The simplest models of viscoelasticity involving fractional derivatives and their connectedness with the Rabotnov fractional order operators*

Yury A. Rossikhin and Marina V. Shitikova
Voronezh State University of Architecture and Civil Engineering
Research Center on Dynamics of Solids and Structures
20-letija Oktjabrja Street 84, 394006 Voronezh, RUSSIA

Dedicated to the 100th Birthday of Academician Yury N. Rabotnov

Abstract – The given paper, the simplest viscoelastic models involving fractional derivatives, namely: fractional derivative Kelvin-Voigt model, Maxwell model, standard linear solid model and Koeller model, which is the generalization of the fractional derivative standard linear solid model, are considered, and their connectedness with the Rabotnov dimensional fractional operators is revealed. It is shown that when the order of the fractional derivative is tending to zero, then some of the enumerated models lose their physical meaning, while the other go over into the models describing pure elastic materials. These results have been achieved only due to the consideration of the resolvent operators, which allow one to express not only the stress in terms of the strain but the strain in terms of the stress as well.

Keywords – Rabotnov fractional order operator, resolvent operators, limiting values of Rabotnov's operator, viscoelastic fractional derivative models, fractional parameter.

I. Introduction

On February 24, 2014, Russian science and fractional calculus community all over the world celebrated the centennial anniversary since the birthday of the Russian Academician Yury Nikolaevich Rabotnov who was an outstanding scientist in the field of Solid Mechanics and who made a basic contribution to the development of the theory of elasticity and plasticity, the theory of shells and stability of elastic and viscoplastic systems, the creep theory of metals and the hereditary theory of elasticity. He actively worked in the new directions of damage and fracture mechanics and mechanics of composite materials.

Rabotnov is a pioneer in the application of fractional operators based on the fractional derivatives in Mechanics of Solids. He suggested such fractional operators, the resolvent operators to which are fractional operators of the same order. Moreover, he developed the algebra of these operators. The main Rabotnov's ideas and results in the field of hereditary mechanics could be found in [1]–[4].

For the first time the fractional operator was introduced by Rabotnov in 1948 in his paper "Equilibrium of an elastic medium with after-effect" published in a Russian academic journal *Prikladnaya Matematika i Mekhanika* [1], which in 2014 has been translated into English and reprinted in *Fractional Calculus and Applied Analysis*, vol. 17, no. 3, pp. 684–696, DOI: 10.2478/s13540-014-0193-1 due to the initiative of Professor Virginia Kiriakova, the Editor-in-Chief.

The authors of this paper wish to thank Prof. Kiriakova on behalf of the Russian Solid Mechanics community for this tremendous contribution in celebrating the centennial jubilee of Academician Rabotnov. Now researchers all over the world interested in fractional calculus application in Mechanics could read this pioneer Rabotnov's paper! It is very important, since all rheological models involving fractional derivatives or fractional integrals could be represented in the form of Boltzmann-Volterra relationships with Rabotnov's fractional operator kernels [5].

It has been emphasized by Rabotnov [2] that the majority of experiments carried out with viscoelastic materials are creep experiments, that is why if a rheological model is written in the form of the stress-strain relationship, then it is necessary to find the reverse connection, i.e. to express the strain ε in terms of the stress σ. Precisely this relation allows one to carry out creep experiments and to determine the physical constants involving in this strain-stress relationship. In other words, it is a need to construct the resolvent operators for each model.

In the present paper, the simplest viscoelastic models involving fractional derivatives are considered, and their relation to the Rabotnov dimensional fractional operator is revealed. It is shown that when the order of the fractional derivative is tending to zero, then some models lose their physical meaning, while the other go over into the models

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describing pure elastic materials.

II. The Simplest Fractional Derivative Viscoelastic Models

Let us consider the simplest viscoelastic fractional calculus models which could be obtained by substituting the integer order derivatives in the conventional models of viscoelasticity by the fractional order time-derivatives using the Riemann-Liouville definition [6]

$$D^\gamma \sigma = \frac{\mathrm{d}}{\mathrm{d}t} \int_0^t \frac{\sigma(t')}{\Gamma(1-\gamma)(t-t')^\gamma}\, \mathrm{d}t', \tag{1}$$

where σ is the stress, $0 < \gamma \le 1$ is the order of the fractional derivative, and $\Gamma(1-\gamma)$ is the Gamma-function.

