Simulation of oil recovery using the Weierstrass elliptic functions

Vladimir I. Astaf'ev and Pavel V. Roters

Dedicated to the 100th Anniversary of the Russian Academicain Yury N. Rabotnov

***Abstract*—**The objective of this research is to simulate the inflow performance of multiple vertical wells producing from a closed reservoir of constant thickness under pseudo-steady state conditions. For this case we represent, like the method of imaginary sources, a closed reservoir as an element of unbounded doubly periodic array of wells and use the elliptic Weierstrass zeta- and sigma-functions to describe this inflow performance. This approach allows us to find the pressure distribution and the fluid velocity in any shape of reservoir; to calculate the productivity index (*PI)* and the Dietz's shape factor (C_A) for any shape of reservoir; to analyze the influence of a reservoir shape on the Dietz's shape factor C_A; to establish the multi-well productivity matrix (*MPM*) for any shape of reservoir; to introduce the multi-well productivity index (*MPI*) and to find the optimal placement of producing wells in a closed reservoir, based on the maximum *MPI* condition.

***Keywords*—** Elliptic functions, oil recovery, productivity index, reservoir shape factor.

I. Introduction

ELLIPTIC functions (single-valued doubly periodic functions of the complex variable $z = x + iy$, the only singularities on the complex z-plane are poles) are widely used in various applications of mechanics and physics. Thus, in solids mechanics elliptic functions are used for perforated plates and shells or plates and shells with a doubly periodic system of cracks [1]-[9] as well as in studies of screw dislocations motion in crystals [10]. In aero- and hydrodynamics, elliptic functions are used to describe the fluid flow through doubly periodic lattice profiles [11], the motion of doubly periodic systems of point vortices [12]-[17].

Unfortunately, the researches on the water flooding started in the 60s years of the 20th century by H.J. Morel-Seitoux [18], [19] and by R.T. Fazlyev [20] are not well developed. Therefore the purpose of the present work is to open the way to use the elliptic functions in the oil industry. As this takes place, doubly periodic systems of oil wells have been selected as the object for the utilization of such functions. Simulation of the oil field recovery via elliptic functions is the main objective of the proposed research.

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II. Elliptic Weierstrass function

The main elliptic function is the Weierstrass *P*-function. It can be represented as follows [21], [22]:

$$P(z) = \frac{1}{z^2} + \sum_{m,n=-\infty}^{\infty}{}' \left\{ \frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right\}, \tag{1}$$

where $\omega = m\omega_1 + n\omega_2 \quad (m,n = 0, \pm 1, \pm 2, ...)$ are the nodes of the period lattice L (Fig. 1). The prime in the sum (1) means that the summation is extended over all pairs of m and n, except m=n=0. The *P*-function defined in (1) has second-order poles at all nodes of the periodic lattice. The main property of the *P*-function is its double periodicity

$$\left. \begin{aligned} P(z+\omega_1) = P(z) \\ P(z+\omega_2) = P(z) \end{aligned} \right\}, \tag{2}$$

where ω_1 and ω_2 are the periods of the Weierstrass *P*-function. The derivatives of the *P*-function are also doubly periodic, and any elliptic function can be represented as a linear combination of the *P*-function and its derivatives.

Integration of the *P*-function (1) can be written as [21], [22]

$$\zeta(z) = -\int P(z)dz = \frac{1}{z} + \sum_{m,n=-\infty}^{\infty}{}' \left\{ \frac{1}{z-\omega} + \frac{1}{\omega} + \frac{z}{\omega^2} \right\}, \tag{3}$$

where $\zeta(z)$ is the Weierstrass zeta-function.

Integration of the zeta-function (3) leads to the Weierstrass sigma-function $\sigma(z)$ [21], [22]

$$\ln \sigma(z) = \ln z + \sum_{m,n=-\infty}^{\infty}{}' \left\{ \ln\left(\frac{z-\omega}{\omega} \right) + \frac{z}{\omega} + \frac{z^2}{2\omega^2} \right\}. \tag{4}$$

In contrast to $P(z)$, the functions $\zeta(z)$ and $\sigma(z)$ do not possess the property of double periodicity. Instead of (2), they satisfy the following conditions of quasi-periodicity [21], [22]:

$$\left.\begin{aligned}\zeta(z+\omega_1)=\zeta(z)+2\eta_1\\ \zeta(z+\omega_2)=\zeta(z)+2\eta_2\end{aligned}\right\}, \qquad (5)$$

$$\left.\begin{aligned}\ln\sigma(z+\omega_1)=\ln\sigma(z)+\eta_1(2z-\omega_1)\\ \ln\sigma(z+\omega_2)=\ln\sigma(z)+\eta_2(2z-\omega_2)\end{aligned}\right\}, \qquad (6)$$

where $\eta_1=\zeta(\omega_1/2)$, and $\eta_2=\zeta(\omega_2/2)$.

The values η_1, η_2, ω_1 and ω_2 are not independent but related by the Legendre identity [21], [22]

$$\eta_1\omega_2-\eta_2\omega_1=\pi i. \qquad (7)$$

The values of the lattice periods ω_1 and ω_2 can be arbitrary complex numbers with the main restriction as $Im(\omega_2/\omega_1) > 0$. This lattice and its base element, which is called as the fundamental parallelogram of periods, are shown in Fig. 1.

The fundamental parallelogram constructed on the basis of ω_1 and ω_2 has the following main parameters: $\tau=\omega_2/\omega_1=\lambda e^{i\theta}$ and $\Delta=\mathrm{Im}(\overline{\omega}_1,\omega_2)=|\omega_1|^2\lambda\sin\theta$ (Fig. 1). Note that pairs (ω_1, ω_2), $(\omega_1+\omega_2, \omega_2)$, $(\omega_2, -\omega_1)$ and $(\omega_1, \omega_2-\omega_1)$ define the same doubly periodic lattice L. Therefore, all the variety of lattices L can be characterized by a dimensionless parameter $\tau=\omega_2/\omega_1$ changing in a fundamental half-domain $D'_0=\{\tau: 0\le \mathrm{Re}\,\tau\le 1/2, |\tau|\ge 1\}$ (Fig. 2).

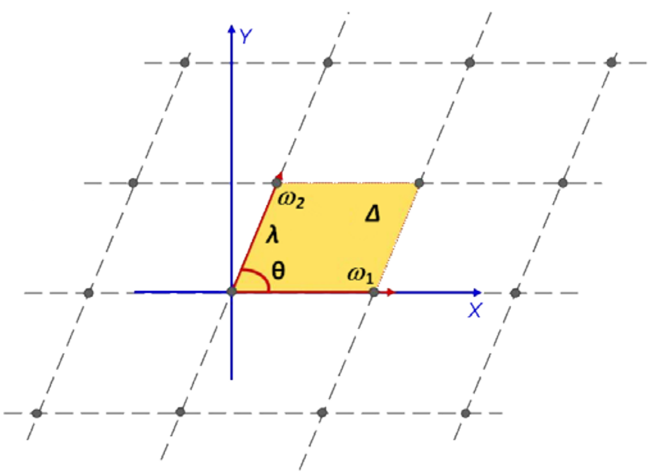


Fig. 1. Doubly periodic lattice L with the periods ω_1 and ω_2 and its fundamental parallelogram.

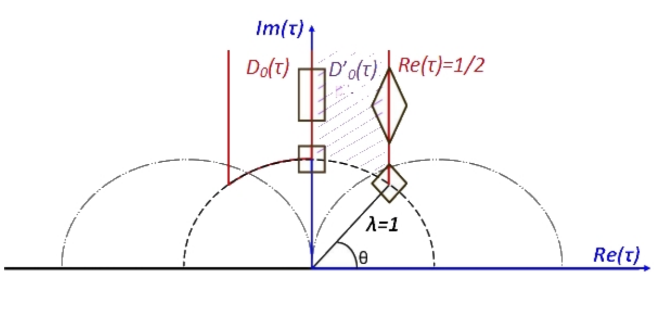


Fig. 2. The fundamental domain D_0 and half-domain D'_0

for the modulus $\tau=\omega_2/\omega_1=\lambda e^{i\theta}$

III. Statement of the Problem

Consider a plane non-stationary filtration of a viscous weakly compressible fluid in a closed reservoir, having the volume $V=h\Delta$, where Δ is the square of the reservoir and h is its thickness. The fluid motion is described by the fluid continuity equation and Darcy's filtration law [23]

$$\frac{m}{\rho}\frac{\partial\rho}{\partial t}+\frac{\partial\upsilon_x}{\partial x}+\frac{\partial\upsilon_y}{\partial y}=0, \qquad (8)$$

$$\upsilon_x=-\frac{k}{\mu}\frac{\partial p}{\partial x},\quad \upsilon_y=-\frac{k}{\mu}\frac{\partial p}{\partial y}, \qquad (9)$$

where ρ and μ are the density and the viscosity of the fluid, respectively, k and m are the permeability and the porosity of the reservoir, respectively, p is the reservoir pressure, v_x and v_y are the components of the fluid filtration velocity.

