Boltzmann Entropy of Thermodynamics versus Shannon Entropy of Information Theory

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***Abstract*—**Some implications of a scale invariant model of statistical mechanics to *Boltzmann* entropy in thermodynamics versus *Shannon* entropy in information theory are investigated. The objective versus subjective nature of entropy as well as the fundamental significance of the choice of *Shannon* measure K as *Boltzmann* constant k is described. In addition, the impact of the results on *Nernst-Planck* statement of the third law of thermodynamics is discussed.

***Keywords*—**Boltzmann entropy of thermodynamic, Shannon entropy of information theory, third law of thermodynamics.

I. INTRODUCTION

STOCHASTIC quantum fields [1-17] and classical hydrodynamic fields [18-29] viewed as ensembles of weakly coupled oscillators resulted in the introduction of a scale-invariant model of statistical mechanics [30] and its application to the fields of thermodynamics [31], fluid mechanics [32-33], statistical mechanics [34], and quantum mechanics [35].

In the present study, some implications of the model to the physical foundation of classical and statistical thermodynamics and *Boltzmann* thermodynamic entropy versus *Shannon* information entropy are examined. Because languages involve distinguishable atoms i.e. alphabets, a model of language as multicomponent thermodynamic mixture is examined thus revealing the connections between *Shannon* information entropy on the one hand and entropy of ideal gas mixtures on the other hand. Finally, some of the implications of the results to *Nernst-Planck* statement of the third law of thermodynamics are discussed.

II. A SCALE-INVARIANT MODEL OF STATISTICAL MECHANICS

The scale-invariant model of statistical mechanics for equilibrium galactic-, planetary-, hydro-system-, fluid-element-, eddy-, cluster-, molecular-, atomic-, subatomic-, kromo-, and tachyon-dynamics corresponding to the scale β = g, p, h, f, e, c, m, a, s, k, and t is schematically shown in Fig.1 [31]. Each statistical field is identified as the "system" and is composed of an ensemble of "elements" described by a distribution function $f_\beta(\mathbf{u}_\beta) = f_\beta(\mathbf{r}_\beta, \mathbf{u}_\beta, t_\beta)\, d\mathbf{r}_\beta d\mathbf{u}_\beta$. Each element is composed of an ensemble of small particles called the "atoms" of the field and are viewed as *point-mass*. The most probable element (system) velocity of the smaller scale *(j)* becomes the velocity of the atom (element) of the larger scale *(j+1)*.

Following the classical methods [36-41] the invariant definition of density ρ_β, and velocity of *atom* $\mathbf{u}_\beta$, *element* $\mathbf{v}_\beta$, and *system* $\mathbf{w}_\beta$ at the scale β are [35, 42]

$$\rho_\beta = n_\beta m_\beta = m_\beta \int f_\beta du_\beta \quad , \quad \mathbf{u}_\beta = \mathbf{v}_{mp\beta-1} \tag{1}$$

$$\mathbf{v}_\beta = \rho_\beta^{-1} m_\beta \int \mathbf{u}_\beta f_\beta d\mathbf{u}_\beta \quad , \quad \mathbf{w}_\beta = \mathbf{v}_{mp\beta+1} \tag{2}$$

Similarly, the invariant definition of the peculiar and diffusion velocities are introduced as

$$\mathbf{V}'_\beta = \mathbf{u}_\beta - \mathbf{v}_\beta \quad , \quad \mathbf{V}_\beta = \mathbf{v}_\beta - \mathbf{w}_\beta \tag{3}$$

such that

$$\mathbf{V}_\beta = \mathbf{V}'_{\beta+1} \tag{4}$$

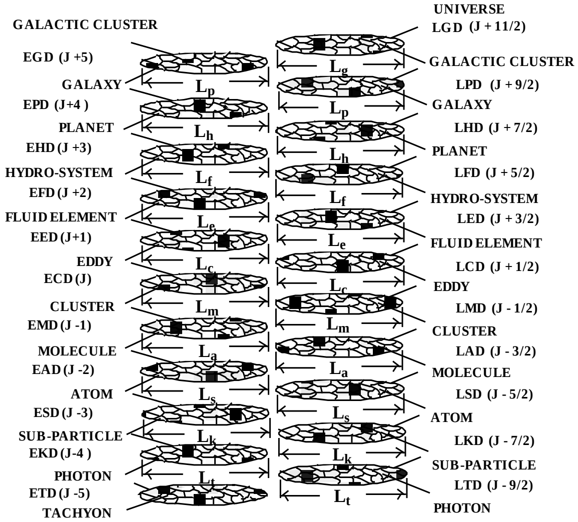

Fig. 1 A scale invariant view of statistical mechanics from cosmic to tachyon scales.

When the model is applied to social structures one arrives at the cascade shown in Fig. 2. Interestingly, as signs for capacity of number of people in elevators in Athens show,

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in Greek a person is referred to as “atom” that is the smallest unit of social structure shown in Fig. 2. The correspondence between statistical fields in Figs. 1-2 clearly show the physical and objective basis of information and its communication between “atoms” or individuals be it in the form of particle exchange or exchange of more complex symbols such as words or numbers. In this sense, the information theory like all other branches of science must of course be a subset of the theory of everything TOE (Fig. 1).

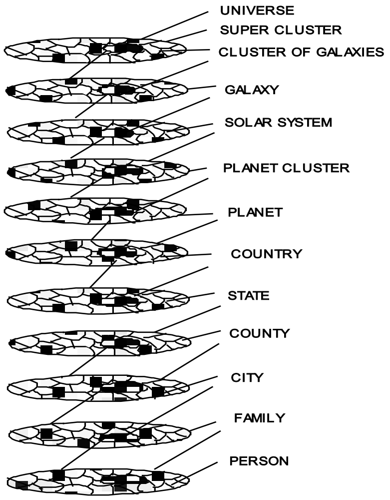

Fig. 2 Hierarchy of social structures from cosmic to individual or atomic scales.

In examining the small-scale limit of the hierarchy in Fig.2, one notes that a living person is composed of a collection of living organs. Similarly, a living organ is composed of a collection of living cells. Therefore it is interesting to speculate if parallel to *Leibnitz* monad there exist such a thing as the smallest unit or “atom” of life out of which all living organisms are constructed. In other words, if all living systems are composed of living elements, is there a limit to such an infinite regression.