A. Fractional Derivative Kelvin-Voigt Model

The fractional derivative Kelvin-Voigt model was introduced by Shermergor in 1966 [7] and it has the form

$$\sigma = E_0 \varepsilon + E_0 \tau_\sigma^\gamma D^\gamma \varepsilon, \tag{2}$$

where ε is the strain, γ $(0 < \gamma \le 1)$ is the fractional parameter, τ_σ is the retardation (creep) time, and E_0 is the relaxed elastic modulus (prolonged modulus of elasticity, or the rubbery modulus).

At $\gamma = 1$, the model (2) goes over into a conventional Kelvin-Voigt model of viscoelasticity.

The equation resolvent to (2) has the form

$$\varepsilon(t) = J_0 \, \frac{1}{1 + \tau_\sigma^\gamma D^\gamma} \, \sigma(t), \tag{3}$$

where $J_0 = E_0^{-1}$ is the prolonged compliance, while the operator

$$\ni_\gamma^* \left(\tau_\sigma^\gamma\right) = \frac{1}{1 + \tau_\sigma^\gamma D^\gamma} \tag{4}$$

is the dimensionless Rabotnov operator [8].

Considering that $D^\gamma I^\gamma = I^\gamma D^\gamma = 1$, we could represent the operator (4) as

$$\ni_\gamma^* \left(\tau_\sigma^\gamma\right) = \frac{I^\gamma \tau_\sigma^{-\gamma}}{1 - \left(-I^\gamma \tau_\sigma^{-\gamma}\right)}, \tag{5}$$

where

$$I^\gamma \sigma = \int_0^t \frac{(t-t')^{\gamma-1}}{\Gamma(\gamma)}\, \sigma(t')\mathrm{d}t' \tag{6}$$

is the fractional integral.

If we suppose that the right part of formula (5) is the sum of an infinite decreasing geometrical progression, the denominator of which is equal to $d = -I^\gamma \tau_\sigma^{-\gamma}$, then we find

$$\ni_\gamma^* \left(\tau_\sigma^\gamma\right) = \sum_{n=0}^{\infty} (-1)^n \tau_\sigma^{-\gamma(n+1)} I^{\gamma(n+1)}, \tag{7}$$

Considering (7), the strain-stress relationship (3) could be rewritten as

$$\varepsilon = J_0 \ni_\gamma^* \left(\tau_\sigma^\gamma\right) \sigma(t), \tag{8}$$

or

$$\varepsilon = J_0 \int_0^t \ni_\gamma \left(t'/\tau_\sigma\right) \sigma(t-t') dt', \tag{9}$$

where

$$\ni_\gamma \left(t/\tau_\sigma\right) = \frac{t^{\gamma-1}}{\tau_\sigma^\gamma} \sum_{n=0}^{\infty} \frac{(-1)^n (t/\tau_\sigma)^{\gamma n}}{\Gamma[\gamma(n+1)]} \tag{10}$$

is the fractional exponential function suggested by Rabotnov [1] in 1948, which at $\gamma = 1$ goes over into conventional exponential function. But when $\gamma \to 0$, it transforms into the δ-like sequence, since $\ni_0 \ (t/\tau_\sigma)$ vanishes to zero at any t $(\Gamma(0) = \infty)$ except the magnitude $t = 0$, at which $\ni_0 \ (0) = \infty$, i. e.,

$$\lim_{\gamma \to 0} \ni_\gamma \left(t'/\tau_\sigma\right) = \delta(t'). \tag{11}$$

That is why at $\gamma = 0$, as it follows from (9),

$$\varepsilon(t) = J_0 \sigma(t), \tag{12}$$

while from the stress-strain relationship (2) at $\gamma = 0$ we have

$$\sigma(t) = 2E_0 \varepsilon(t). \tag{13}$$

From the comparison of contradictive formulas (12) and (13) it follows that the generalized Kelvin-Voigt model (2) could not be utilized at $\gamma = 0$, since it loses the physical meaning in this case.

B. Fractional Derivative Maxwell Model

The same conclusion, as it has been done above for the fractional derivative Kelvin-Voigt model, could be made for the fractional derivative Maxwell model

$$J_\infty \left(\sigma + \tau_\varepsilon^\gamma D^\gamma \sigma\right) = \tau_\varepsilon^\gamma D^\gamma \varepsilon, \tag{14}$$

where $J_\infty = E_\infty^{-1}$ is the instantaneous compliance, E_∞ is the non-relaxed (instantaneous, or the glassy modulus) modulus of elasticity, and τ_ε is the relaxation time.

