic} = \left[\frac{\alpha V}{\omega_n \sqrt{1-\zeta^2}} \cdot e^{-\zeta\omega_n \frac{\pi}{\omega_n \sqrt{1-\zeta^2}}} \cdot \cos(\pi) \right] \cdot h_{TRACK} \Rightarrow$$

$$\Rightarrow Q_{dynamic} = \frac{\alpha V h_{TRACK}}{\omega_n \sqrt{1-\zeta^2}} \cdot e^{-\zeta \frac{\pi}{\sqrt{1-\zeta^2}}} \quad (34)$$

Since:

$$\omega_n = \sqrt{\frac{h_{TRACK}}{m_{NSM}}}$$

The equation (36) is transformed:

$$Q_{dynamic} = \frac{\alpha V h_{TRACK}}{\sqrt{\frac{h_{TRACK}}{m_{NSM}}} \cdot \sqrt{1-\zeta^2}} \cdot e^{-\zeta \frac{\pi}{\sqrt{1-\zeta^2}}} \Rightarrow$$

$$\Rightarrow Q_{dynamic} = k \cdot \alpha \cdot V \cdot \sqrt{m_{NSM} \cdot h_{TRACK}} \quad (35a)$$

where:

$$k = \frac{e^{-\zeta \frac{\pi}{\sqrt{1-\zeta^2}}}}{\sqrt{1-\zeta^2}} \quad (35b)$$

The dynamic increase of the load is proportional to the speed V and to the square root of the product of the Non Suspended Mass m_{NSM} times the dynamic stiffness coefficient of track h_{TRACK}. Furthermore the dynamic component of the load due to a deformed, bent joint or welding, $Q_{dynamic}$ decreases when the damping coefficient ζ increases and the relation between ζ and k is given in the table 1 below:

Table 1: solution of a cosine form differential equation

ζ=	0,05	0,1	0,15	0,2	0,25	0,3	0,35	0,4	0,45	0,5
k=	5,43	2,31	1,30	0,82	0,55	0,38	0,26	0,19	0,13	0,09

The equation (31) could take the form of a sinusoidal solution of the form (with appropriate choice of the initial conditions):

$$z(t) = \frac{\alpha V}{\omega_n \sqrt{1-\zeta^2}} \cdot e^{-\zeta\omega_n t} \cdot \sin\left(\omega_n \sqrt{1-\zeta^2} \cdot t\right) \quad (36a)$$

In this solution the relation between ζ and k is (the arc is equal to $\pi/2$):

$$k = \frac{e^{-\zeta \frac{\frac{\pi}{2}}{\sqrt{1-\zeta^2}}}}{\sqrt{1-\zeta^2}} \quad (36b)$$

and the Table 2 depicts the relation between ζ and k.

Table 2: solution of a sine form differential equation

ζ=	0,05	0,1	0,15	0,2	0,25	0,3	0,35	0,4	0,45	0,5
k=	0,93	0,86	0,80	0,74	0,69	0,64	0,59	0,55	0,51	0,46

From both Tables 1 and 2 we derive the necessity that the joints should be maintained in a very soft support (low stiffness coefficient of track) and simultaneously an increased damping coefficient. This implication leads towards very “soft” fastenings, with low static stiffness coefficient, in order to achieve a very low total static stiffness coefficient of the track. This is a special demand for the high speed railway tracks.

VII. A BATTERED/ BENT JOINT BETWEEN TWO RAILS WITH DEFORMED EDGES OF PARABOLIC FORM

We consider the case of a joint (or welding), which is consisted of two parabolic curves formed by the ends of two consecutive rails, as in Figs. 14a and 14b.

The total depth of the defect is α and the total length of the deformed edges of the rails ℓ. In order to approach the issue we examine mathematically two parabolic arcs as in Fig. 15a.

The line JD has its D edge on the directrix (not designed here) of the parabola and intersects the horizontal axis (*xx*) at the point A. Since F is the focus of the parabola then FB=BD and the points D and F have the equal y-coordinates but of opposite sign. Moreover JD=JF, since the point D is on the directrix. Besides that the angles FĴB and BĴD are equal. The x-coordinate of B is the jalf of the x-coordinates of the points J, A, D. Consequently AB=BC=ℓ/4.