III. STOCHASTIC NATURES OF PLANCK AND BOLTZMANN CONSTANTS AND DE PRETTO NUMBER 8338

Because at thermodynamic equilibrium the mean velocity of each particle, *Heisenberg-Kramers* virtual oscillator [43], vanishes $<\mathbf{u}_\beta> = 0$ the translational kinetic energy of particle oscillating in two directions (x+, x–) is expressed as

$$\begin{aligned}\varepsilon_\beta &= m_\beta \langle u^2_{\beta x+}\rangle / 2 + m_\beta \langle u^2_{\beta x-}\rangle / 2 \\ &= m_\beta \langle u^2_{\beta x+}\rangle = \bar{p}_\beta \langle \lambda^2_\beta\rangle^{1/2} \langle \nu^2_\beta\rangle^{1/2}\end{aligned} \tag{5}$$

where $\bar{p}_\beta = m_\beta \langle u^2_{\beta x+}\rangle^{1/2}$ is the root-mean-square momentum of particle and $<u^2_{\beta x+}> = <u^2_{\beta x-}>$ by *Boltzmann* equipartition principle. At any scale β, the result in (5) can be expressed in terms of either frequency or wavelength

$$\varepsilon_\beta = m_\beta \langle u^2_\beta\rangle = \bar{p}_\beta \langle \lambda^2_\beta\rangle^{1/2} \langle \nu^2_\beta\rangle^{1/2} = h_\beta \nu_\beta \tag{6a}$$

$$\varepsilon_\beta = m_\beta \langle u^2_\beta\rangle = \bar{p}_\beta \langle \nu^2_\beta\rangle^{1/2} \langle \lambda^2_\beta\rangle^{1/2} = k_\beta \lambda_\beta \tag{6b}$$

when the definition of stochastic *Planck* and *Boltzmann* factors are introduced as [33]

$$h_\beta = \bar{p}_\beta \langle \lambda^2_\beta\rangle^{1/2} \tag{7a}$$

$$k_\beta = \bar{p}_\beta \langle \nu^2_\beta\rangle^{1/2} \tag{7b}$$

At the important scale of EKD (Fig. 1) corresponding to *Casimir* [44] vacuum composed of photon gas, the universal constants of *Planck* [45, 46] and *Boltzmann* [31] are identified from equations (6)-(7) as

$$h = h_k = m_k c \langle \lambda^2_k\rangle^{1/2} = 6.626\times 10^{-34}\ \text{J-s} \tag{8a}$$

$$k = k_k = m_k c \langle \nu^2_k\rangle^{1/2} = 1.381\times 10^{-23}\ \text{J/K} \tag{8b}$$

Next, following *de Broglie* hypothesis for the wavelength of matter waves [2]

$$\lambda_\beta = h / \bar{p}_\beta \tag{9a}$$

the frequency of matter waves is defined as [31]

$$\nu_\beta = k / \bar{p}_\beta \tag{9b}$$

When matter and radiation are in the state of thermodynamic equilibrium equations (9a) and (9b) can be expressed as

$$h_\beta = h_k = h \quad , \qquad k_\beta = k_k = k \tag{10}$$

The definitions in equations (8a) and (8b) result in the gravitational mass of photon [31]

$$m_k = (hk / c^3)^{1/2} = 1.84278\times 10^{-41}\ \text{kg} \tag{12}$$

that is much larger than the reported [47] value of 4×10^{-51} kg. The finite gravitational mass of photons was anticipated by *Newton* [48] and is in accordance with *Einstein-de Broglie* theory of light [49-53]. *Avogardo-Loschmidt* number was predicted as [31]

$$N^o = 1/(m_k c^2) = 6.0376\times 10^{23} \tag{13}$$

leading to the modified value of the universal gas constant

$$R^o = N^o k = 8.338\ \text{kJ/(kmol-K)} \tag{14}$$

Also, by (13) the atomic mass unit becomes

$$\begin{aligned}\mathrm{amu} &= \mathrm{m_k c^2} \\ &= (\mathrm{hkc})^{1/2} = 1.6563\times 10^{-27}\ \text{kg/kmol} \qquad (15)\end{aligned}$$

Since all baryonic matter is known to be composed of atoms, equations (12) and (15) suggest that all matter in the universe is composed of light [54]. From (8a)-(8b) the wavelength and frequency of photon in vacuum $\langle\lambda_k^2\rangle^{1/2}\langle\nu_k^2\rangle^{1/2} = c$ are

$$\lambda_k = \langle\lambda_k^2\rangle^{1/2} = 1/R^o = 0.119935\ \text{m} \quad ,$$
$$\nu_k = \langle\nu_k^2\rangle^{1/2} = 2.49969\times 10^9\ \text{Hz} \qquad (16)$$

In a recent study [35] a modified definition of thermodynamic temperature $T' = 2T$ was introduced that resulted in the modified value of *Joule-Mayer* mechanical equivalent of heat J [35]

$$J = 2J_c = 2\times 4.169 = 8338 \quad \text{Joule/kcal} \qquad (17)$$

where the classical value $J_c = 4.169 \simeq 4.17$ [kJ/kcal] is the average of two values $J_c = (4.15, 4.19)$ reported by *Pauli* [55]. The number in (17) is thus identified as the universal gas constant in (14) when expressed in appropriate MKS system of units

$$R^o = kN^o = J = 8338 \qquad \text{Joule/(kmol-K)} \qquad (18)$$

The modified value of the universal gas constant in (14) was recently identified [56] as *De Pretto* number 8338 that appeared in the mass–energy equivalence equation of *De Pretto* [57]

$$E = mc^2 \quad \text{Joule} \quad = mc^2/8338 \quad \text{kcal} \qquad (19)$$

Unfortunately, the name of *Olinto De Pretto* in the history of evolution of mass energy equivalence is little known. Ironically, *Einstein's* best friend *Michele Besso* was a relative and close friend of *Olinto De Pretto's* brother *Augusto De Pretto*. The relativistic form of (19) was first introduced in 1900 by *Poincaré* [58]

$$E = m_r c^2 \qquad (20)$$

where $m_r = m_o/\sqrt{1 - v^2/c^2}$. Since the expression (19) is the only equation in the paper by *De Pretto* [57], the exact method by which he arrived at the number 8338 is not known even though one possible method was recently suggested [56]. The important contributions by *Hasenöhrl* [59] and *Einstein* [60] as well as the equivalence principle, equivalence of the rest or gravitational mass and the inertial mass were discussed in a recent study [54].

According to (15) the atomic mass $\hat{m}_j$ of any chemical element j is the product of an integral number representing its molecular weight $\tilde{n}_j$ and the universal constant called atomic mass unit defined in (15) such that $\hat{m}_j \approx \tilde{n}_j(\mathrm{amu}) = \tilde{n}_j m_k c^2 = \tilde{n}_j\sqrt{hkc}$ in harmony with the perceptions of *Sommerfeld* [61]

> **"Our spectral series, dominated as they are by integral quantum numbers, correspond, in a sense, to the ancient triad of the lyre, from which the Pythagoreans 2500 years ago inferred the harmony of the natural phenomena; and our quanta remind us of the role which the Pythagorean doctrine seems to have ascribed to the integers, not merely as attributes, but as the real essence of physical phenomena."**

as well as the ideas of *Weyl* [62].