In the case of quasi-state fluid filtration it is assumed that the function $p(x,y,t)$ can be represented as the sum $p(x,y,t)=\overline{p}(t)+\delta p(x,y)$ (or $\partial p/\partial t=d\overline{p}/dt$), where $\overline{p}(t)=\frac{1}{\Delta}\iint\limits_{\Delta}p(x,y,t)dxdy$ is the average reservoir pressure, and $\delta p(x,y)$ does not depend on the time (accordingly to the quasi-state condition). Then the value of $\frac{m}{\rho}\frac{\partial\rho}{\partial t}$ in (8) can be rewritten as $\frac{m}{\rho}\frac{\partial\rho}{\partial p}\frac{d\overline{p}}{dt}=mC_p\frac{d\overline{p}}{dt}$, where $C_p=\frac{1}{\rho}\frac{\partial\rho}{\partial p}=\mathrm{const}$ is the constant coefficient of the compressibility. From the condition of the overall material balance of a fluid in a closed reservoir of the volume $V=h\Delta$ with a producing well of the rate Q, it can be written as

$$mC_p\frac{d\overline{p}}{dt}=-\frac{Q}{h\Delta}. \qquad (10)$$

As a result, the equation of material balance (10) together with the continuity equation (8), rewritten in the form

$$\frac{\partial\upsilon_x}{\partial x}+\frac{\partial\upsilon_y}{\partial y}=\frac{Q}{h\Delta}, \qquad (11)$$

and the Darcy law equations (9) forms a complete system of four equations for four unknown functions $\delta p(x,y)$, $\overline{p}(t)$, v_x, and v_y. These equations describe the process of quasi-steady fluid flow in closed reservoir of volume $V=h\Delta$ with a producing well of the fluid rate $Q(t)$.

IV. THE GENERAL SOLUTION OF THE PROBLEM

The general solution of equations (9)–(11) in the quasi-state case is easily obtained if, instead of the variables x and y, the complex and complex conjugate variables $z = x + iy$ and $\bar{z} = x - iy$ are introduced, as well as instead of the component of the velocity vector v_x и v_y, the complex $V = \upsilon_x + i\upsilon_y$ and complex conjugate $\bar{V} = \upsilon_x - i\upsilon_y$ functions of flow rate are used. In this case, equations (9) and (11) take the form

$$V = -\frac{2k}{\mu}\frac{\partial p}{\partial \bar{z}}, \qquad \bar{V} = -\frac{2k}{\mu}\frac{\partial p}{\partial z}, \tag{12}$$

$$\frac{\partial V}{\partial z} + \frac{\partial \bar{V}}{\partial \bar{z}} = \frac{Q}{h\Delta} \ . \tag{13}$$

Equations (12) and (13) allow one to represent the required functions of the complex velocity $V(z,\bar{z})$ and the pressure $p(z,\bar{z})$ in the following form:

$$V = \frac{Q}{2h\Delta}z - \overline{F'(z)}, \qquad \bar{V} = \frac{Q}{2h\Delta}\bar{z} - F'(z), \tag{14}$$

$$p = \frac{\mu}{2k}\left(F(z) + \overline{F(z)} - \frac{Q}{2h\Delta}z\bar{z} + C\right), \tag{15}$$

where $F(z)$ is an unknown analytical function and C is an arbitrary constant.

To find $F(z)$, it is necessary to use the following boundary conditions:

- the equality of the pressure p on the well contour with radius r_w to downhole pressure p_w

$$p\big|_{z = r_w e^{i\theta}} = p_w, \tag{16}$$

- the equality of the fluid flow through the well contour with radius r_w to flow rate Q

$$h\int_o^{2\pi} \upsilon_n r_w d\theta = -Q, \tag{17}$$

- the absence of fluid inflow through the reservoir boundary (the closed reservoir)

$$\upsilon_n\big|_{\Gamma} = 0. \tag{18}$$

The value of the normal component of the velocity vector v_n can be written as $\upsilon_n ds = \mathrm{Im}(\bar{V}dz)$, where $z = z(s)$ is the equation of the reservoir boundary in the complex variables. Then from the boundary conditions (16) and (17) (considering that the ratio of the well square πr_w^2 to the reservoir square Δ is small) it is followed that the general form of the unknown function $F(z)$ and the constant C are

$$F(z) = \frac{Q}{2\pi h}(\ln z + F_0(z)), \qquad F_0(z) = \sum_{n=1}^{\infty} a_n z^n, \tag{19}$$

$$C = \frac{2k}{\mu}p_w - \frac{Q}{2\pi h}\ln r_w^2. \tag{20}$$

Thus, in the case of a closed reservoir with the square of the drainage area Δ, boundary of drainage area Γ and a single producing well of flow rate Q, placed at the origin $z=0$, the general solution of (9)–(11) in the complex variables will be as follows:

$$\bar{V} = -\frac{Q}{2\pi h}\left(\frac{1}{z} + F_0'(z) - \frac{\pi}{\Delta}\bar{z}\right), \tag{21}$$

$$p = p_w + \frac{Q\mu}{4\pi kh}\left(\ln\frac{z\bar{z}}{r_w^2} + F_0(z) + \overline{F_0(z)} - \frac{\pi}{\Delta}z\bar{z}\right). \tag{22}$$

In the general solution (21)–(22) there is an unknown analytical function of complex variable $F_0(z)$. To find $F_0(z)$, it is necessary to use the boundary condition (18) , i.e., the condition of the absence of inflow through the outside reservoir boundary. For the known boundary of the reservoir $z=z(s)$, the unknown function $F_0(z)$ is found by solving the following boundary value problem:

$$\mathrm{Im}\,F_0(s) = \frac{2\pi}{\Delta}\Sigma(s) - \theta(s), \tag{23}$$

where s is the arc length of a point P on the boundary Γ, $\theta(s)$ is the polar angle of the point P, and $\Sigma(s)$ is the square of the sector with the arc length s (Fig. 3).

Methods for solving equation (23) for different types of boundaries are well known [24].

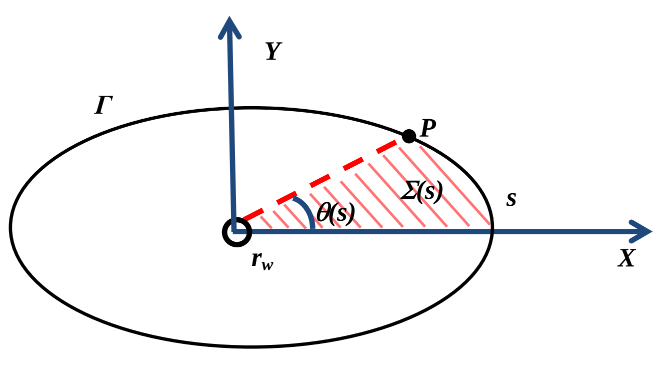


Fig. 3. Closed reservoir with the boundary Γ and producing well with the radius r_w. in the origin.

V. THE MAIN SOLUTION IN THE CASE OF A DOUBLY PERIODIC SYSTEM OF WELLS

The solution, presented in (21)-(22), applies to the case where a single well is placed in a closed reservoir with the known boundary of reservoir. In multi-well field systems of oil development, the drainage area for each well depends on the relative positions of wells, and the magnitude of their flow rates (drill-hole interference [23]).

In order to generalize the constructed solution (21)-(22) for multi-well systems with an unknown drainage area of each well, consider the case when the field is developing by a system of producing wells with the equal flow rates Q for all wells, placed at the nodes of the doubly periodic lattice L (Fig. 1). In this case, the function $F'(z)$ in (14), which determines the nature of the fluid flow to the wells, should have at all nodes of the lattice L the first-order poles with the equal flow rates Q of the each well. Mathematically, this means that near the every node $\omega = m\omega_1 + n\omega_2$ on the complex z-plane the desired function $F'(z)$ must have the form

$$F'(z) = \frac{Q}{2\pi h}\frac{1}{z-\omega} + F_0'(z-\omega)\,. \tag{24}$$

The representation (24) should be considered together with the condition of repeatability of the fluid flow in the vicinity of the each well, i.e., the condition of double periodicity of the function $F'(z)$. Such function was constructed in [25] and was written as follows:

$$F'(z) = \frac{Q}{2\pi h}(\zeta(z) + \alpha z), \tag{25}$$

where $\alpha = \left(\beta\bar{\omega} - 2\zeta(\omega/2)\right)/\omega$, and $\beta = \pi/\Delta$. In view of the Legendre identity (7), the parameter α is independent of m and n and can be computed for $\omega = \omega_1$ only.

