Unstable Constitutive Law in Continuum Mechanics

D. A. Indeitsev, D. Yu. Skubov, L.V. Shtukin, and D.S. Vavilov

Abstract—The present paper deals the possibility of obtaining stress-strain relation with a decreasing segment through considering two-component medium. The basic idea lies in the assumption that a crystalline structure consists of two lattices connected by nonlinear interaction force. This force depends on the relative displacement of the lattices. When the displacement reaches its critical value, the system passes to the unstable position leading to a complicated dynamics. In this article the problem of static loading of two-component medium is considered. The problem is limited to one space dimension. The analytic solution obtained by applying the Galerkin procedure is compared with results of numerical calculations.

Keywords— Kinematic loading, structural conversion, unstable constitutive law.

I. INTRODUCTION

IN recent papers on experimental study of a high-speed deformation under a shock wave loading [1], [2] it was found that inner structure can exert a serious impact on material, resulting in increasing such important characteristics such as yield stress and hardness. These changes indicate an important role of structural transformations, which are caused by a local loss of stability by crystalline lattice leading to appearance of inclusions. Obviously, the process of structural conversion make some contribution in overall deformation, but despite a large number of experimental data it is very difficult to estimate it. The influence of microstructure on material properties still remains unclear. Classical continuum mechanics doesn't take the microstructure into consideration. The relation between the conservation laws and the properties of material is established only by the constitutive law, and the possibility of structural transformations should be somehow reflected in it. For instance, in many works devoted to phase transitions in solids [3],[4] it is accepted that description of this process is based on a non-monotone stress-strain relation. In case of one-dimensional problem the constitutive curve is often presented with a piece-wise function including a decreasing segment that separates two stable phases A and B (see Fig.1).

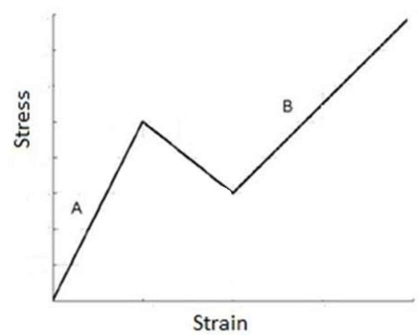

Fig. 1 Constitutive curve

The presence of decreasing branch means that after reaching the critical value of deformation material can not resist to external load, and after passing the unstable segment this ability is recovered, leading to modification of material properties [5],[6]. The solution of dynamic problems with such constitutive law requires the introduction of a new degree of freedom indicating the position of phase border [7]. Integrating mass and momentum balance equations through the border allows to obtain the additional jump conditions, which are necessary to determine the stress distribution. In the present paper we would like not to postulate the non-monotone stress-strain relation, but to obtain it as a result of solving a problem of static loading, which is widely used for determination of elastic properties[8].

II. TWO-COMPONENT MEDIUM. BASIC EQUATIONS

Let us consider the model of material with a complicated crystalline structure consisting of two lattices. Let $\mathbf{V}_k$ be the particle velocity of the component (k=1,2). Then the mass balance equations for each component are given by

$$\frac{\partial \rho_1}{\partial t} + \nabla \cdot \left(\rho_1 \mathbf{V}_1 \right) = \chi_{m1}$$
$$\frac{\partial \rho_2}{\partial t} + \nabla \cdot \left(\rho_2 \mathbf{V}_2 \right) = \chi_{m2} \quad (1)$$

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Functions χ_{m1} and χ_{m2} characterize the rate of mass production. If the total density of material $\rho = \rho_1 + \rho_2$ is preserved, then the equality $\chi_{m1} = -\chi_{m2}$ should be satisfied. Otherwise, the balance of mass won't be obeyed. The conventional form of mass balance equation can be obtained by adding the equations (1)