IV. OBJECTIVE VERSUS SUBJECTIVE NATURE OF THERMODYNAMIC ENTROPY

Possible subjective versus objective nature of entropy has been subject of much debate ever since 1948 when *Shannon* [63-64] used the name entropy in his information theory. The objective nature of thermodynamic entropy is evident form its classical definition first introduced by *Clausius*

$$dS = dQ_{rev}/T \qquad (21)$$

In other words, direct connection between entropy *S* and heat *Q* clearly establishes the objective nature of the former.

The misunderstandings concerning possible subjective nature of entropy could be due to its statistical definition first introduced by *Boltzmann*

$$S = k\ln W \quad , \qquad S_j = k\ln W_j \qquad (22)$$

The number of complexions W_j for distributing N_j indistinguishable particles among g_j distinguishable particle cluster j [42] or energy level j introduced by *Boltzmann* [65-67] and *Planck* [45] is

$$W_j = \frac{(N_j + g_j - 1)!}{N_j!\,(g_j - 1)!} \qquad (23)$$

The total number of complexions for statistically independent energy levels is

$$W = \prod_j W_j \qquad (24)$$

According to the scale invariant model of statistical mechanics the degeneracy g_j denotes number of distinguishable clusters (elements) in a particular energy level j of the system [35]. However, the smallest cluster contains only a single particle and is therefore considered to be full since no other particle can be added to this smallest cluster. Because an empty cluster has no physical significance, the total number of *available* clusters of energy level j will be $(g_j - 1)$. Therefore, *Planck-Boltzmann* formula (23) is the exact number of complexions for distributing N_j indistinguishable oscillators amongst $(g_j - 1)$ distinguishable *available* clusters. Hence, the scale invariant model of statistical mechanics (Fig. 1) provides new perspectives on the probabilistic nature of (23) and the problem of distinguishability discussed by *Darrigol* [68].

If following the ideas of *Onnes* and *Ehrenfest* the quantity $(N_j + g_j - 1)$ is viewed as the total number of symbols composed of N_j indistinguishable particles and $(g_j - 1)$ available distinguishable clusters one may consider the inverse of (23) as probability of complexion j

$$P_j = \frac{N_j!\,(g_j - 1)!}{(N_j + g_j - 1)!} = \frac{1}{W_j} \tag{25}$$

such that (22) and (24) result in

$$S_j = -k \ln P_j \qquad , \qquad S = -k \ln P \tag{26}$$

when the probability of all complexions are assumed to be independent of one another.

Under the realistic assumptions

$$g_j \gg N_j \qquad , \qquad N_j \gg 1 \tag{27}$$

it is known that the number of complexions for *Bose-Einstein* statistics in (23) simplifies such that all three types namely *Corrected Maxwell-Boltzmann*, *Bose-Einstein*, and *Fermi-Dirac* statistics will have [69]

$$W_j = N_j^{N_j} / N_j! \qquad , \qquad W = N^N / N! \tag{28}$$

Following the classical methods [69] the maximization of W in (23) will result in the *Boltzmann* distribution [42]

$$N_j = g_j e^{-\beta(\varepsilon_j - \alpha\hat{\mu}_j)} = g_j e^{-(\varepsilon_j - \alpha\hat{\mu}_j)/kT} \tag{29}$$

The constant coefficient α multiplying *Gibbs* chemical potential $\hat{\mu}_j$ was shown to be unity $\alpha = 1$ [42].

The exact nature of *Boltzmann* thermodynamic entropy is best revealed by looking at the simple case of monatomic ideal gas on which the present study is focused. Since it is known that physical particles are not mathematical points and hence possess a finite spatial extent, the energy due to rotation of particles cannot be properly neglected as emphasized by *Clausius* [70]. Also, because particles are not absolutely incompressible such as back holes, the energy due to vibration (pulsation) of particles should also be accounted for as pointed out by *Clausius* [70]

> **"In liquids, therefore, an oscillatory, a rotatory, and a translator motion of the molecules take place, but in such a manner that these molecules are not themselves separated from each other, but even in the absence of external forces, remain within a certain volume"**

In recent studies [42, 71] it was shown that when one identifies four degrees of freedom respectively associated with particle translational, rotational, vibrational, and potential energies and applies *Boltzmann* equipartition principle (28) leads to

$$W_{j4} = W_{jt} W_{jr} W_{jv} W_{jp} = (N_j^{N_j} / N_j!)^4 \approx e^{4N_j} \tag{30}$$

Substitution for total number of complexions from (30) into the *Boltzmann* formula (22) gives

$$S_j = 4N_j k \qquad , \qquad S = 4Nk \tag{31}$$

in exact agreement with the result obtained for photon gas in *Planck* equilibrium radiation [42].

According to (31) entropy is directly related to the number of *Heisenberg-Kramers* virtual oscillators [43] and its dimension [J/(kg.K] arises solely from that of *Boltzmann* constant k defined in (10). For example, when 1 kg water is evaporated at constant atmospheric pressure and constant temperature T = 373 K from liquid to vapor phase, since the mean energy per oscillator $\overline{\varepsilon}_j = kT$ is constant the increase in thermal energy equal to the latent heat of vaporization $L_v = h_g - h_f \approx 2260$ kj/kg occurs through increase of the number of oscillators hence entropy by (31). In other words the same mass of water at constant temperature of T = 373 K contains much larger number of oscillators in gas phase than in liquid as is to be expected. Identification of thermodynamic entropy as number of oscillators in (31) clearly establishes its objective nature.

Next, what is known as information entropy H according to *Shannon* [67-68] information theory and expressed by *Shannon* formula

$$H = -K \sum_j p_j \ln p_j \tag{32}$$

where K is a positive constant is examined. The above definition is based on conditional probability for occurrence of event j given a message A [69]

$$\hat{i} = K \ln[p_j(j/A)/p_j] \tag{33}$$

Only with maximum value of the conditional probability

$$p_j(j/A) = 1 \tag{34}$$

does equation (33) result in maximum information

$$\hat{i}_{max} = -K \ln p_j \tag{35}$$

leading to the number-averaged or the mean value of information [69]

$$\frac{I}{N} = \sum_j \hat{i}_{max} p_j = -K \sum_j p_j \ln p_j \tag{36}$$

Comparison of (36) and (32) shows that *Shannon* information entropy H should correspond to information conveyed per symbol.