Thus, the nature of the fluid flow in a doubly periodic system of wells located at the nodes of the lattice L is defined by the following function [25]:

$$\bar{V} = \upsilon_x - i\upsilon_y = -\frac{Q}{2\pi h}\left(\zeta(z) + \alpha z - \beta\bar{z}\right). \tag{26}$$

The distribution of pressure in the reservoir with the doubly periodic system of wells is determined by the following relationship [26]:

$$p = p_w + \frac{Q}{2\pi kh}\left[\mathrm{Re}\left(\ln\sigma(z) + \alpha\frac{z^2}{2}\right) - \beta\frac{z\bar{z}}{2} - \ln r_w\right]. \tag{27}$$

In view of the dual periodicity and oddness of the function $\bar{V}(z,\bar{z})$, this function is equal to zero at the points of half-periods $\omega_1/2$, $\omega_2/2$, $(\omega_1+\omega_2)/2$. These points are often called "critical," because the both components of the velocity vector v_x и v_y are equal to zero. These points define the shape of the drainage boundary for every well. Figures 4-8 show the streamlines of fluid flow to the wells in their drainage area for the every well of rhombic lattice with a different angle θ (Fig. 1) changing from $\pi/2$ to $\pi/6$.

Critical points on the Figs 4-8 are marked by crosses in the circles with a dark brown border. Dark circles represent producing wells located at the nodes of the lattice L, the dotted line shows the boundaries of the fundamental parallelograms, and the solid lines label the contours of the drainage areas.

Reference to Figs 4-8 shows also how the drainage area is transformed when the angle θ changes from $\theta=\pi/2$ (the square drainage area) to $\theta=\pi/3$ (the regular hexagonal drainage area), and then with a further decrease of the angle θ, to the rectangular drainage area with the ratio of drainage sides as $\cos\theta/\sin\theta$.

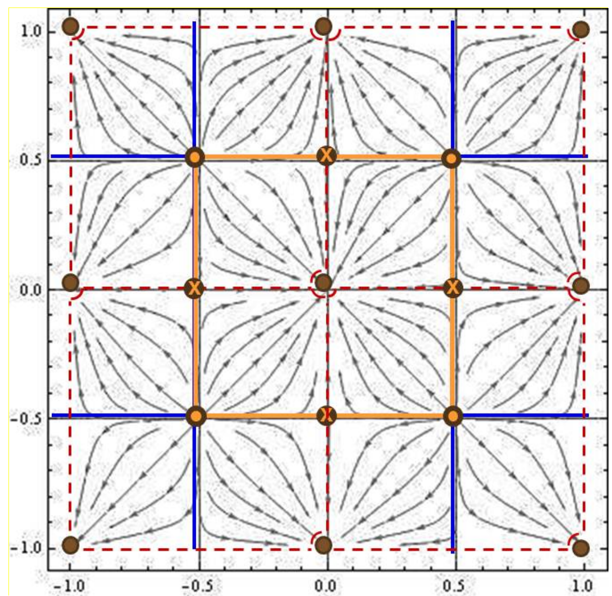


Fig. 4. The nature of fluid flow in a square drainage area for $\theta=\pi/2$.

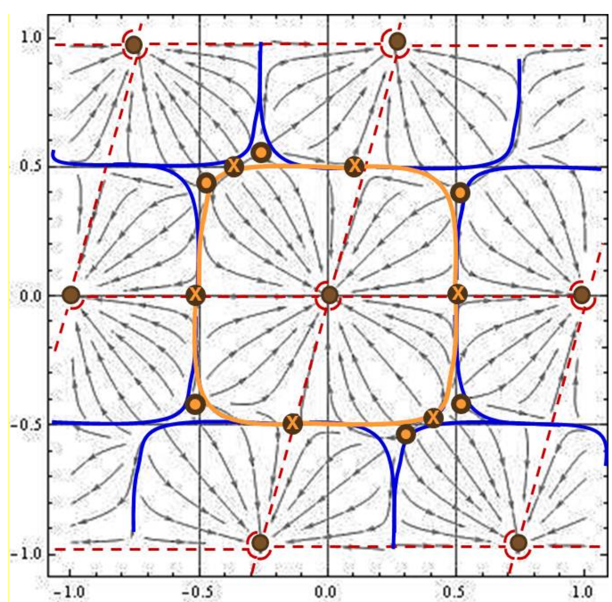


Fig. 5. The nature of fluid flow in a rhombic lattice for $\theta=5\pi/12$.

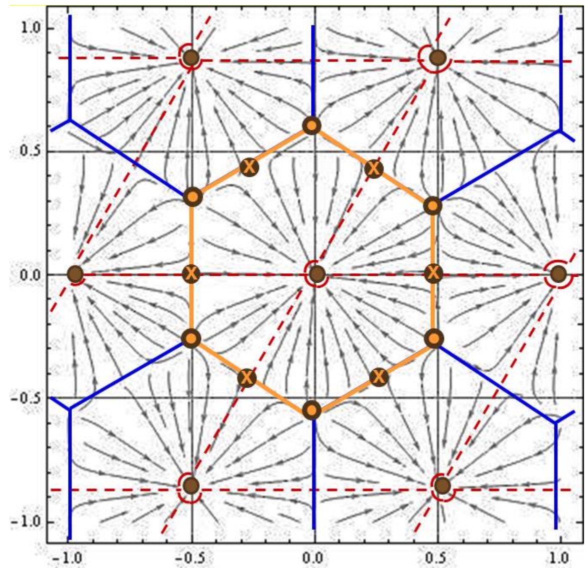


Fig. 6. The nature of fluid flow in a rhombic lattice for $\theta=\pi/3$.

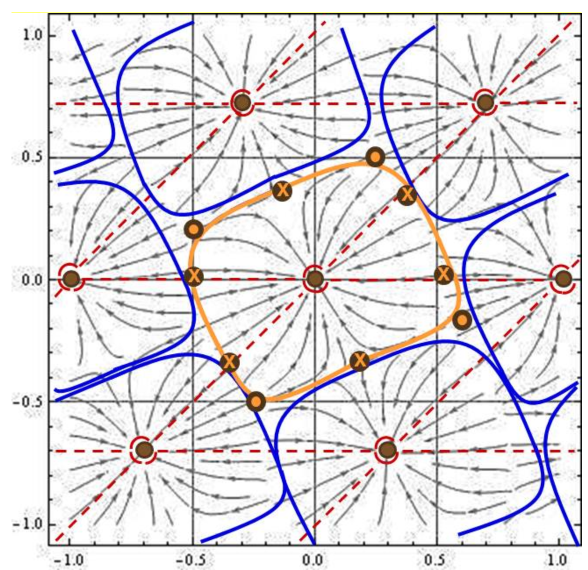


Fig. 7. The nature of fluid flow in a rhombic lattice for $\theta=\pi/4$.

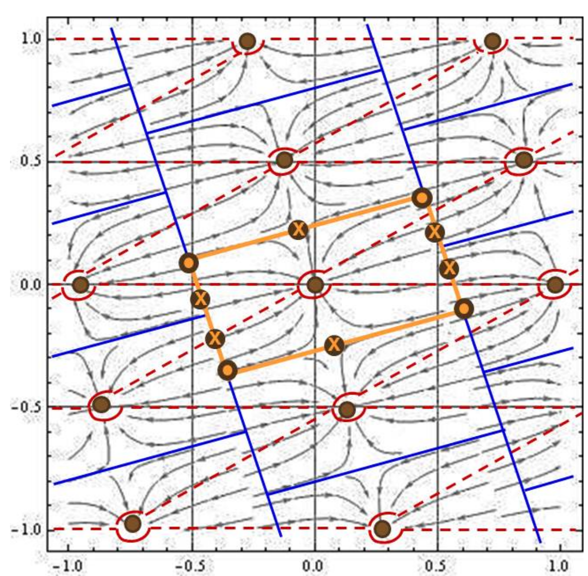


Fig. 8. The nature of fluid flow in a rhombic lattice for $\theta=\pi/6$.

Using expression (27) for the pressure distribution, the average pressure $\overline{p}(t)$ in the fundamental parallelogram could be calculated, which, considering the doubly periodicity of the function $p(x,y,t)$, coincides with the average reservoir pressure. The average pressure $\overline{p}(t)$ takes the following form [26]:

$$\overline{p}-p_w=\frac{Q\mu}{2\pi kh}\left(\ln R-\ln r_w\right), \tag{28}$$

where R is the reduced radius of the well drainage area, i.e. the radius of an equivalent circular closed drainage area of a well, when the same flow rate Q is achieved at the same pressure depression $\overline{p}(t)-p_w$.