The stress-strain relationship (14) could be rewritten in another form if we apply the operator I^γ to the left and right hand-side parts of (14) and consider that $I^\gamma D^\gamma = D^\gamma I^\gamma = 1$. As a result we obtain the strain-stress relationship corresponding to the generalized Maxwell model written in terms of the fractional integral, which was for the first time introduced by Shermergor in 1966 [7]

$$\varepsilon(t) = J_\infty \left[\sigma(t) + \tau_\varepsilon^{-\gamma} \int_0^t \frac{t'^{\gamma-1}}{\Gamma(\gamma)}\, \sigma(t-t')\mathrm{d}t'\right]. \tag{15}$$

Expressing the value of $\sigma(t)$ from (14), we have

$$\sigma(t) = E_\infty \frac{\tau_\varepsilon^\gamma D^\gamma}{1 + \tau_\varepsilon^\gamma D^\gamma} \varepsilon(t), \tag{16}$$

or

$$\sigma(t) = E_\infty \left(1 - \frac{1}{1 + \tau_\varepsilon^\gamma D^\gamma}\right) \varepsilon(t). \tag{17}$$

Considering (4) and (7), (17) could be rewritten in the following form:

$$\sigma(t) = E_\infty \left[\varepsilon(t) - \int\limits_0^t \ni_\gamma (t'/\tau_\varepsilon) \, \varepsilon(t - t')\mathrm{d}t'\right]. \tag{18}$$

Tending γ to 0 in (15) and (18) and considering that

$$\lim_{\gamma \to 0} \frac{t'^{\gamma-1}}{\Gamma(\gamma)} = \delta(t'), \tag{19}$$

$$\lim_{\gamma \to 0} \ni_\gamma (t'/\tau_\varepsilon) = \delta(t'), \tag{20}$$

we find

$$\varepsilon(t) = 2J_\infty \sigma(t), \tag{21}$$

$$\sigma(t) \equiv 0, \tag{22}$$

whence it follows that $\varepsilon \equiv 0$ as well.

C. Fractional Derivative Standard Linear Solid Model

The other situation is with the fractional derivative standard linear solid model, which for the first time was suggested by Meshkov in 1967 [9], who was a postdoctoral researcher of Academician Rabotnov in those times,

$$J_0 \left(\sigma + \tau_\varepsilon^\gamma D^\gamma \sigma\right) = \varepsilon + \tau_\sigma^\gamma D^\gamma \varepsilon, \tag{23}$$

where

$$\left(\frac{\tau_\varepsilon}{\tau_\sigma}\right)^\gamma = \frac{E_0}{E_\infty} = \frac{J_\infty}{J_0}. \tag{24}$$

It should be noted that formula (24) was derived by Shermergor [7] and Meshkov [9] from the comparison of the resolvent operators describing the stress-strain and strain-stress relationships. This formula is very important from the physical point of view, since it provides the coupling between the rheological parameters of the model to ensure its physical validity.

Let us first express $\sigma(t)$ in terms of $\varepsilon(t)$ from (23). As a result we obtain

$$\sigma(t) = E_0 \frac{1 + \tau_\sigma^\gamma D^\gamma}{1 + \tau_\varepsilon^\gamma D^\gamma} \varepsilon(t), \tag{25}$$

or with due account for (24)

$$\sigma(t) = E_0 \frac{1 + E_\infty E_0^{-1} \tau_\varepsilon^\gamma D^\gamma}{1 + \tau_\varepsilon^\gamma D^\gamma} \varepsilon(t). \tag{26}$$

Further we represent (26) in the form

$$\sigma(t) = E_\infty \frac{E_0 E_\infty^{-1} + \tau_\varepsilon^\gamma D^\gamma + 1 - 1}{1 + \tau_\varepsilon^\gamma D^\gamma} \varepsilon(t), \tag{27}$$

or after dividing the numerator by the denominator

$$\sigma(t) = E_\infty \left[1 - \nu_\varepsilon \frac{1}{1 + \tau_\varepsilon^\gamma D^\gamma}\right] \varepsilon(t), \tag{28}$$

or

$$\sigma(t) = E_\infty \left[1 - \nu_\varepsilon \ni_\gamma^* (\tau_\varepsilon^\gamma)\right] \varepsilon(t), \tag{29}$$

where $\nu_\varepsilon = \triangle E E_\infty^{-1}$, and $\triangle E = E_\infty - E_0$ is the defect of the modulus, i.e., the value characterizing the decrease in the elastic modulus from its nonrelaxed value to its relaxed value.