It is possible to arrive at an expression for thermodynamic entropy similar to *Shannon* formula in (32) and (36) for information entropy. To do this however, the total number of complexions W must be expressed in terms of distribution of particles amongst various quantum states [69, 72]

$$W = N!/\prod_j N_j! \tag{37}$$

rather than amongst various energy levels as in (24). Substitutions for W from (37) and the atomic probability

$$p_j = N_j / N \tag{38}$$

into *Boltzmann* equation (22) lead to [69]

$$S = k \ln W = -Nk \sum_j p_j \ln p_j \tag{39}$$

or entropy per particle

$$\frac{S}{N} = -k \sum_j p_j \ln p_j \tag{40}$$

Besides the difference between the “measure” K versus *Boltzmann* universal constant k the thermodynamic entropy in (40) is identical to *Shannon* formula (32). One notes however that the thermodynamic entropy of *Boltzmann* (22) is based on W in (24) whereas entropy (40) is based on W in (37). The nature of equivalence of (32) and (40) will be further examined in the next Section.

V. INFORMATION THEORY AND ITS CONNECTIONS TO THERMODYNAMICs OF IDEAL MIXTURES

The application of scale invariant model of statistical mechanics schematically shown in Fig. 1 to languages in general and information theory of *Shannon* in particular suggests the following hierarchy

$$\text{Alphabet} = A_j = (\text{atom})_\beta$$
$$\text{Word} = W_j = (\text{elemnt})_\beta = (\text{atom})_{\beta+1}$$
$$\text{Sentence} = S_j = (\text{system})_\beta = (\text{element})_{\beta+1} = (\text{atom})_{\beta+2}$$

$$\text{Paragraph} = P_j = (\text{system})_{\beta+1} = (\text{element})_{\beta+2}$$
$$\text{Essay} = E_j = (\text{sytem})_{\beta+2} = (\text{element})_{\beta+3}$$
$$\text{Book} = B_j = (\text{system})_{\beta+3}$$
$$\cdots \tag{41}$$

Each language is constructed from a set of *distinguishable* alphabets or atoms A_j such as the English alphabets

A_1 = A, A_2 = B, A_3 = C, ….. A_{26} = Z, A_{27} = Blank

The above small number of distinguishable alphabets could be combined to form a very large number of *distinguishable* words W_j such as

W_1 = Democracy, W_2 = Freedom, …

One can thus create an entire hierarchy of entities, such as sentences, essays, books, libraries, … in accordance with (41).

Because of their *distinguishable* atomic species languages could be best modeled as multi-component mixture in thermodynamics. Therefore, one begins to examine ideal mixtures by considering the entropy of an ideal monatomic gas in classical thermodynamics given by

$$T d\bar{s} = d\bar{u} + p d\bar{v}, \tag{42}$$

or

$$d\bar{s} = \bar{c}_v \frac{dT}{T} + p \frac{d\bar{v}}{T} \quad . \tag{43}$$

According to (31) entropy is related to the number of oscillators. The number of *Heisenberg-Kramers* [43] virtual oscillators depends on volume *V* and on thermodynamic temperature since T is identified as *de Broglie* wavelength by (6b)-(7b) and hence relates to the atomic volume $\hat{v} = \lambda^3$ [71]. Therefore, for an ideal gas $p\bar{v} = R^o T$ the temperature derivative in (43) is expressed in terms of derivative of number of moles as

$$d\bar{S} = \bar{c}_v \frac{dT}{T} + p \frac{d\bar{v}}{T} = -3R^o \frac{d\bar{N}}{\bar{N}} + R^o \frac{d\bar{v}}{\bar{v}} \tag{44}$$

With the multiplicative factor of two in the modified definition of thermodynamic temperature $T' = 2T$ due to harmonic motion in two coordinate directions (x+, x-) at thermodynamic equilibrium, the specific heat at constant volume becomes $\overline{c}_v = 3R^o$ rather than $\overline{c}_v = 3R^o / 2$ as discussed in [42, 71]. By integration of (44) for species j from its pure state to its mixture state one arrives at

$$\begin{aligned}\overline{S}_j - \overline{S}_o &= -3R^o \ln \frac{N_j}{N} - R^o \ln \frac{p_j}{p} \\ &= -3R^o \ln \frac{N_j}{N} - R^o \ln \frac{N_j}{N} \\ &= -4R^o \ln \frac{N_j}{N} = -4N^o k \ln \frac{N_j}{N} \,. \end{aligned} \qquad (45)$$

Equation (45) with the atomic probability defined as in (38)

$$\hat{p}_j \equiv \frac{N_j}{N} \qquad (46)$$

and reference entropy $\overline{S}_{jo} = 0$ of a pure state $\hat{p}_j = 1$ gives

$$\overline{S} = N^o \hat{s}_j = -4N^o k \ln \hat{p}_j \qquad (47)$$

that results in the atomic entropy

$$\hat{s}_j = -4k \ln \hat{p}_j \qquad (48)$$

The factor of 4 in (48) is associated with the four degrees of freedom discussed in the previous Section and will be further clarified in the following.

From (48) the entropy of all type j atomic species becomes

$$S_j = N_j \hat{s} = -4k \ln \hat{p}_j^{\,N_j} = -k \ln P_j \qquad (49)$$

where

$$P_j = \prod_{N_j} \hat{p}_j = \hat{p}_j^{\,N_j} \qquad (50)$$

Next, entropy of the mixture is obtained by adding the entropy of all species from (49) to get

$$\begin{aligned} S = \sum S_j &= -4k \sum N_j \ln \hat{p}_j \\ &= -4kN \sum \hat{p}_j \ln \hat{p}_j \end{aligned} \qquad (51)$$

Comparison of (51) with *Shannon* formula (32) shows three important differences. First is the constant measure K in (32) versus *Boltzmann* constant k in (51) to be further discussed in the sequel. Next is the factor of 4 in (51) arising from the four degrees of freedom discussed above equation (30) and manifested in (45). Finally, the total number of atoms of all species

$$N = \sum N_j \qquad (52)$$

in (51) that does not appear in (32). *Shannon* entropy of information theory (32) and mixture entropy of thermodynamics (51) become mathematically equivalent if one chooses the *Shannon* "measure" as K = k and H in (32) is identified as atomic entropy of the mixture from (51)

$$\hat{s} = S / (4N) = -k \sum \hat{p}_j \ln \hat{p}_j \qquad (53)$$

The results in (52) and (32) could be compared with the classical form of molar entropy of ideal mixture [72]

$$S_{mix} = \sum N_j \hat{s}_j = -k \sum N_j \ln x_j \qquad (54)$$

Hence

$$S_{mix} = -kN \sum \frac{N_j}{N} \ln x_j = -kN \sum x_j \ln x_j \qquad (55)$$

involving species mole fraction

$$x_j = \frac{\overline{N}_j}{\overline{N}} = \frac{N_j}{N} = \frac{\overline{v}_j}{\overline{v}} = \hat{p}_j \qquad (56)$$

The result in (54) unlike (32) does not involve any averaging and represents direct sum of the entropy contributions of all species. The averaging process occurs through division by total number of atoms of all species N in (55). The difference by factor of 4 between (51) versus (55) is due to the inclusion of all four degrees of freedom in the former.