The reduced radius R has been defined in [26] as follows:

$$R=\Delta^{1/2}\Big/\left(4\pi^2\,\mathrm{Im}\,\tau\left|(qq_1)^{1/6}\prod_{n=1}^{\infty}(1-q^{2n})^2(1-q_1{}^{2n})^2\right|^{1/2}\right), \tag{29}$$

where $\tau=\omega_2/\omega_1=\lambda e^{i\theta}$, $q=e^{i\pi\tau}$, and $q_1=e^{-i\pi/\tau}$ are the Jacobi parameters.

Similar to (28), the equation for the pressure depression $\overline{p}(t)-p_w$ for the wells with different drainage areas was obtained by Dietz [27] in the following form:

$$\overline{p}-p_w=\frac{Q\mu}{4\pi kh}\ln\left(\frac{4\Delta}{\gamma C_A r_w^2}\right), \tag{30}$$

where $\gamma=1.781$, C_A is a dimensionless form factor for different drainage area (Dietz shape factor), and Δ is the square of the drainage area.

Comparing (28) and (30), the following relationship between the Dietz shape factor C_A and the reduced radius of the drainage area R could be established:

$$\gamma C_A=4\Delta/R^2=16\pi^2\,\mathrm{Im}\,\tau\left|(qq_1)^{1/6}\prod_{n=1}^{\infty}(1-q^{2n})^2(1-q_1{}^{2n})^2\right|. \tag{31}$$

It should be noted that the value of the Dietz shape factor C_A was calculated using the method of "imaginary sources" for a very limited number of drainage areas and only for the rectangular and triangular shape of drainage areas [27], while the analytic expression (29) for the parameter R [26] allows us to calculate this value (and, hence, C_A using the formula (31)) for any form of drainage areas.

In the case of a rectangular lattice ($\theta=\pi/2$, $\tau=i\lambda$), equation (31) can be represented as follows [26]:

$$\gamma C_A=16KK'(2kk_1)^{2/3}, \tag{32}$$

where $K = K(k)$ and $K' = K(k_1)$ are the complete elliptic integrals, k and $k_1 = \sqrt{1-k^2}$ are the main and additional modules of these integrals [21]-[22]. The value of k in the case of a rectangular lattice is associated with the value of the lattice parameter λ as follows [26]:

$$\lambda = K(\sqrt{1-k^2}) / K(k). \tag{33}$$

In particular, for a square lattice, when λ=1, the values of k and k_1 are $k_1 = k = 1/\sqrt{2}$. Considering that $K(1/\sqrt{2}) = K'(1/\sqrt{2}) = 1.8541$, we obtain from (32) that C_A=30.8832. According to Dietz [27], this coefficient for a square drainage area is equal to 30.88.

In the case of a rectangular lattice with λ>1, the parameter C_A can be calculated using a simple approximate expression for small values of k, constructed on the basis of (32) and the asymptotic representations for $K = K(k)$, $K' = K(k_1)$ as follows [26]:

$$\gamma C_A = 16\pi^2 \lambda \exp(-\pi\lambda/3)/\sqrt[3]{2}. \tag{34}$$

The results of calculations of the parameter C_A using (31) are presented graphically in Fig. 9 and summarized in Table 1. In Table 1, the numerical values of the parameter C_A, obtained by Dietz [27] and Earlougher *et al.* [28] using the method of "imaginary sources" are also presented.

The analysis based on the $C_A(\lambda)$ dependence (Fig. 9) reveals that the square lattice with λ=1 is the best among the all rectangular lattices placement of wells. The productivity index *PI* of such a lattice

$$PI = \frac{Q}{p - p_w} = \frac{4\pi kh}{\mu}\left(\ln\left(\frac{4\Delta}{\gamma C_A r_w^{\,2}}\right)\right)^{-1} \tag{35}$$

is maximal among the set of all rectangular lattices having the same square of the drainage area Δ.

Then, the main question there arises: whether the square lattice location of wells is the most productive lattice among the all possible ways of placing wells in the design field?

To answer this question, we must refer to the initial expression (31) for the dependence of the C_A Dietz shape factor on the lattice parameters and to find the value $\tau = \omega_2/\omega_1 = \lambda e^{i\theta}$ corresponding to the maximal magnitude of C_A. Based on the symmetrical dependence of the parameter C_A from the Jacobi parameters $q = e^{i\pi\tau}$ and $q_1 = e^{-i\pi/\tau}$, we can conclude that the maximal value of the parameter C_A will be achieved in the case $|q| = |q_1|$. This condition is satisfied only for rhombic lattices, i.e., when $\tau = e^{i\theta}$ and $|q| = |q_1| = e^{-\pi\sin\theta}$. In this case, equation (31) takes the following form:

$$\gamma C_A(\theta) = 16\pi^2 \sin\theta \left| q^{1/3} \prod_{n=1}^{\infty} (1 - q^{2n})^4 \right|. \tag{36}$$

Numerical calculations according to (36) show that the derivative of $C_A(\theta)$ in the range $0 < \theta < \pi$ is equal to zero at $\theta=\pi/3$, $\theta=\pi/2$ and $\theta=2\pi/3$, i.e., the angles $\theta=\pi/3$, $\theta=\pi/2$ and $\theta=2\pi/3$ are the extreme points of the function $C_A(\theta)$, at which it takes on its maximal and minimal (local) values.

The results of $C_A(\theta)$ calculations on the base of (36) for the rhombic lattices are shown in Fig. 10 and are presented in Table 1. From these results it is clear that optimum productivity index *PI* corresponds to rhombic lattices with the angles $\theta = \pi/3$ and $\theta = 2\pi/3$, which give a hexagonal drainage area for wells (Fig. 6). A square lattice with the angle $\theta = \pi/2$ and a square drainage area (Fig. 4) gives a slightly lower value of the productivity index *PI*.

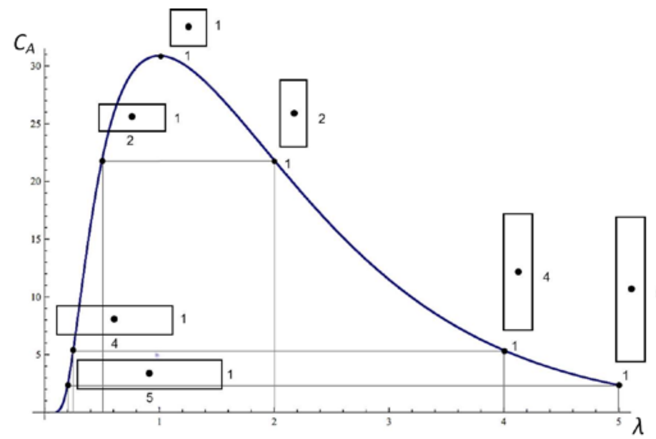


Fig. 9. A plot of the Dietz shape factor C_A vs λ in the case of rectangular lattices.

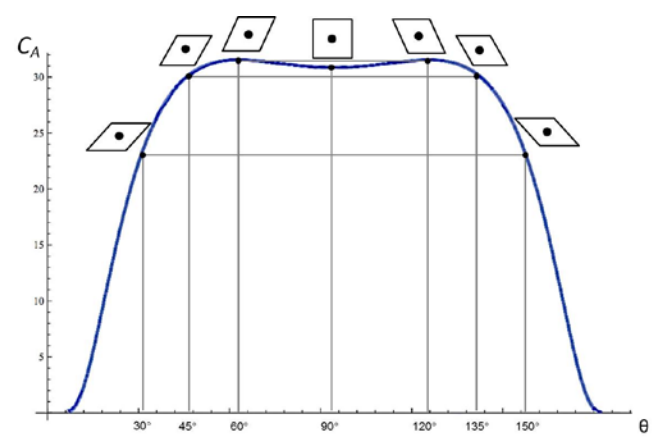


Fig. 10. A plot of the Dietz shape factor C_A vs angle θ in the case of rhombic lattices.

VI. The Main Solution in the Case of a Doubly Periodic System of Multi-Well Clusters

Consider a doubly periodic multi-well cluster, i.e., a system of n vertical wells with flow rates Q_1, Q_2, ... Q_n, located at the points (x_1, y_1), (x_2, y_2), ... (x_n, y_n) and repeated double-periodically with the main periods $x=\omega_1$ and $y=\omega_2$ (Fig. 11).

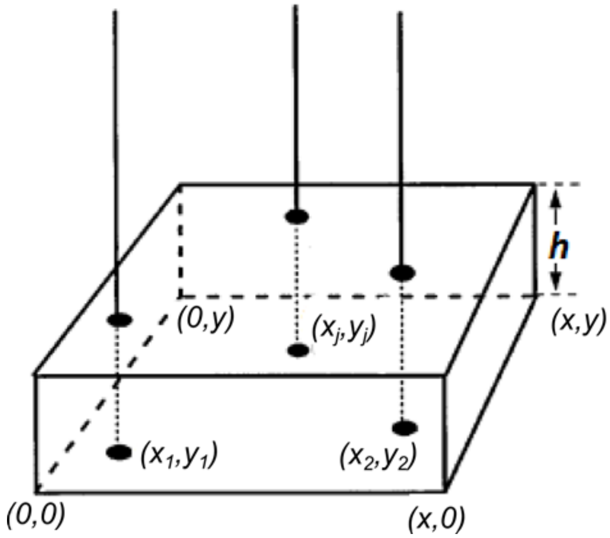


Fig. 11. Doubly periodic multi-well cluster.