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) = 0. \quad (2)$$

Here $\mathbf{V} = \frac{(\rho_1 \mathbf{V}_1 + \rho_2 \mathbf{V}_2)}{\rho_1 + \rho_2}$ signifies the center of mass velocity. The dynamic equation of the system has the form [9]

$$\nabla \cdot \boldsymbol{\sigma} + \rho \mathbf{F} = \rho_1 \frac{d_1 \mathbf{V_1}}{dt} + \rho_2 \frac{d_2 \mathbf{V}_2}{dt} + \chi_{m1} \left(\mathbf{V_1} - \mathbf{V}_2 \right), \quad (3)$$

where $\boldsymbol{\sigma}$ is the stress tensor and $\mathbf{F}$ is the mass density of external force. Assuming that $\boldsymbol{\sigma} = \boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2$, it is convenient to rewrite equation (3) in the form of two equations

$$\begin{aligned} \nabla \cdot \boldsymbol{\sigma}_1 + \rho_1 \mathbf{F_1} - \mathbf{Q} &= \rho_1 \frac{d\mathbf{V}_1}{dt} + \chi_{m1} \mathbf{V_1} \\ \nabla \cdot \boldsymbol{\sigma}_2 + \rho_2 \mathbf{F}_2 + \mathbf{Q} &= \rho_2 \frac{d\mathbf{V}_2}{dt} + \chi_{m2} \mathbf{V}_2 \end{aligned}, \quad (4)$$

where $\rho \mathbf{F} = \rho_1 \mathbf{F}_1 + \rho_2 \mathbf{F}_2$. Here the stress in each lattice is denoted as $\boldsymbol{\sigma}_k$ and $\mathbf{Q}$ signifies the interaction force between the components. The operator $\frac{d}{dt}$ signifies taking the material derivative, i.e. $\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{V_k} \cdot \nabla$.

Equations (4) are the basic ones for two-component medium. They will be used for the further analysis, but for the sake of simplicity we need to make several assumptions. First of all, let us suppose that there is no mass production (or destruction) and there are no external forces acting on the system. Also, we will consider only small deformations. Thus, material derivatives can be replaced by partial ones. Finally, we will limit the problem by only one dimension in space. Taking into account all these assumptions, we can present equations (4) in the following form

$$\begin{aligned} \frac{\partial \sigma_1}{\partial x} - Q &= \rho_{01} \frac{\partial V_1}{\partial t} \\ \frac{\partial \sigma_2}{\partial x} + Q &= \rho_{02} \frac{\partial V_2}{\partial t} \end{aligned}, \quad (5)$$

where ρ_{0k} is density in the equilibrium. Beside dynamic equations, we also need to establish the relation between stress and displacement. If deformations in each component are small, we can assume the validity of Hooke's Law. In this case the equations (5) are written as

$$\begin{aligned} E_1 \frac{\partial^2 u_1}{\partial x^2} - Q &= \rho_{01} \frac{\partial^2 u_1}{\partial t^2} \\ E_2 \frac{\partial^2 u_2}{\partial x^2} + Q &= \rho_{02} \frac{\partial^2 u_2}{\partial t^2} \end{aligned}. \quad (6)$$

Here u_k denotes displacement of the components and E_k is Young's modulus. The form of expression for interaction force depends on the properties of medium which have a dominant role in the investigated phenomena. The process of conversion of crystalline structure during phase transition can be conceived by using a rheological model [10] depicted in Fig. 2.

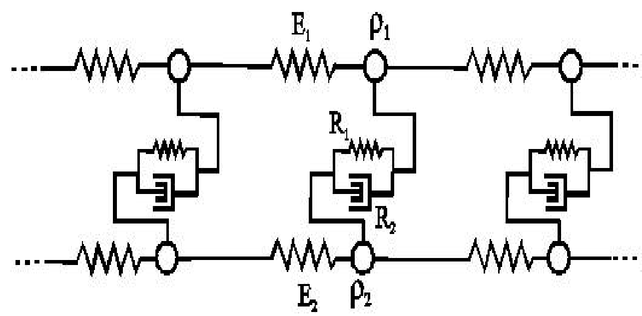

Fig. 2 Rheological Model

Suppose that as a result of the longitudinal displacement of the chains with respect to each other, material points can occupy a new stable equilibrium position corresponding to another phase. Taking into account the periodic structure of the lattice, the simplest expression for interaction force can be chosen as follows