The classical definition of entropy by *Boltzmann* is based on the most probable number of complexions according to equations (22)-(24). However, because *Boltzmann* distribution function (29) itself is derived by maximization of W, it is possible to obtain a modified formula for determination of entropy directly from (29) that after simplification for ideal gas and approximate normalization is expressed as

$$N_j = Ne^{-\hat{\varepsilon}_j / kT} \qquad (57)$$

The simplification in (57) is due to the vanishing of chemical potential $\hat{\mu}_j = 0$ of ideal gas at equilibrium in accordance with the equilibrium conditions *G* = d*G* = 0 [42]. For a single atom with energy $\hat{\varepsilon}_j = kT$ (57) results in atomic probability per degree of freedom

$$\hat{p}_{\mathrm{j}} = \hat{p}_{\mathrm{jt}} \equiv \mathrm{N}_{\mathrm{j}} / \mathrm{N} = \mathrm{e}^{-1} \tag{58}$$

Although approximate, the result in (58) when substituted in (48) gives atomic entropy per degree of freedom exactly equal to *Boltzmann* constant k as is to be expected.

When all four degrees of freedom associated with translational, rotational, vibrational, and potential energy are taken into account along with *Boltzmann* equipartition principle the total atomic probability becomes

$$\hat{p}_{\mathrm{j4}} = \hat{p}_{\mathrm{jt}} \hat{p}_{\mathrm{jr}} \hat{p}_{\mathrm{jv}} \hat{p}_{\mathrm{jp}} = (\mathrm{N}_{\mathrm{j}} / \mathrm{N})^4 \approx \mathrm{e}^{-4} \tag{59}$$

By (58) one can express (59) as

$$\hat{p}_{\mathrm{j4}} = (\mathrm{N}_{\mathrm{j}} / \mathrm{N})^4 = \mathrm{e}^{-4\hat{\varepsilon}_{\mathrm{j}}/\mathrm{kT}} = \mathrm{e}^{-\hat{h}_{\mathrm{j}}/\mathrm{kT}} \tag{60}$$

where the atomic *transition enthalpy* $\hat{h}_{\mathrm{j}}$ of specie j of ideal monatomic gas is given by [42, 71]

$$\hat{h}_{\mathrm{j}} = \hat{u}_{\mathrm{j}} + \mathrm{p}_{\mathrm{j}} \hat{\mathrm{v}} = \hat{u}_{\mathrm{jt}} + \hat{u}_{\mathrm{jr}} + \hat{u}_{\mathrm{jv}} + \hat{u}_{\mathrm{jp}} = 4\mathrm{kT} \tag{61}$$

The results in (57), (58), (60), and (61) lead to the introduction of *atomic thermodynamic entropy* as

$$\hat{s}_{\mathrm{j}} = -\mathrm{k} \ln \hat{p}_{\mathrm{j4}} = -4\mathrm{k} \ln \hat{p}_{\mathrm{j}} = 4\mathrm{k} \tag{62}$$

The summation of the atomic entropy (62) over all atoms of specie j results in

$$S_{\mathrm{j}} = \sum \hat{s}_{j} = \mathrm{N}_{\mathrm{j}} \hat{s} = -4\mathrm{k} \ln P_{\mathrm{j}} = 4\mathrm{kN}_{\mathrm{j}} \tag{63}$$

where the probability of element composed of N_{j} atoms is

$$P_{\mathrm{j}} = \prod_{\mathrm{N}_{\mathrm{j}}} \hat{p}_{\mathrm{j}} = (\mathrm{N}_{\mathrm{j}} / \mathrm{N})^{\mathrm{N}_{\mathrm{j}}} = \hat{p}_{\mathrm{j}}^{\mathrm{N}_{\mathrm{j}}} = e^{-\mathrm{N}_{\mathrm{j}}} \tag{64}$$

Finally, summation of entropy (63) over all species results in the total mixture or system entropy

$$S = \sum S_{j} = \sum \mathrm{N}_{\mathrm{j}} \hat{s} = \mathrm{N}\hat{s} = -4\mathrm{k} \ln P = 4\mathrm{kN} \tag{65}$$

when the total probability *P* is defined as

$$P = \prod_{\mathrm{j}} P_{\mathrm{j}} = \prod_{\mathrm{j}} \hat{p}_{\mathrm{j}}^{\mathrm{N}_{\mathrm{j}}} = \hat{p}_{\mathrm{j}}^{\sum \mathrm{N}_{\mathrm{j}}} = \hat{p}_{\mathrm{j}}^{\mathrm{N}} = \mathrm{e}^{-\mathrm{N}} \tag{66}$$

In summary, the results (62), (63), and (65) respectively correspond to *Boltzmann* expressions of entropy at “atomic”, “element”, and “system” scales $(\hat{s}_{\mathrm{j}}, S_{\mathrm{j}}, S)$ with the corresponding probabilities $(\hat{p}_{\mathrm{j4}}, P_{\mathrm{j4}}, P_4)$ for four degrees of freedom

$$\hat{p}_{\mathrm{j4}} = \hat{p}_{\mathrm{j}}^{\,4} = \mathrm{e}^{-4\hat{\varepsilon}_{\mathrm{jt}}/\mathrm{kT}} = \mathrm{e}^{-\hat{\mathrm{h}}_{\mathrm{j}}/\mathrm{kT}} = \mathrm{e}^{-4} \tag{67a}$$

$$P_{\mathrm{j4}} = \hat{p}_{\mathrm{j}}^{4\mathrm{N}_{\mathrm{j}}} = \mathrm{e}^{-\mathrm{N}_{\mathrm{j}}\hat{\mathrm{h}}_{\mathrm{j}}/\mathrm{kT}} = \mathrm{e}^{-H_{\mathrm{j}}/\mathrm{kT}} = \mathrm{e}^{-4\mathrm{N}_{\mathrm{j}}} \tag{67b}$$

$$P_4 = \prod_{\mathrm{j}} P_{\mathrm{j}}^4 = \mathrm{e}^{-\sum H_{\mathrm{j}}/\mathrm{kT}} = \mathrm{e}^{-H/\mathrm{kT}} = \mathrm{e}^{-4\mathrm{N}} \tag{67c}$$