In the case of one well in a cluster, the solution for the velocity and the pressure fields in the complex variable $z=x+iy$ was built in [25] and [26] in the form of (26)-(27).

Using the superposition method, the solution for the multi-well cluster can be represented as the follows [29]:

$$\overline{V}(x,y) = -\sum_{k=1}^{n} \frac{Q_k}{2\pi h}\left(\zeta(z-z_k)+\alpha(z-z_k)-\beta(\overline{z}-\overline{z}_k)\right), \quad (37)$$

$$\overline{p}(t)-p(x,y)=\sum_{k=1}^{n} q_k \left(\ln R - \right. \quad (3$$
$$\left. -\left[\operatorname{Re}(\ln\sigma(z-z_k)+\alpha\frac{(z-z_k)^2}{2})-\beta\frac{|z-z_k|^2}{2}\right]\right), \quad (38)$$

where $q_k = Q_k\mu / 2\pi kh$ is the generalized flow rate of the k-well in the cluster. In the case of rectangular lattice, the similar solution was introduced by Ozkan [30] and was used by Valko *et al.* [31] to calculate the multi-well productivity index (*MPI*). The solutions (37) and (38) are more general and are valid for arbitrary doubly lattices.

Equation (38) allows one to represent the connection between the pressure drawdown of the i-well in the cluster $\Delta p_i = \overline{p}(t) - p_w^i$ (i=1, 2,… n) and the generalized flow rate of the k-well in the cluster q_k in the following form [29]:

$$\Delta p_i = \sum_{k=1}^{n} q_k \ln(R/R_{ik}), \quad (39)$$

$$\ln R_{ik} = \operatorname{Re}\left(\ln\sigma(z_i-z_k)+\alpha\frac{(z_i-z_k)^2}{2}\right)$$
$$-\beta\frac{|z - z_k|^2}{2}, (i\neq k); \ln R_{ii} = \ln r_w. \quad (40)$$

The values of $\ln(R/R_{ik})$ are the elements of the influence matrix a_{ik}, as they reflect the influence of the k-well flow rate q_k on the pressure drawdown Δp_i of the i-well. Valko *et al.* [31] calculated the elements of the influence matrix a_{ik} only for the rectangular lattice, while (40) generalizes their equation for any doubly periodic lattices.

The expression for productivity index (*PI*) $J_i = q_i/\Delta p_i$ of the i-well in the multi-well cluster and the total multi-well productivity index $J=(q_1+q_2+...q_n)/(p_1+p_2+...p_n)$ (*MPI*) for the whole cluster has the form

$$J_i^{-1} = \sum_{k=1}^{n} s_{ki}\ln(R/R_{ik}), \qquad J^{-1} = \sum_{k=1}^{n} s_k J_k^{-1} = \sum s_k a_k, \quad (41)$$

where $s_{ki} = q_k/q_i$, $s_k = q_k/(q_1+q_2+...q_n)$ is the generalized flow rate of the k-well to the i-well and to the cluster at the whole, and $a_k = \sum_{i=1}^{n}\ln(R/R_{ik})$.

Representing the productivity index of the k-well as $J_k^{-1} = (1/2)\ln(4\Delta_k/\gamma C_A^{(k)} r_w^2)$, where $C_A^{(k)}$ is the shape factor of the k-well in the multi-well cluster, we obtain the following expression for the $C_A^{(k)}$ calculation [29]:

$$\frac{C_A^{(k)}}{C_A} = s_k \prod_{\substack{i=1\\ i\neq k}}^{n}\left(\frac{\gamma C_A R_{ik}^2}{4\Delta}\right)^{s_{ik}}. \quad (42)$$

Thus, the problem of optimization of field development by doubly periodic multi-well cluster is reduced to finding the optimal placement of wells in a doubly periodic lattice with the aim to maximize the *MPI* value J.

VII. The Examples

As examples let us consider the character of flow for some clusters in square and rhombic lattices.

A. *Two-well cluster in the square lattice.*

As it was shown in [29], for the two-well cluster and lattices that are close to a rectangular form, the maximum value of C_A would be achieved at the point $(\omega_1+\omega_2)/2$, but for the rhombic lattice at the point $(\omega_1+\omega_2)/3$.

For the square lattice with equal flow rates of the both wells ($q_2=q_1$) and the location of the second well at the point $(\omega_1+\omega_2)/2$, the streamlines are shown in Fig. 12. The shape factor for both wells calculated by (42) is equal to C_A=30.881.

For the rhombic lattice with equal flow rates of the both wells and the location of the second well at the point $(\omega_1+\omega_2)/3$, the streamlines are shown in Fig. 13. For comparison, the streamlines for the rhombic lattice when the second well is at the point $(\omega_1+\omega_2)/2$ are shown in Fig. 14.

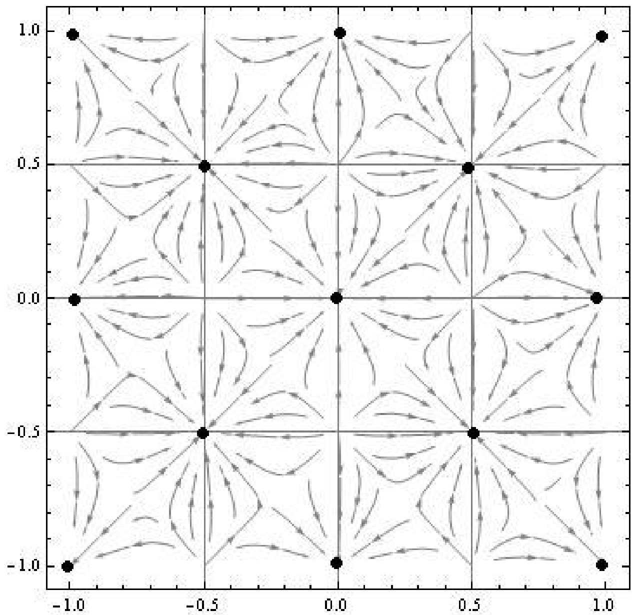


Fig. 12. The nature of fluid flow in a square lattice for two-well cluster with z_1=0 and z_2=(ω_1+ω_2)/2.

B. Two-well cluster in the rhombic lattice.

Comparison of Fig. 13 and Fig. 14 shows that in the case of the second well is placed at the point (ω_1+ω_2)/3, the drainage area for both wells takes the form of an equilateral triangle and the shape factor of the triangle drainage area becomes C_A=27.321. Placing the second well at the point (ω_1+ω_2)/2 results in a rectangular shape of the drainage area with an aspect ratio of 1: √ 3 and the shape factor C_A=25.035.

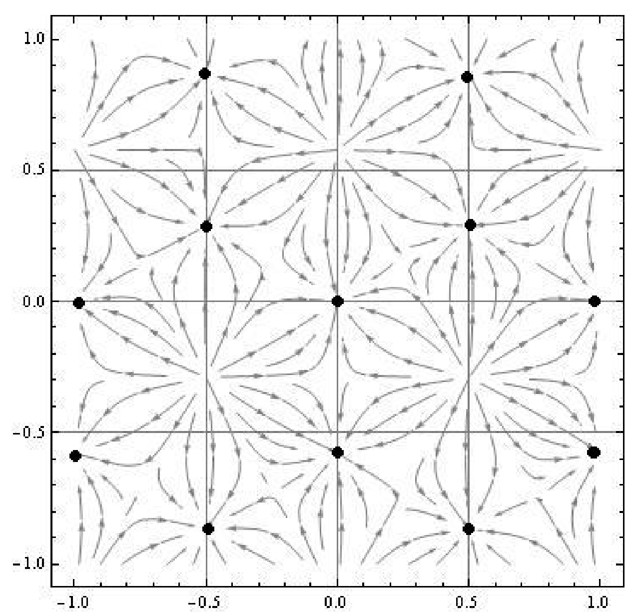


Fig. 13. The nature of fluid flow in a rhombic lattice for two-well cluster with z_1=0 and z_2=(ω_1+ω_2)/3.