$$Q(z) = K \sin \lambda z, \quad (7)$$

where K defines its maximum value. The relative displacement is denoted as $z = u_1 - u_2$ and λ signifies the reciprocal of the period in crystalline lattice. In essence, function z plays the role of a latent degree of freedom, which can't be recorded during experiment. It can be revealed only through its impact on the center of mass. Basing upon these considerations [11] it is convenient to rewrite equations (6) with respect to the center of mass displacement $U=\dfrac{\rho_1 u_1+\rho_2 u_2}{\rho_1+\rho_2}$ and relative displacement z

$$
\begin{aligned}
&U_{xx}-\frac{m^2}{c_1^2(1-\alpha(1-m^2))}U_{tt}=\frac{(1-\alpha)\alpha(1-m^2)}{1-\alpha(1-m^2)}z_{xx}\\
&z_{xx}-\frac{(1-\alpha(1-m^2))}{c_1^2}z_{tt}=\beta Q(z)+\frac{1-m^2}{c_1^2}U_{tt}
\end{aligned}
\quad . \quad (8)
$$

In these equations $\alpha=\dfrac{\rho_1}{\rho_1+\rho_2}$ is the ratio of the first component's density to the total density of material. The parameter $m=\dfrac{c_1}{c_2}$ stands for the ratio of the sound velocities $c_k=\sqrt{\dfrac{E_k}{\rho_{0k}}}$ and the coefficient $\beta=\dfrac{E_1+E_2}{E_1E_2}$ depends on rigidity of the components. If z is negligibly small, the problem reduces to the linear wave equation with respect to the center of mass.

III. KINEMATIC LOADING. STATEMENT OF THE PROBLEM

In the present paper we discuss the possibility of obtaining the constitutive relation with decreasing segment (see Fig. 1). For this purpose we consider a static loading of two-component rod of length l, which is described by system of equations (8). It is obvious that the static problem reduces to one equation with respect to z

$$z_{xx}-\beta Q(z)=0\,, \quad (9)$$

which should be supplemented by the appropriate boundary conditions. Suppose that one of the butts of the rod is fixed and another one moves accordingly to the prescribed function of time $U_0(t)$. These conditions can be written down in the following form

$$u_1=u_2\big|_{x=0}=0,\ U=\left.\frac{\rho_{01}u_1+\rho_{02}u_2}{\rho_{01}+\rho_{02}}\right|_{x=l}=U_0(t)\,. \quad (10)$$

From system of equations (6) it is clearly seen that three boundary conditions are not enough to determine the solution. So, we need an additional assumption which allows us to find one more condition. Let us assume that the stress distribution among the components is proportional to their densities

$$E_k\frac{\partial u_k}{\partial x}=\frac{\rho_{0k}\sigma}{\rho_{01}+\rho_{02}}\quad . \quad (11)$$

The assumption (11) is a moot point, but at least it doesn't contradict to the postulate of stress summation. The equation (9) and conditions (10) and (11) form a full problem of kinematic loading. Of course, all boundary conditions should be rewritten with respect to z. Our aim is to determine the relation between σ and $U_0(t)$. After introducing dimensionless variables: $\xi=\dfrac{x}{l}$, $w=\dfrac{z}{l}$, $P=\dfrac{\sigma}{E_1+E_2}$, $\varepsilon_0=\dfrac{U_0}{l}$ the problem can be presented in the form :

$$
\begin{aligned}
&w_{xx}-\gamma\sin(\lambda l w)=0\\
&w_{\xi=0}=0,\ \left.\frac{\partial w}{\partial\xi}\right|_{\xi=1}=\frac{Pr}{E_1E_2\rho_0},\\
&w\big|_{\xi=1}=\frac{\rho_0}{r}\left(\varepsilon_0-P\right).
\end{aligned}
\quad (12)
$$