Exact correspondence between *Shannon* and *Boltzmann* statistics respectively based on distribution among quantum states versus energy levels can be established by substituting from (58) in (67b) to obtain

$$\frac{1}{P_{\mathrm{j}}} = (\mathrm{N} / \mathrm{N}_{\mathrm{j}})^{\mathrm{N}_{\mathrm{j}}} = \frac{\mathrm{N}^{\mathrm{N}_{\mathrm{j}}}}{\mathrm{N}_{\mathrm{j}}^{\mathrm{N}_{\mathrm{j}}}} = \frac{\mathrm{e}^{\mathrm{N}_{\mathrm{j}}} \mathrm{N}_{\mathrm{j}}^{\mathrm{N}_{\mathrm{j}}}}{\mathrm{N}_{\mathrm{j}}^{\mathrm{N}_{\mathrm{j}}}}$$

$$= \frac{\mathrm{N}_{\mathrm{j}}^{\mathrm{N}_{\mathrm{j}}}}{(\mathrm{N}_{\mathrm{j}} / \mathrm{e})^{\mathrm{N}_{\mathrm{j}}}} \approx \frac{\mathrm{N}_{\mathrm{j}}^{\mathrm{N}_{\mathrm{j}}}}{\mathrm{N}_{\mathrm{j}}!} \approx \frac{(\mathrm{N}_{\mathrm{j}} + \mathrm{g}_{\mathrm{j}} - 1)!}{\mathrm{N}_{\mathrm{j}}!(\mathrm{g}_{\mathrm{j}} - 1)!} = \mathrm{W}_{\mathrm{j}} \tag{68}$$

in accordance with (25). Similarly, by (58) the system probability in (67c) becomes

$$\frac{1}{P} = \prod_{\mathrm{j}} \hat{p}_{\mathrm{j}}^{\mathrm{N}_{\mathrm{j}}} = \frac{\mathrm{N}^{\mathrm{N}}}{\mathrm{N}_{\mathrm{j}}^{\mathrm{N}}} = \frac{\mathrm{N}^{\mathrm{N}}}{(\mathrm{N} / \mathrm{e})^{\mathrm{N}}}$$

$$= \frac{\mathrm{N}^{\mathrm{N}}}{\mathrm{N}!} \approx \frac{(\mathrm{N} + \mathrm{g} - 1)!}{\mathrm{N}!(\mathrm{g} - 1)!} = \mathrm{W} \tag{69}$$

that is in consistent with (24) and (68).

According to (62) the atomic entropy of all monatomic species are the same being four times the universal *Boltzmann* constant

$$\hat{s}_{\mathrm{j}} = \hat{s} = 4\mathrm{k} \tag{70}$$

The result in (70) is due to the fact that when particles of monatomic ideal gas are in equilibrium with the radiation field of photon gas within which they reside particles must satisfy both *de Broglie* matter-wave relation in (9a) as well as matter-wave frequency relation in (9b). Therefore (7b) (9b) (10), and (72) result in

$$\hat{s}_{\mathrm{j}} = 4\mathrm{k}_{\mathrm{j}} = 4\nu_{\mathrm{j}} \overline{\mathrm{p}}_{\mathrm{j}} = 4\mathrm{k} = \hat{s} \tag{71}$$

such that at thermodynamic equilibrium the *Boltzmann* factors (7b) of all particles satisfy (10) and hence are equal to the universal *Boltzmann* constant defined in (8b). A similar behavior occurs for atomic mass $\hat{m}_{\mathrm{j}}$ of different elements being an integral numbers $\tilde{n}_j$ times a universal constant namely the atomic mass unit given in (15) such that

$$\hat{m}_j \approx \tilde{n}_j(\text{amu}) = \tilde{n}_j m_k c^2 = \tilde{n}_j \sqrt{hkc} \qquad (72)$$

as discussed in Section III.

VI. PHYSICAL NATURE OF BOTH THERMODYNAMICs AND INFORMATION ENTROPY

Since *Boltzmann* entropy represents a measure of randomness of particles in thermodynamic system a decrease of entropy from its maximum value could be viewed as information and by (26) expressed as

$$I_{j\max} = S_{\max} - S_j = kN_j \ln(P_j / P_{\max}) \qquad (73)$$

Hence, in harmony with ideas of *Brillouin* [73], reduction of uncertainty from its maximum value to a lower value at state j could be interpreted as an increase in certainty thus constituting information as schematically shown in Fig. 3.

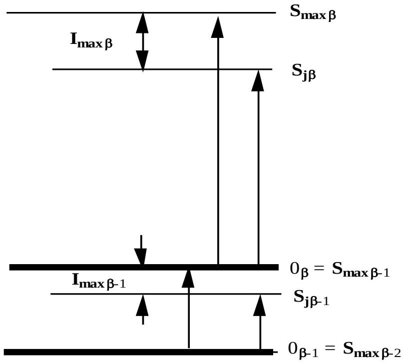

Fig. 3 Scale-invariant definition of thermodynamic information defined as reduction of absolute uncertainty or entropy $I_{\max} = S_{\max\beta} - S_{j\beta}$.

The nature of approach to the state of zero entropy $S_\beta = 0$ at scale β shown in Fig. 3 according to *Nernst-Planck* third law of thermodynamics will be discussed in the next Section. The region of negative entropy $S_\beta < 0$ at scale β ($S_{\beta-1} > 0$) in Fig. 3 on the other hand, reminiscent of *Brillouin* [73] concept of negentropy, could be identified with negative number of particles and hence associated with *Dirac* anti-matter. Thus, one could have a particle (N = 1), no particle or *Casimir* [44] vacuum (N = 0), or an anti-particle (N = –1). For example, in case of electron gas particle and anti-matter (hole) correspond to N = (–1, 1) and refer to electron and positron.

The occurrence of the universal constant of *Boltzmann* k in (40) has fundamental significance. According to equations (7)-(8) *Planck* and *Boltzmann* constants are two fundamental constants of nature that relate to spatial and temporal aspect of *Casimir* [44] vacuum fluctuations. Because all conceivable information must be transmitted by some physical entity such as electron, photon, or neutrino, etc. through an equally physical background field of space-time, it is reasonable to expect that both constants (h, k) will play a central role is transmission of information. Therefore, it is not advisable to modify *Boltzmann* thermodynamic entropy in (40) to achieve correspondence with *Shannon* information entropy in (32). Instead one can choose *Shannon* "measure" of information K to be identical to *Boltzmann* constant K = k and simply define both dimensionless thermodynamic and information entropy as

$$H / K = S / (Nk) = -\sum_j p_j \ln p_j \qquad (74)$$

in exact correspondence with one another given the equivalence of their respective probabilities in (38).