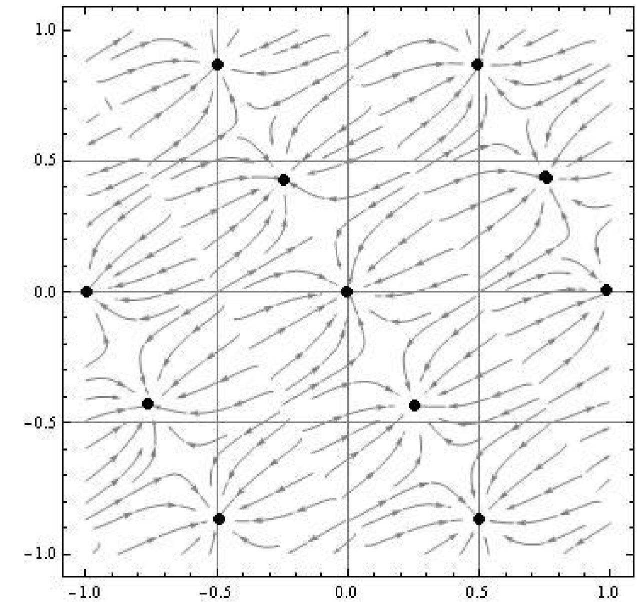


Fig. 14. The nature of fluid flow in a rhombic lattice for two-well cluster with z_1=0 and z_2=(ω_1+ω_2)/2.

C. Three-well cluster in the square lattice.

As it is shown in [29], for the three-well cluster the maximum productivity index J is achieved not at the critical points, i.e., at the points of half-periods $\omega_1/2$, $\omega_2/2$ and (ω_1+ω_2)/2, but at the points, where R_{12}, R_{13} and R_{23} are equal. There exist only three pairs of points, namely: (ω_1/3, 2ω_1/3), (ω_2/3, 2ω_2/3) and ((ω_1+ω_2)/3, 2(ω_1+ω_2)/3), where R_{12}=R_{13}=R_{23}. In this case, the values of $C_A^{(1)}$, $C_A^{(2)}$, and $C_A^{(3)}$ are equal too. In accordance with (42), they can be calculated as

$$\frac{C_A^{(1)}}{C_A}=\frac{C_A^{(2)}}{C_A}=\frac{C_A^{(3)}}{C_A}=\frac{1}{3}\left(\frac{\gamma C_A R_{12}^2}{4\Delta}\right)^2 \qquad (43)$$

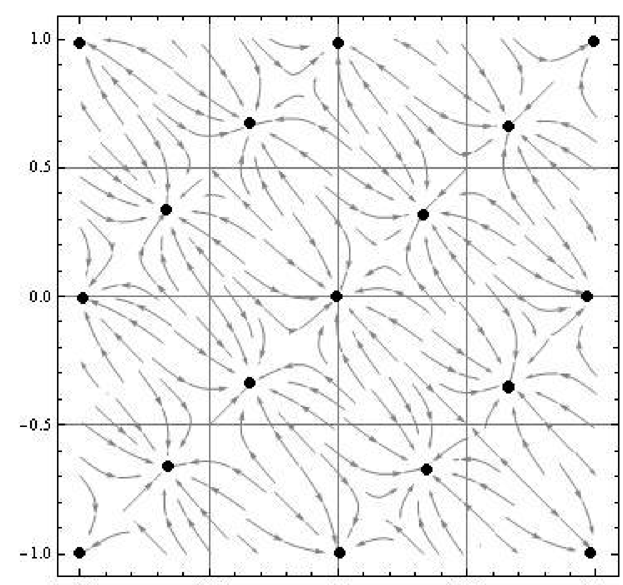


Fig. 15. The nature of fluid flow in a square lattice for three-well cluster with z_1=0, z_2=(ω_1+ω_2)/3 and z_3=2(ω_1+ω_2)/3.

For the square lattice with equal flow rates (q_3=q_2=q_1) and the second and the third wells located at the points (ω_1+ω_2)/3 and 2(ω_1+ω_2)/3, the streamlines are shown in the Fig. 15.

For the comparison, the streamlines of flow in the square lattice, wherein the second and the third wells are located at critical points ω_1/2 and ω_2/2, are shown in the Fig. 16. In this case, the values of R_{12} and R_{13} are different, therefore, the drainage areas for the second and the third wells as well the shape factors C_A are different too.

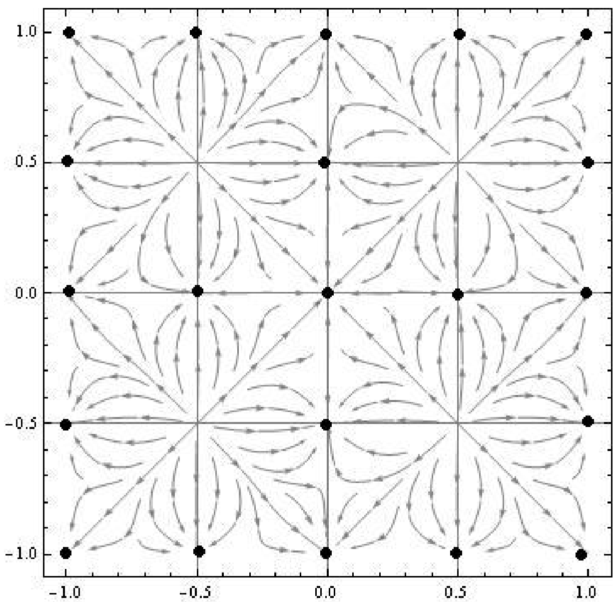


Fig. 16. The nature of fluid flow in a square lattice for three-well cluster *with* z_1=0, z_2=ω_1/2 and z_3=ω_2/2.

D. Three-well cluster in the rhombic lattice.

Similarly to the case of the flow in a square lattice, let us consider the nature of the flow in a rhombic lattice with equal flow rates (q_3=q_2=q_1) and the second and the third wells located at the points (ω_1+ω_2)/3 and 2(ω_1+ω_2)/3 along the rhomb diagonal. In this case, the streamlines are shown in Fig. 17, from which it is evident that such a placement of the wells in the rhombic lattice is optimal and provides the maximum value of the shape factor C_A =31.548.

The nature of the flow in the rhombic lattice, when the second and the third wells are located at critical points ω_1/2 and ω_2/2 of the parallelogram period is shown in Fig. 18, whence it can be seen that each well has a rhombic drainage area with the value of the shape factor C_A=26.493.

E. The closed square reservoir.

The above-proposed approach to modeling the oil recovery using the Weierstrass elliptic functions is based on a priori assignment of the parallelogram period and the placement of the wells in this parallelogram. The drainage area for every well and for the entire cluster as a whole thus may not be straightforward and is strongly depended on the flow rates and the well locations in the cluster. However, the elliptic functions can be used in modeling of oil recovery for predetermined drainage area like a rectangle or right triangle.

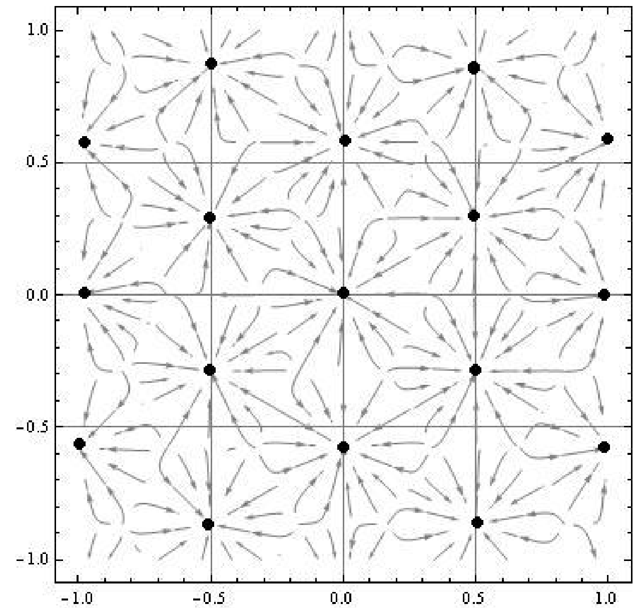


Fig. 17. The nature of fluid flow in a rhombic lattice for three-well cluster with z_1=0, z_2=(ω_1+ω_2)/3 and z_3=2(ω_1+ω_2)/3.