Here the following notation is used: $r=E_2\rho_{01}-E_1\rho_{02}$, $\gamma=Kl\beta$, $\rho_0=\rho_{01}+\rho_{02}$. Eliminating P from equations (12) one can obtain the problem with mixed-boundary conditions. Note that parameter r corresponds to the difference between the sound velocities of the components. If dimensionless parameter λl is small, the problem becomes a linear one. Its analytical solution is given by

$$P=E_{ef}\varepsilon_0,\ E_{ef}=\frac{1}{1+\dfrac{r^2\tanh\sqrt{\tilde\gamma}}{\rho_0E_1E_2\sqrt{\tilde\gamma}}}, \quad (13)$$

where $\tilde\gamma=\gamma\lambda l$ The relation (13) can be regarded as Hooke's law with equivalent Young modulus equal to E_{ef}, which depends on parameters r and $\tilde\gamma$. If the sound velocities of the components are equal, we deal with spring's parallel connection. The same situation happens, if the force between the components becomes infinitely strong. The function $F(\tilde\gamma)=\dfrac{\tanh(\sqrt{\tilde\gamma})}{\sqrt{\tilde\gamma}}$ tends to zero as the argument $\tilde\gamma$ goes to infinity.

IV. Galerkin Procedure. Numerical Solution.

For nonlinear problem we apply the Galerkin method, taking for simplicity only one basic function $f(x) = x$ and searching solution in the form $z = Ax$. After multiplying equation (13) by $f(x)$ and integrating it between the limits $x = 0$ and $x = 1$, we arrive at the following algebraic system of equations

$$\begin{aligned} &\frac{\gamma}{\eta}\cos(\eta) - \frac{\gamma}{\eta^2}\sin(\eta) - A + \frac{P\beta r}{\rho_0} = 0 \\ &A = \frac{(E_1 + E_2)\rho_0}{r}(\varepsilon_0 - P) \end{aligned}, \qquad (14)$$

where $\eta = \lambda l A$. Boundary conditions (12) are already taken into account. The first equation of system (14) can be written as $F(A, P)$ =0, where F is implicit function. Obtaining the stress-strain relation analogous to the one shown in Fig. 1 requires that the derivative of stress with respect to the prescribed elongation of the rod equals to zero at critical points. Between these points the derivative should be negative. Using the implicit function allows to present the last requirement in the following form:

$$\frac{d\varepsilon_0}{dP} = -\frac{\dfrac{\partial F}{\partial A}\dfrac{\partial A}{\partial P} + \dfrac{\partial F}{\partial P}}{\dfrac{\partial F}{\partial A}\dfrac{\partial A}{\partial \varepsilon_0}} < 0 \ . \qquad (15)$$

At critical points the denominator is equal to zero. This condition can be brought to the form

$$\left(\frac{2}{\eta^2} - 1\right)\sin(\eta) - \frac{2}{\eta}\cos(\eta) = \frac{\eta}{\lambda l \gamma}. \qquad (16)$$

The equation (16) has roots only if the inequality $\eta < \lambda l \gamma$ is fulfilled. On the other hand, relation (15) leads us to another inequality $\eta > \dfrac{\gamma\lambda l}{2 + \dfrac{r^2}{E_1 E_2 \rho_0^2}}$. Thus, we have a strong constraint on the roots of equation (16)

$$\frac{\gamma\lambda l}{2 + \dfrac{r^2}{E_1 E_2 \rho_0^2}} < \eta < \gamma\lambda l \ . \qquad (17)$$

Although the gap is rather narrow, it is not very difficult to find the parameters, satisfying inequality (17). One of the examples is demonstrated in Fig. 3. The calculations are performed with the following parameters:

$$E_1 = 1\cdot 10^{10}\,N,\ \mathrm{E}_2 = 5\cdot 10^{10}\,N,\ \rho_{01} = \rho_{02} = 3\cdot 10^3\,kg/m,$$
$$K = 5\cdot 10^{11}\,N/m,\ l = 1\cdot 10^{-3}m,\ \lambda = 1\cdot 10^5 m^{-1}.$$

These parameters do not correspond to any real material. They were chosen arbitrarily to illustrate the solution. Dots in Fig. 3 indicate the results of numerical calculation performed by using finite difference method with the same parameters.