The central role of *Boltzmann* constant and its intimate relation to entropy is best revealed by examination of *Planck* energy spectrum for equilibrium radiation [35, 42]. Because of the closure of gap between ideal gas theory and radiation theory [42], the entropy of equilibrium photon gas is given by the same relation as ideal gas namely [42]

$$T_j S_j = 4N_j kT_j = 4N_j \overline{\varepsilon}_{tj} = 4N_j \frac{\hat{u}_j}{3} = \frac{4}{3} U_j \qquad (75)$$

In (75) the mean energy of each oscillator $\hat{\varepsilon}_{tj} = kT_j$ involves the product of two parts the first part k that relates to atomic entropy by (62) and the second part T that relates to *de Broglie* wavelength of particle by (6b) such that

$$\overline{\varepsilon}_\beta = \overline{p}_\beta \langle v_\beta^2 \rangle^{1/2} \langle \lambda_\beta^2 \rangle^{1/2} = k_\beta \lambda_\beta = kT_\beta \qquad (76)$$

By identifying k as atomic entropy per degree of freedom according to (62) the conjugate role played by (k, T) in thermodynamic will be also played by $(\hat{s}, T)$ such that their product $T\hat{s}$ gives the total oscillator energy or atomic enthalpy $T\hat{s} = 4kT = \hat{h}$ in accordance with (75).

Now that the nature of constant k in *Boltzmann* thermodynamic entropy has been identified, possible role of k in *Shannon* information entropy will be examined. First, it is noted that all communications and exchange of information must involve some kind of physical entity such as a particle like molecule, atom, electron, photon, neutrino, … involving molecular-dynamics, … electro-dynamics, chromo-dynamics, … statistical fields. Communication also involves ensembles of different symbols such as alphabets A_j, alphabet clusters or words W_j word clusters or sentences $S_{j\ \ldots}$ as encountered in the information theory of *Shannon* [67, 68]. Hence, parallel to system of ideal gas, one seeks a spectrum of "alphabet cluster" or "word" sizes that will maximize the entropy i.e. total number of symbols N of the system such that the number of *distinguishable complexions* W expressed by a corresponding *Boltzmann-Planck* equation (24) is maximized. In other words, under "*equilibrium*" distribution alphabet-cluster sizes will correspond to maximum W and hence entropy and therefore maximum N by (31) to be used to transmit maximum number of bits (1, 0) and hence information.

It is now clear that in order to insure an exact correspondence between *Boltzmann* and *Shannon* entropy one must choose the "measure" K = k and write *Shannon* formula (32) as

$$\mathrm{H} = -\mathrm{k}\sum_{\mathrm{j}} p_{\mathrm{j}} \ln p_{\mathrm{j}} \qquad (77)$$

Concerning the positive constant K in his formula (32) *Shannon* writes [63- 64]

> **"The choice of the coefficient K is a matter of convenience and amounts to the choice of a unit of measure"**

Because of the central roles of the conjugate pairs (k, T) and $(\hat{s}, \mathrm{T})$ in thermodynamic discussed after (75) the choice of the *measure* K = k in (77) is of fundamental significance. Since thermodynamic temperature is identified as a length scale by (76), choosing the measure K = k insures that $\mathrm{T}S = 4\mathrm{kNT} = H$ relates to the total energy for communication of N symbols in harmony with the perceptions of *Shannon* described in the above quotation. In summary, the introduction of *Boltzmann* constant k in (77) is necessary because it allows the entropy of information to be related to energy through *de Broglie* wavelength of particles being used to transmit the information.

Recently, it was suggested by *Ben-Naim* [74] that to achieve exact correspondence between *Boltzmann's* entropy in thermodynamics in (40) and *Shannon's* entropy in information theory in (32) the dimension of thermodynamics absolute temperature be changed

> **"One should redefine a new absolute temperature; denote it tentatively as $\overline{\mathrm{T}}$ to replace kT. The new temperature $\overline{\mathrm{T}}$ would have the units of energy and there would be no need for Boltzmann constant. The equation for entropy will be simply S = lnW, [21] and entropy would be rendered dimensionless"; "This will automatically expunge the Boltzmann constant $\mathrm{k_B}$ from the vocabulary of physics"**

According to the definitions in equations (5)-(8) of Section 3 *Planck* h and *Boltzmann* k constants are associated with the stochastically stationary aspects of *Casimir* [75] vacuum fluctuations with the dimensions

$$\varepsilon_\beta = \mathrm{h}\langle \nu_\beta^2 \rangle^{1/2} = \mathrm{h}_\beta \nu_\beta \quad \text{[J-s].[Hz] = [J]} \qquad (89a)$$

$$\varepsilon_\beta = \mathrm{k}\langle \lambda_\beta^2 \rangle^{1/2} = \mathrm{k}_\beta \lambda_\beta \quad \text{[J/m].[m]= [J]} \qquad (89b)$$

Therefore, neither *Planck* constant h nor what he defined as *Boltzmann* constant k could be expunged from the vocabulary of physics. Also, according to (6b) *Kelvin* absolute temperature is identified as a length scale associated with mean wavelength of thermal oscillations.

$$\varepsilon_\beta = \mathrm{k}\langle \lambda_\beta^2 \rangle^{1/2} = \mathrm{kT} \qquad (80)$$

The identification of dimension of absolute thermodynamic temperature as [meter] rather than [degree K] is a step towards clarification of the exact physical nature of this intensive property.

Because temperature is a measured quantity ultimately it is the theory that must determine the significance of such a variable and its dimension according to *Planck* [76].

> **"Every measurement first acquires its meaning for physical science through the significance which a theory gives it"**

Therefore, following *Planck's* suggestion, it is reasonable to change the dimension of absolute thermodynamic temperature T from [degree K] to [meter]. The exact correspondence between H and $\mathrm{I_{max}}$ could be achieved through the definition of dimensionless thermodynamic information entropy H in (74).