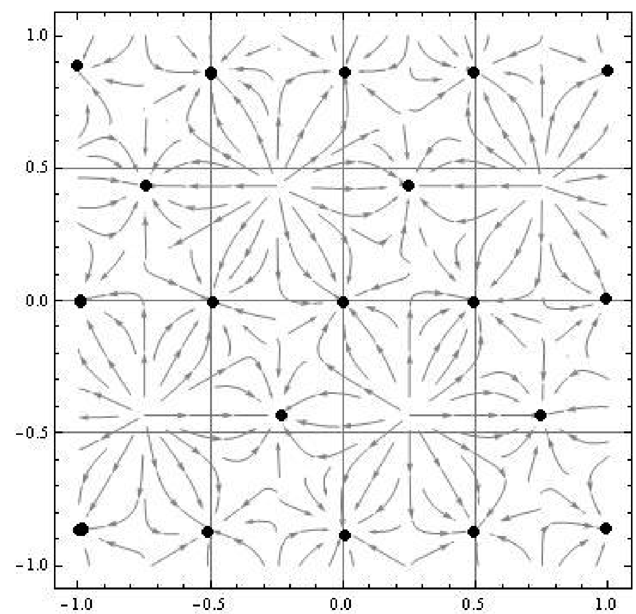


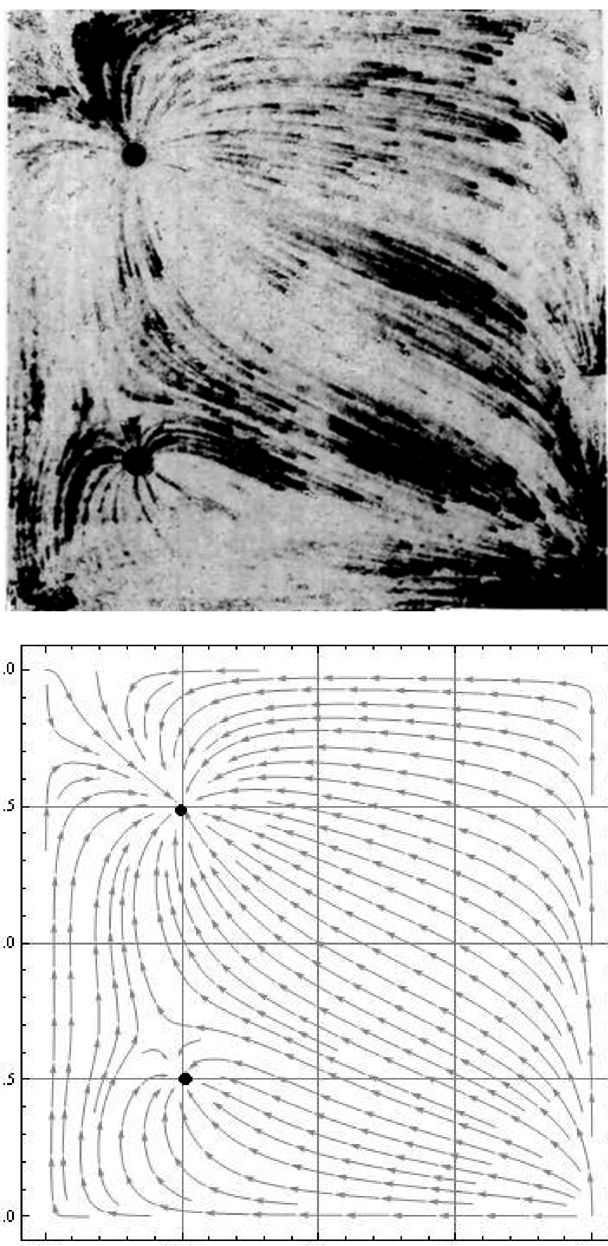
Fig. 18. The nature of fluid flow in a rhombic lattice for three-well cluster with z_1=0, z_2=ω_1/2 and z_3=ω_2/2.

Consider a closed square reservoir (x_e, y_e) with n wells located at the points (x_1, y_1), (x_2, y_2), … (x_n, y_n) with flow rates Q_1, Q_2, ... Q_n (Fig. 11). Condition of impermeability of reservoir boundaries is the condition of zero normal velocity components at the boundary, i.e., $v_y = 0$ on the horizontal boundaries y=0 and y=y_e of the rectangle and $v_x = 0$ on the vertical boundaries x=0 and x= x_e. Using the method of image sources, this condition will be satisfied if we consider the infinite double-periodic rectangular lattice with the periods $\omega_1 = 2x_e$ and $\omega_2 = 2iy_e$, where 4 groups of n wells with coordinates $((x_1, y_1)$, (x_2, y_2), … $(x_n, y_n))$; $((x_1, -y_1)$,

$(x_2,-y_2)$, … $(x_n,-y_n)$); $((-x_1,y_1)$, $(-x_2,y_2)$, … $(-x_n,y_n))$ and $(-x_1,-y_1)$, $(-x_2,-y_2)$, … $(-x_n,-y_n)$) are placed.

Therefore, the velocity and the pressure functions can also be written in the form (37) and (38), and the summation is taken over all *4n* wells. The productivity index *J* for the whole reservoir and the productivity index J_k for *k*-well in the reservoir as well the shape factor $C_A^{(k)}$ could also be calculated via (41) and (42).

Figure 19a shows the experimental streamlines data from [32] for a closed reservoir with a square drainage area 2x2 and two wells placed at the points (0.5, 0.5) and (0.5, 1.5), respectively. The ratio q_1:q_2 of the flow rates is 1:4.1.

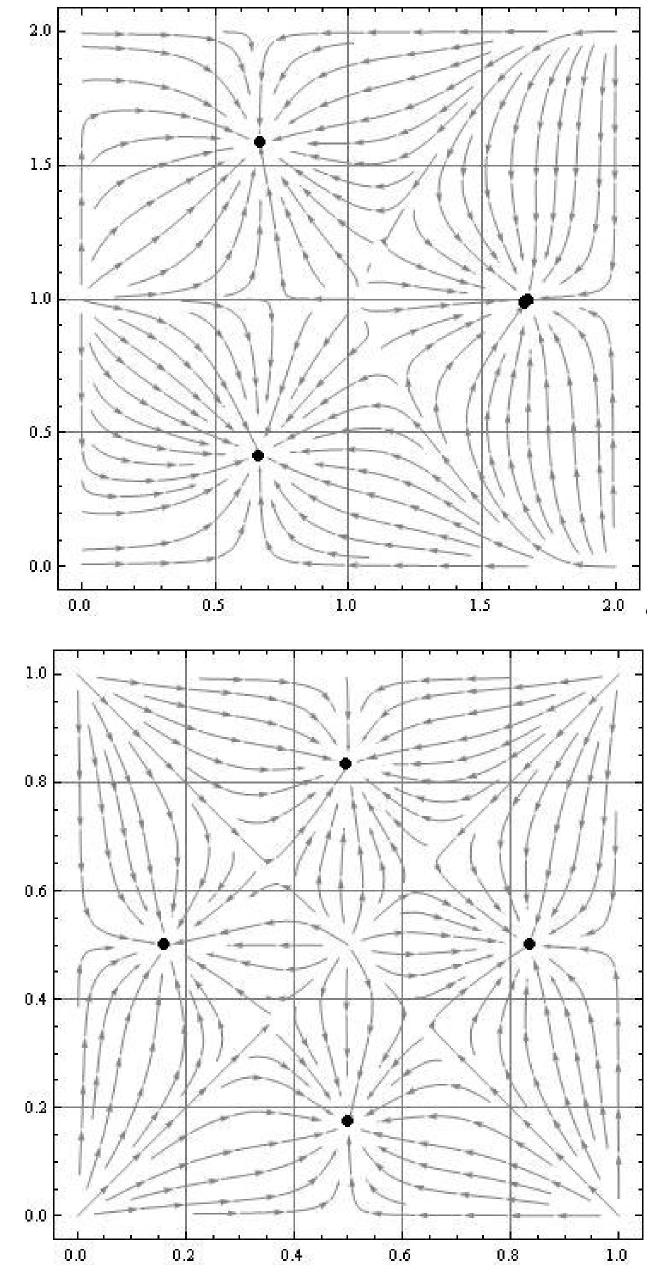


a) b)

Fig. 19 a) the experimental streamlines [32] and b) the streamlines calculated according to (41) in a closed square reservoir 2x2 with q_1=1 и q_2=4.1.

The streamlines for this reservoir calculated according to (41) are shown in the Fig. 19b. Reference to Figs 19a and 19b shows the complete agreement between the experimental and calculated streamlines data.

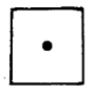
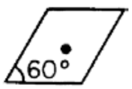
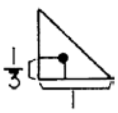
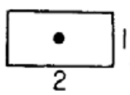
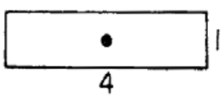
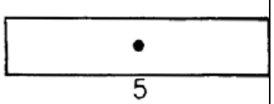
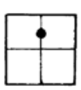
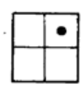
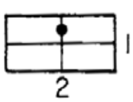
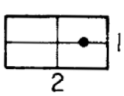
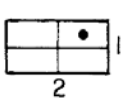
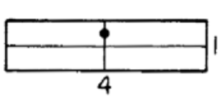
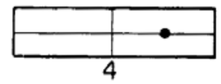
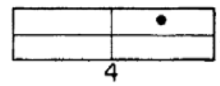
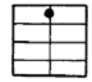
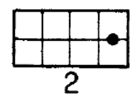
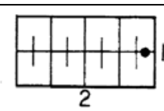
Figure 20 shows the nature of the fluid flow in a square reservoir 2x2 with a symmetric placement of three (a) or four (b) wells of the same flow rate. The first case can further be interpreted as a rectangular reservoir 2x1, when one of the wells is located within the reservoir, and the second is on the boundary with the 2:1 ratio of the flow rates.



a) b)

Fig. 20. The nature of the fluid flow in a closed square reservoir with a symmetrical placement in the reservoir of three (a) or four (b) wells.