Considering (4) and (7), (29) could be rewritten in the following form:

$$\sigma(t) = E_\infty \left[\varepsilon(t) - \nu_\varepsilon \int\limits_0^t \ni_\gamma (t'/\tau_\varepsilon) \, \varepsilon(t - t')\mathrm{d}t'\right]. \tag{30}$$

Then we could express $\varepsilon(t)$ in terms of $\sigma(t)$ from (23). As a result we obtain

$$\varepsilon(t) = J_0 \frac{1 + E_\infty E_0^{-1} \tau_\varepsilon^\gamma D^\gamma}{1 + \tau_\sigma^\gamma D^\gamma} \sigma(t), \tag{31}$$

or with due account for (24)

$$\varepsilon(t) = J_0 \frac{1 + J_\infty J_0^{-1} \tau_\sigma^\gamma D^\gamma}{1 + \tau_\sigma^\gamma D^\gamma} \sigma(t). \tag{32}$$

Then we represent (32) in the form

$$\varepsilon(t) = J_0 \frac{J_0 J_\infty^{-1} + \tau_\sigma^\gamma D^\gamma + 1 - 1}{1 + \tau_\sigma^\gamma D^\gamma} \sigma(t), \tag{33}$$

or after dividing the numerator by the denominator

$$\varepsilon(t) = J_\infty \left[1 + \nu_\sigma \frac{1}{1 + \tau_\sigma^\gamma D^\gamma}\right] \sigma(t), \tag{34}$$

or

$$\varepsilon(t) = J_\infty \left[1 + \nu_\varepsilon \ni_\gamma^* (\tau_\sigma^\gamma)\right] \sigma(t), \tag{35}$$

where $\nu_\sigma = \triangle J J_\infty^{-1}$, and $\triangle J = J_0 - J_\infty$.

Considering (4) and (7), relationship (35) could be rewritten in the following form:

$$\varepsilon(t) = J_\infty \left[\sigma(t) + \nu_\sigma \int\limits_0^t \ni_\gamma (t'/\tau_\sigma) \, \sigma(t - t')\mathrm{d}t'\right]. \tag{36}$$

At $\gamma \to 0$, expressions (30) and (36) take, respectively, the form

$$\sigma(t) = E_\infty (1 - \nu_\varepsilon) \varepsilon(t), \tag{37}$$

$$\varepsilon(t) = J_\infty (1 + \nu_\sigma) \sigma(t). \tag{38}$$

Considering that

$$1-\nu_\varepsilon = E_0 E_\infty^{-1} = \tau_\varepsilon^\gamma \tau_\sigma^{-\gamma}, \tag{39}$$

$$1+\nu_\sigma = J_0 J_\infty^{-1} = \tau_\sigma^\gamma \tau_\varepsilon^{-\gamma}, \tag{40}$$

from relationships (37) and (38) we have

$$\sigma(t) = E_0 \varepsilon(t), \tag{41}$$

$$\varepsilon(t) = J_0 \sigma(t). \tag{42}$$

But, as it is seen from formula (24),

$$E_0 = E_\infty, \qquad J_0 = J_\infty,$$

and hence at $\gamma = 0$ the fractional derivative standard linear solid model goes over into the correct model of a pure elastic body.

Based on formulas (30) and (36), it is possible to write the connection between the resolvent operators

$$\frac{1}{1-\nu_\varepsilon \ni_\gamma^* (\tau_\varepsilon^\gamma)} = 1 + \nu_\sigma \ni_\gamma^* (\tau_\sigma^\gamma). \tag{43}$$

In order to prove the relationship (43), it is necessary to multiply its right hand-side part by the denominator of the fraction in its left hand-side, to consider the theorem of multiplication of Rabotnov's operators [10]

$$\ni_\gamma^* (\tau_\varepsilon^\gamma) \ni_\gamma^* (\tau_\sigma^\gamma) = \frac{\tau_\varepsilon^\gamma \ni_\gamma^* (\tau_\varepsilon^\gamma) - \tau_\sigma^\gamma \ni_\gamma^* (\tau_\sigma^\gamma)}{\tau_\varepsilon^\gamma - \tau_\sigma^\gamma}, \tag{44}$$

as well as take the formulas

$$\frac{\nu_\varepsilon \tau_\sigma^\gamma}{\tau_\sigma^\gamma - \tau_\varepsilon^\gamma} = \frac{\nu_\sigma \tau_\varepsilon^\gamma}{\tau_\sigma^\gamma - \tau_\varepsilon^\gamma} = 1$$

into account.