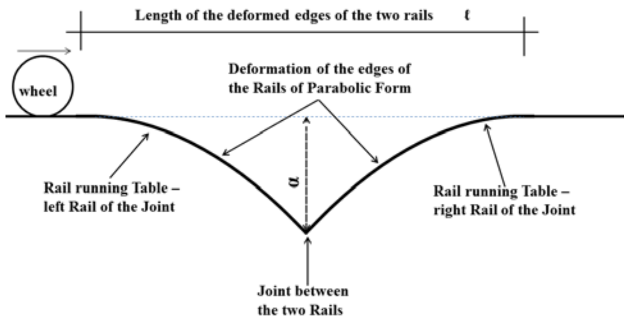

Figure 14a (8a) A Joint between two successive rails, with deformations of their ends in a parabolic form, with saggitta of the defect α.

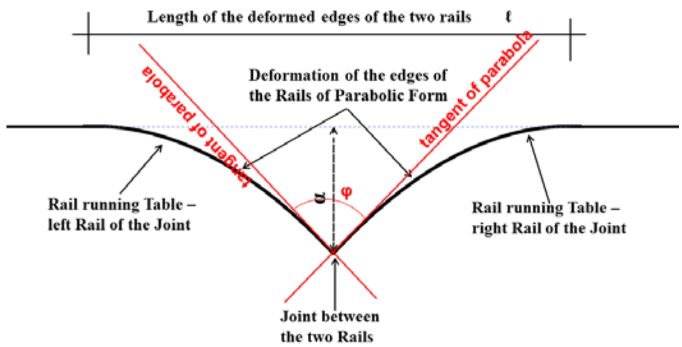

Figure 14b (8b) A Joint between two consecutive rails, with deformations of their ends in a parabolic form, and the tangents of the parabolas at the joint, forming an angle φ.

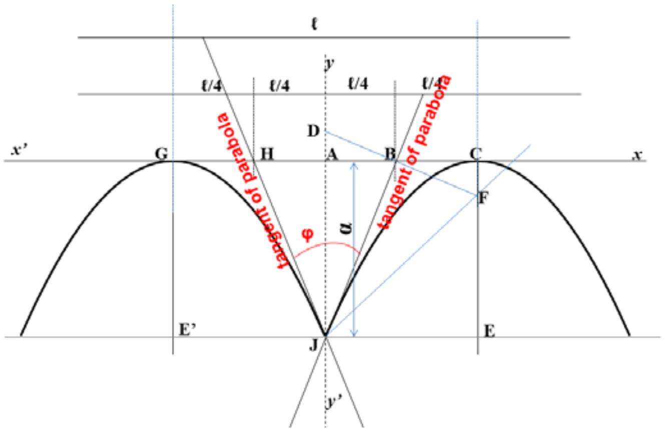

Figure 15a (9α) Two parabolas (of the same mathematical equation) intersecting each other at a random point.

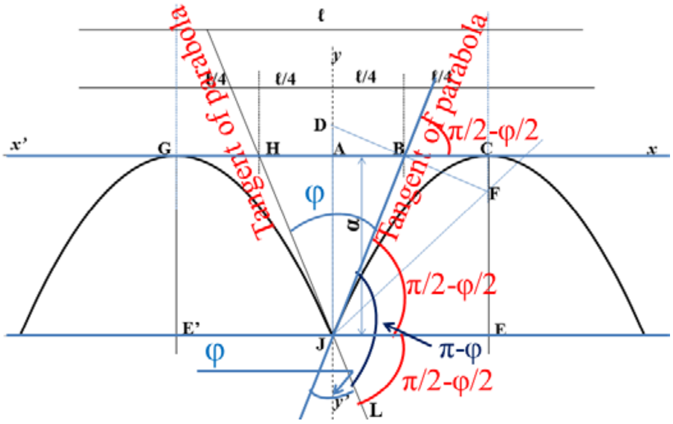

Figure 15b Two parabolas (of the same mathematical equation) intersecting each other at a random point, with the relations among the angles.