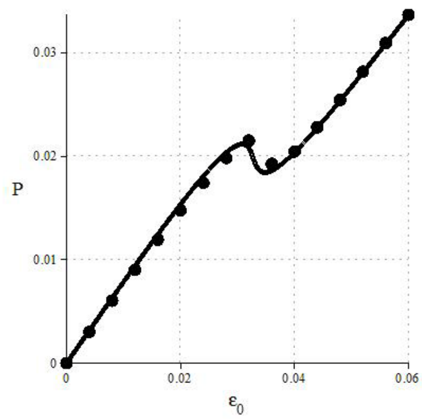

Fig. 3 Constitutive relation

This method was used to verify the solution obtained by Galerkin procedure. Two different approaches correlate well with each other.

V. Conclusion

The present paper deals with the problem of structural conversions of material. We consider the model of two-component medium which implies introducing an additional degree of freedom and writing a separate equation for it. We demonstrate that in one-dimensional case this approach allows to obtain a non-monotone stress-strain relation, which is widely used for description of phase transitions in solids. One of the controversial points here is the form of expression for nonlinear force. It may depend on different factors and it is a complicated problem to pick out the most important ones. The boundary condition (8) also seems to be rather contentious. There are many doubts about the restriction (14). Does it have any physical sense or is it just a mathematical trick? We assume that it is necessary to investigate dynamic equations (5) to answer this question. Finally, it not clear yet whether it is possible to extend this approach for two and three-dimensional problems.

References

[1] Garkushin G. V., Ignatova, O. N. Kanel, G. I. Meyer, L. W., and Razorenov S. V. (2010). Submicrosecond strength of ultrafine-grained materials. Mechanics of solids, 45(4), pp.624-632.

[2] Razorenov S. V., Garkushin G. V., and Ignatova O. N. (2012). Resistance to dynamic deformation and fracture of tantalum with different grain and defect structures. Physics of the Solid State, 54(4), pp. 790-797.

[3] Slepyan L. I. (2001). Feeding and dissipative waves in fracture and phase transition: II. Phase-transition waves. Journal of the Mechanics and Physics of Solids, 49(3), pp. 513-550.

[4] Făciu C., and Molinari A. (2006). On the longitudinal impact of two phase transforming bars. Elastic versus a rate-type approach. Part I: The elastic case. International journal of solids and structures, 43(3), pp. 497-522.

[5] Padevet P., Tesarek P., Plachy T. (2011) Evolution of Mecanical Properties of gypsum in time. International J. of Mechanics, 1(5) pp.1-9.

[6] Z. Boukira, A. Limam. Experimetal damage analysis of concrete structures using the vibration signature-PartI: diffuse damage (porosity), 1(6), pp. 17-27.

[7] Gavrilov S. N., and Shishkina E. V. (2010). On stretching of a bar capable of undergoing phase transitions. Continuum Mechanics and Thermodynamics, 22(4), pp. 299-316.

[8] L. Pagnotta. (2008) Recent Progress in identification methods for the elastic characterization of materials,4(2), pp.129-140.

[9] Zhilin P. A. (2006). Advanced problems in mechanics. St. Petersburg.

[10] Indeitsev D. A., Naumov V. N., Semenov B. N. (2007). Dynamic effects in materials of complex structure. Mechanics of Solids, 42(5), pp. 672-691.

[11] Aero E. L., and Bulygin A. N. (2008). The Nonlinear Theory of Localized Waves in a Complex Crystalline Lattice as a Discrete Continual Model. Vych. Mekh. Sploshn. Sred, 1(1), pp. 14-30.