III. Koeller Model and the Generalized Rabotnov Model

The Koeller model [11] is the immediate generalization of the fractional derivative standard linear solid model (23) via involving in the left and right sides by n fractional time-derivative terms, instead of two terms in the model (23):

$$\sum_{i=0}^{n} a_i D^{i\gamma} \varepsilon = \sum_{j=0}^{n} b_j D^{j\gamma} \sigma, \tag{45}$$

where $D^{i\gamma}$ and $D^{j\gamma}$ are the Riemann-Liouville derivatives (1), and a_i and b_j are some coefficients.

Expressing the strain ε in terms of the stress σ from (45) and vise versa, we have

$$\varepsilon = \frac{\sum_{j=0}^{n} b_j D^{j\gamma}}{\sum_{i=0}^{n} a_i D^{i\gamma}}\, \sigma, \tag{46}$$

$$\sigma = \frac{\sum_{i=0}^{n} a_i D^{i\gamma}}{\sum_{j=0}^{n} b_j D^{j\gamma}}\, \varepsilon, \tag{47}$$

Suppose that equations

$$\sum_{j=0}^{n} b_j Z^j = 0, \tag{48}$$

$$\sum_{i=0}^{n} a_i Y^i = 0, \tag{49}$$

possess only simple real negative roots $Z_j = -t_j^{-\gamma}$ $(j = 1, ..., n)$ and $Y_i = -\tau_i^{-\gamma}$ $(i = 1, ..., n)$. Then dividing in (46) and (47) the polynomials standing in numerators by those in denominators, and further decomposing the proper fractions obtained in the remainder into simple fractions with due account for the assumptions for the roots of (48) and (49), we have

$$\sigma = E_\infty \left[1 - \sum_{j=1}^{n} m_j \ni_\gamma^* \left(t_j^\gamma\right) \right] \varepsilon, \tag{50}$$

$$\varepsilon = J_\infty \left[1 + \sum_{i=1}^{n} n_i \ni_\gamma^* \left(\tau_i^\gamma\right) \right] \sigma, \tag{51}$$

where $J_\infty = b_n a_n^{-1}$, $E_\infty = a_n b_n^{-1}$, τ_i^γ $(i = 1, 2, ..., n)$ are retardation times, t_j^γ $(j = 1, 2, ..., n)$ are relaxation times, $n_i = g_i \tau_i^\gamma$ and $m_j = e_j t_j^\gamma$ are constants,

$$g_i = \sum_{k=0}^{n-1} \left(\frac{b_k}{b_n} - \frac{a_k}{a_n} \right) \tau_i^{-k\gamma} (-1)^k \left[\prod_{\substack{l=1\\(l\neq i)}}^{n} \left(\tau_l^{-\gamma} - \tau_i^{-\gamma} \right) \right]^{-1}$$

$$e_j = \sum_{k=0}^{n-1} \left(\frac{b_k}{b_n} - \frac{a_k}{a_n} \right) t_j^{-k\gamma} (-1)^k \left[\prod_{\substack{l=1\\(l\neq j)}}^{n} \left(t_l^{-\gamma} - t_j^{-\gamma} \right) \right]^{-1}$$

Moreover, it could be shown that the models (50) and (51) are resolvent one only if the following equalities are valid:

$$1 + \sum_{j=1}^{n} \frac{m_j t_j^{-\gamma}}{\tau_i^{-\gamma} - t_j^{-\gamma}} = 0, \tag{52}$$

$$1 + \sum_{i=1}^{n} \frac{n_i \tau_i^{-\gamma}}{\tau_i^{-\gamma} - t_j^{-\gamma}} = 0. \tag{53}$$

From the n-th order Eqs. (53) we could define n magnitudes of $t_j^{-\gamma}$ $(j = 1, ..., n)$, while knowing $t_j^{-\gamma}$ from the set of n Eqs. (52) we find the values m_j $(j = 1, ..., n)$.

If we suppose now that constants m_j and $t_j^{-\gamma}$ $(j = 1, ..., n)$ are known, and it is a need to determine constants n_i and $\tau_i^{-\gamma}$ $(i = 1, ..., n)$. In this case, from the n-th order Eqs. (52) we could define n magnitudes of $\tau_i^{-\gamma}$, while knowing $\tau_i^{-\gamma}$, we could find the values of n_i from the set of n Eqs. (53).