The inclination of BJ=α/(ℓ/4)=4α/ℓ. The angle BĴE is 4α/ℓ rad. The angle HĴB between the two tangents of the two parabolas at the point J is equal to (see Fig. 15b):

HĴB = π – 2*(BĴE)= -2*(BĴE) = 8α/ℓ (36)

in absolute values.

Consequently, if we consider a damping ζ=0, then the term k=1 of the equations (35b) and (36b) and the term α in the equations (35a) and (35b) is equal to the angle HĴB of the equation (36). Therefore:

$$Q_{dynamic} = \frac{8\cdot\alpha}{\ell}\cdot V\cdot\sqrt{m_{NSM}\cdot h_{TRACK}} \qquad (37)$$

The equation (37) will not be valid, if the speed of the railway vehicle is higher than a critical value of $V_{Critical}$, for which $Q_{dynamic}$ overpasses a $Q_{dynamic^{-}max}$.

From equation (37) we derive:

$$V_{Critical} = \frac{Q_{dynamic}}{h_{TRACK}} \cdot \frac{1}{8\alpha} \cdot \sqrt{\frac{h_{TRACK}}{m_{NSM}}} \cdot \ell = z \cdot \frac{1}{8\alpha} \cdot \sqrt{\frac{h_{TRACK}}{m_{NSM}}} \cdot \ell \Rightarrow$$

$$\Rightarrow V_{Critical} = 0,25 \cdot \sqrt{\frac{h_{TRACK}}{m_{NSM}}} \cdot \ell \qquad (38)$$

per mm of depth at the joint (that is for z = 1 mm) and for an angle α=0,5 rad.

For a track equipped with rail UIC60, monoblock sleepers of prestressed concrete B70, fastening W14, ballast two years old on track and subgrade of NBS1 type ($\rho_{subgrade}$=100 kN/mm), then a rigidity h_{TRACK}= 85 kN/mm is derived and supposing m_{NSM}=1,5+0,3 t = 1800/9,81 kg-mass, then:

$$V_{Critical} = 0,25 \cdot \sqrt{\frac{9,81\frac{m}{\sec^2}*85000\frac{\frac{1}{10}kg-mass}{\frac{1}{1000}m}}{1800kg-mass}} \cdot \ell \Rightarrow$$

$$\Rightarrow V_{Critical} = 0,25 \cdot \sqrt{\frac{9,81*8.500.000}{1800}\frac{1}{\sec^2}} \cdot \ell \Rightarrow$$

$$V_{Critical} = 53,808 \cdot \ell \qquad (39)$$

Where ℓ in [m] and $V_{Critical}$ in [m/sec]. For a joint of 2 m length, then $V_{Critical}$= 387,5 km/h is calculated.

For a more general approach about the Energy-Efficient Train Control and Speed Constraints, see [24]. An approach from a more mathematical point of view is cited in [25].

VIII. The Cases of Undulatory Wear and of an Isolated Defect of Track

Besides the analysis in the case of a bent joint or welding on track, a detailed analysis of the actions on a railway track, due to an isolated defect along the track derived from the differential equation (6), is presented in [26]. An analysis for the case of undulatory wear as well as for defects of random form are presented in [2], as derived also from the differential equation (6).

IX. Conclusions

In the present paper the dynamic component of the acting load on a railway track is investigated through the second order differential equation of motion of the Non Suspended Masses of the Vehicle and specifically through the transient term of the solution. The reaction/ action on each support point (sleeper) of the rail is also calculated for a bent or battered joint or welding of the rail. The case of a deformed or bent joint or welding is theoretically analyzed through the second order differential equation of motion and the solution is investigated. The necessity that the joints –and consequently the railway track– should present a very low static stiffness coefficient and simultaneously an increased damping coefficient, is derived by the analysis. This can be mainly achieved through the development of very “soft” fastenings and elastic pads.

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