Table 1. The values of the Dietz shape factor C_A according Eq. (31) or Eq. (42), as well calculated by Dietz [27] and Earlougher [28].

The shape of drainage area	C_A calculated by		
	Eqs. (31), (42)	[27]	[28]
	30.881	30.9	30.8828
	31.548	31.6	-
	27.321	27.6	-
	26.493	27.1	-
	21.918	21.9	-
	21.836	22.6	21.8309
	5.378	5.38	5.3780
	2.359	2.36	2.3606
	12.984	12.9	12.9851
	4.522	4.57	4.5132
	10.837	10.8	10.8374
	4.522	4.86	4.5141
	2.081	2.07	2.0769
	2.689	2.72	
	0.232	0.232	0.2318
	0.116	0.115	0.1155
	3.335	3.39	3.3351
	3.157	3.13	3.1573
	0.583	0.607	0.5813
	0.112	0.111	0.1109
	0.100	0.098	-

VIII. Conclusions

This paper presents the analytical solution for modeling the oil fields development with the use of the Weierstrass elliptic functions. The velocity and the pressure fields defined by relationships (26) and (27) allow us

1) to determine the drainage areas for each well depending on its placement on a doubly periodic lattice of wells,
2) to describe the nature of the fluid flow (streamlines) in the drainage area for each well in a doubly periodic lattice of wells,
3) to find the pressure distribution within the drainage area as well as in the whole reservoir,
4) to determine the dependence between the productivity index *PI* of the drainage area and the reduced radius *R* of the drainage area,
5) to establish the relationship between the reduced radius of the drainage area *R* and the Dietz shape factor C_A,
6) to calculate the Dietz shape factor C_A for any shape of drainage area.

The approach proposed has been extended for the case of multi-well producing systems (doubly periodic multi-well clusters). In this case, the productivity index (*PI*) has been found for each well of the multi-well cluster, as well the multi-well productivity index (*MPI*) of this cluster as a whole.

The dependences (31) and (42) for *PI* and *MPI* allow us to calculate the shape factor for each well of the multi-well cluster and to optimize the placement of the wells in this cluster. It is shown that in any case the most optimal structure of the multi-well cluster is the hexagonal one with the shape factor C_A=31.548.

The calculated values of the shape factor C_A for different forms of drainage areas are presented in Table 1. These values correspond to the results given by Dietz [27] and Earlougher *et al*. [28] for the rectangular and triangular drainage areas.

REFERENCES

[1] W. T. Koiter, "Some general theorems on doubly-periodic and quasi-periodic functions," *Proc. Kon. Ned. Acad. Wt. Amsterdam,* A, vol. 157, pp. 120-128, 1959.

[2] W. T. Koiter, "Stress distribution in an infinite elastic sheet with a doubly-periodic set of holes," in *Boundary problems in differential equations.* Madison: Univ. of Wisconsin Press, pp. 191-213, 1960.

[3] E. I. Grigolyuk and L. A. Filshtinsky, *Perforated Plates and Shells* (in Russian), Moscow: Nauka, 556 pp., 1970.

[4] L. A. Fil'shtinskii, "Doubly-periodic problem of the theory of elasticity for an isotropic medium weakened by congruent groups of arbitrary holes," *J. Appl. Math. Mech.,* vol. 36(4), pp. 682-690, 1972 (Engl. transl.).

[5] V. M. Fomin, "Antiplane waves in an elastic medium with biperiodic system of cavities," *J. Appl. Math. Mech.,* vol. 62(3), pp. 443-451, 1998 (Engl. transl.)

[6] V. V. Panasuk, M. P. Savruk and A. P. Datsishin, *Stress Distribution around Cracks in Plates and Shells* (in Russian), Kiev: Naukova Dumka, pp. 149-162, 1976.

[7] N. I. Iokamidis and P. S. Theokaris, "Array of periodic curvilinear cracks in an infinite isotropic mediym," *Acta Mechanica,* vol. 28, pp. 239-254, 1977.

[8] V. M. Mirsalimov, *Two-dimensional Elastic-Plastic Problems* (in Russian), Moscow: Nauka, pp. 56-66, 129-140, 1987.

[9] A. M. Linkov and V. F. Koshelev, "Complex variables BIE and BEM for a plane doubly-periodic system of ftaws," *J. Chin. Inst. Eng.,* vol. 22, pp. 709-720, 1999.

[10] J. D. Eshelby, "The continuum theory of lattice defects," in *Progress in Solid State Physics,* vol. 3, F. Seitz and D. Turnbull, Eds. New York: Academic Press, pp. 79-144, 1956.

[11] L. I. Sedov, *Two-dimensional Problems in Hydrodynamics and Aerodynamics,* New York: Wiley, ch. III, 1965.

[12] V. K. Tkachenko, "On vortex lattices," *Soviet Physics, JETF,* vol. 22, pp. 1282-1286, 1966 (Engl. transl.).

[13] V. K. Tkachenko, "Stability of vortex lattices," *Soviet Physics, JETF,* vol. 23, pp. 1049-1056, 1966 (Engl. transl.).

[14] K. A. O'Neil, "On the Hamiltonian dynamics of vortex lattices," *J. Math. Phys,.* vol. 30, pp. 1373–1379, 1989.

[15] H. Aref, "Three-vortex motion with zero total circulation: Addendum," *J. Appl. Math. Phys. (ZAMP),* vol. 40, pp. 495-500, 1989.

[16] H. Aref and M. A. Stremler, "On the motion of three point vortices in a periodic strip," J. Fluid Mech., vol. 314, pp. 1-25, 1996.

[17] M. A. Stremler and H. Aref, "Motion of three vortices in a periodic parallelogram," *J. Fluid Mech*., vol. 392, pp. 101-128, 1999.

[18] H. J. Morel-Seytoux, "Analytical-numerical method in waterflooding predictions," *SPE Journal,* vol. 5, pp. 247-258, 1965.

[19] H. J. Morel-Seytoux, "Unit mobility ratio displacement calculations for pattern floods in homogeneous medium," *SPE Journal,* vol. 6, pp. 217-227, 1966.

[20] R. T. Fazlyev, A*real Waterflooding of Oilfields* (in Russian), Moscow: Nedra, 256 pp., 1979.

[21] N. I. Akhiezer, *Elements of the Theory of Elliptic Functions,* Providence, RI: American Mathematical Society (translated from Russian as *AMS Translations of Mathematical Monographs. Vol. 79).* 1990.

[22] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions: with Formulas, Graphs and Mathematical Tables,* New York: Dover, 1965.

[23] M. Muskat, *The Flow of Homogeneous Fluids through Porous Media,* New York: McGraw-Hill, 1937.

[24] F. D. Gakhov, *Boundary-value problems.* Oxford: Pergamon Press, 1966. (Engl. transl.).

[25] V. I. Astaf'ev and P. V. Roters, "Modeling of doubly periodic system of producing wells" (in Russian), *Vestnik of Samara State University,* No. 4 (78), pp. 5-11, 2010.

[26] V.I. Astaf'ev and P.V. Roters, "Modeling of doubly periodic system of producing wells. 2. Productivity index", *Vestnik of Samara State University,* No. 8 (89), pp. 118-127, 2011 (in Russian).

[27] D.N. Dietz, "Determination of average reservoir pressure from build-up surveys", *J. Pet. Tech.,* vol. 17, pp. 955-959, Aug. 1965.

[28] R.C. Earlougher, Jr., H.J. Ramey, Jr., F.G. Miller and T.D. Mueller, "Pressure distributions in rectangular reservoirs", *J. Pet. Tech.,* vol. 20, pp. 199-208, Feb. 1968.

[29] V.I. Astaf'ev and P.V. Roters, "Modeling and optimization of oil recovery by doubly periodic multi-well clusters", *Vestnik of Samara State University,* No. 9/2 (110), pp. 171-184, 2013 (in Russian).

[30] E. Ozkan, "Performance of horizontal wells", Ph.D. dissertation, The Graduate School, The University of Tulsa, Tulsa, OK, 1988.

[31] P.P. Valko, L.E. Doublet and T.A. Blasingame, "Development and application of the multiwell productivity index (MPI)", *SPE Journal,* vol. 5, pp. 21-31, March 2000.

[32] C.S. Matthews and H.C. Lefkovits, "Studies on pressure distribution in bounded reservoirs at steady state", *Petroleum Transactions, AIME,* vol. 204, pp. 182-189, 1955.