The analysis of relationships (52) and (53) shows that the following constrains are implied on the relaxation and retardation times [10]

$$\tau_k^{-\gamma} < t_k^{-\gamma} < \tau_{k+1}^{-\gamma}, \qquad \tau_n^{-\gamma} < t_n^{-\gamma}. \tag{54}$$

Note that relationships (52) and (53) differ a little from those presented in [2], since in this paper we use fractional operators in dimensionless form, what has allowed us to generalize relationship (24) for the case of the generalized Rabotnov model [10]

$$\left(\frac{\prod\limits_{i=1}^{n} \tau_i}{\prod\limits_{j=1}^{n} t_j} \right)^{\gamma} = 1 + \sum_{i=1}^{n} n_i, \tag{55}$$

$$\left(\frac{\prod\limits_{j=1}^{n} t_j}{\prod\limits_{i=1}^{n} \tau_i} \right)^{\gamma} = 1 - \sum_{j=1}^{n} m_j. \tag{56}$$

Formulas (55) and (56) are the immediate extension of formulas (39) and (40).

To prove formula (55), we adopt relationship (53) rewritten in the following form:

$$1 + \sum_{i=1}^{n} \frac{n_i}{1 - x\tau_i^{\gamma}} = 0, \tag{57}$$

where $x = t^{-\gamma}$.

Reducing all terms of (55) to the common denominator, we are led to the equation of the nth order:

$$x^n + c_1 x^{n-1} + c_2 x^{n-2} + ... + c_n = 0, \tag{58}$$

where

$$c_n = \frac{1 + \sum\limits_{i=1}^{n} n_i}{(-1)^n \prod\limits_{i=1}^{n} \tau_i^{\gamma}}. \tag{59}$$

Utilizing one of the Viet formulas concerning the roots of the algebraic nth-order equation, i.e.,

$$c_n = (-1)^n \prod_{j=1}^{n} t_j^{-\gamma}, \tag{60}$$

and substituting c_n by (60), as a result we are led to the relationship (55).

In a similar way, to prove formula (56) we adopt relationship (52) rewritten in the following form:

$$1 - \sum_{j=1}^{n} \frac{m_j}{1 - y t_j^{\gamma}} = 0, \tag{61}$$

where $y = \tau^{-\gamma}$.

Reducing all terms of (61) to the common denominator, we are led to the equation of the nth order similar to (58), where

$$c_n = \frac{1 - \sum\limits_{j=1}^{n} m_j}{(-1)^n \prod\limits_{j=1}^{n} t_j^{\gamma}}. \tag{62}$$

Considering the Viet formula concerning the roots of the algebraic nth-order equation

$$c_n = (-1)^n \prod_{i=1}^{n} \tau_i^{-\gamma}, \tag{63}$$

and equating relationships (62) and (63) to each other, as a result we are led to formula (56).

Tending γ to 0 in (55) and (56) yields, respectively:

$$\sum_{i=1}^{n} n_i = 0, \tag{64}$$

and

$$\sum_{j=1}^{n} m_j = 0. \tag{65}$$

Now tending the fractional parameter $\gamma \to 0$ in the generalized Rabotnov resolvent models (50) and (51) with due account for (20), (64) and (65), we are led to two unrepugnant equalities (41) and (42) describing the pure elastic behavior of the material.

IV. Conclusion

The simplest viscoelastic models involving fractional derivatives, namely: fractional derivative Kelvin-Voigt model, Maxwell model, standard linear solid model and Koeller model, which is the generalization of the fractional derivative standard linear solid model, are considered, and their connectedness with the Rabotnov dimensional fractional operators is revealed.

Using the resolvent operators for each model, which allow one to express not only the stress in terms of the strain but the strain in terms of the stress as well, it has been shown that not all of the simplest fractional derivative models retain their physical meaning when the fractional parameter vanishes to zero.

The role of the resolvent operators is shown for each of the models under consideration, since they allow one to define the physical meaning of the parameters involving in these models.

For the Koeller model, it is demonstrated its connectedness with the generalized Rabotnov model involving the sum of Rabotnov fractional operators under the certain restrictions for the coefficients entering into the Koeller model.

For the generalized Rabotnov models, formulas coupling the relaxation and retardation times with the elastic moduli entering in these models have been deduced.

In conclusion it should be emphasized once again that nowadays the ideas of the Russian Academician Rabotnov are still widely used worldwide for solving intricate static and dynamic problems dealing with behavior of hereditarily elastic bodies [8].

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