Arguments have also been made that the very name *entropy* first coined by its discoverer *Clausius* should be changed [77]

> **"It is also time to change not only the units of entropy to make it dimensionless, but the term "entropy" altogether. Entropy, as is now recognized, does not mean "transformation, or "change" or "turn". It does mean information. Why not replace the term that means "nothing" as Cooper noted, and does not even convey the meaning it was meant to convey when selected by Clausius? Why not replace it with a simple, familiar, meaningful, and precisely defined term "information?"**

It is emphasized however that contrary to the above quotation the name entropy has been chosen most appropriately by *Clausius* to represent precisely what it should namely "transformation". According to the result in (65) for ideal gas, entropy relates to the number of *Planck* [45] or *Heisenberg-Kramers* oscillators [43]. Therefore, the second law of thermodynamics suggests that in all natural processes there is a tendency to transform energy and hence motion from ordered motions of a few large-scale oscillators to random motions of many small-scale oscillators thus increasing the entropy of the system. In other words entropy according to its macroscopic definition by *Clausius* is directly related to heat

$$\mathrm{d}S = \mathrm{d}Q_{\mathrm{rev}} / \mathrm{T} \qquad (81)$$

Therefore, all types of energy dissipation induced by plastic deformation, friction, viscosity,… will lead to *transformation* of ordered (correlated) motions into disordered (uncorrelated) random motions that is heat thus leading to increased entropy by (81). It seems that if we follow the changes of dimensions and notations suggested by *Ben–Naim* [74, 77] the state of our scientific terminology may become void and hence approach what he calls "*Tohu Vavohu*" [74] meaning total chaos.

VII. IMPACT ON THIRD LAW OF THERMODYNAMICS

According to *Nernst-Planck* statement of the third law of thermodynamics, entropy must approach zero as absolute thermodynamic temperature approaches zero [72]. Another statement of the third law is [78]

“It is impossible to reach absolute zero using a finite number of processes”

or the absolute zero temperature cannot be reached with a finite number of steps. Examination of Fig. 1 clearly shows that the very definitions of space and time that is applicable across the hierarchies of embedded statistical fields require careful investigations. Recently, it was suggested [42] that the description of the hierarchy of statistical fields shown in Fig. 1 could be expressed in terms of a logarithmic coordinates schematically shown in Fig. 4.

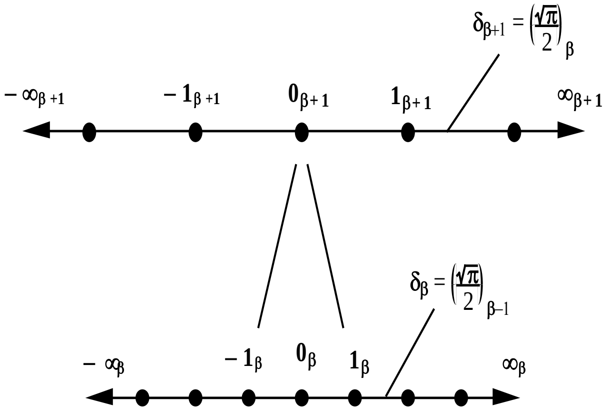

Fig. 4 Hierarchy of normalized coordinates associated with embedded statistical fields [42].

Because according to (76) thermodynamic temperature is identified as a length scale from the invariant normalized coordinate system shown in Fig.4 one arrives at a hierarchy of absolute zero temperatures [42]

$$\begin{aligned} &\cdots \\ &\mathrm{T}_\beta = 0 \Leftrightarrow \mathrm{T}_{\beta-1} = 1 \Leftrightarrow \mathrm{T}_{\beta-2} = \infty \\ &\mathrm{T}_{\beta-1} = 0 \Leftrightarrow \mathrm{T}_{\beta-2} = 1 \Leftrightarrow \mathrm{T}_{\beta-3} = \infty \\ &\mathrm{T}_{\beta-2} = 0 \Leftrightarrow \mathrm{T}_{\beta-3} = 1 \Leftrightarrow \mathrm{T}_{\beta-4} = \infty \\ &\cdots \end{aligned} \tag{82}$$

Furthermore, entropy of an ideal gas is a measure of the number of oscillators by (65). Therefore, from *Nernst-Planck* statement of the third law and (82) one arrives at the hierarchy of “absolute” zero temperature and corresponding “absolute” zero entropy as schematically shown in Fig. 5. The hierarchy shown in Fig. 5 is associated with the hierarchy of vacua at *equilibrium* conditions and the associated entropy and enthalpy leading to vanishing of the invariant *Gibbs* free energy

$$G_{v\beta} = H_{v\beta} - \mathrm{T}_{v\beta} S_{v\beta} = 0_\beta \tag{83}$$

From (75) and (76) the product $\mathrm{T}_\beta S_\beta = H_\beta$ and $\mathrm{T}_\beta \hat{s}_\beta = \hat{h}_\beta$ represent the system and atomic energy. Therefore as temperature T approaches zero the atomic energy $\hat{\varepsilon}_\beta = \mathrm{kT}_\beta$, the atomic internal energy $\hat{u}_\beta = 3\mathrm{kT}_\beta$, and the atomic enthalpy $\hat{h}_\beta = 4\mathrm{kT}_\beta$ of particles will all approach zero. However, such approach will occur through infinite steps corresponding to hierarchies of embedded statistical fields (Fig. 1) associated with ever smaller scales in accordance with (82) and as schematically shown in Figs. 4 and 5.

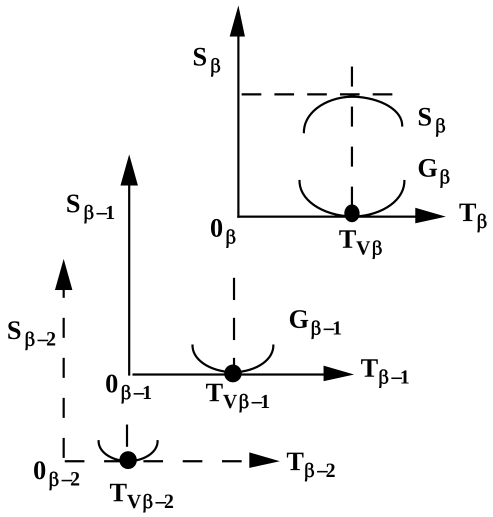

Fig. 5 Hierarchy of absolute “zeros” $\mathrm{T}_\beta = 0_\beta$ and “vacuum” $\mathrm{T}_{v\beta}$ temperatures and associated entropies approached at equilibrium corresponding to minimum Gibbs free energy G_β or maximum entropy S_β.

According to Fig. 5, the vacuum-vacuum of scale β is identified as *Casimir* [75] vacuum of scale β−1 as described in Section 10 of [42].

VIII. CONCLUDING REMARKS

A scale-invariant model of statistical mechanics was applied to describe the nature of the connection between *Boltzmann* thermodynamic entropy and *Shannon* information entropy. Also, the application of the model to statistical fields associated with social structures was described. In addition, the connections between *Shannon* information entropy and thermodynamic entropy of ideal gas mixtures were examined. A modified definition of thermodynamic entropy of ideal monatomic gas was introduced relating entropy to the number of oscillators in harmony with photon gas in *Planck* equilibrium radiation. Finally, the impact of the model on *Nernst-Planck* statement of the third law of thermodynamics was briefly discussed.

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