# Quasianalytic evaluation of stability of nonlinear dynamic systems oscillations

Lelya Khajiyeva, Askat Kudaibergenov, Askar Kudaibergenov and Almatbek Kydyrbekuly

***Abstract*** – The stability of nonlinear systems is studied by the method of partial discretization. The steady motion of a mechanical system is defined here as its motion without resonant oscillations. In order to avoid resonance oscillations in the operating modes of the system a quasianalytic evaluation of the behavior of the solution of the perturbed state equation is made. The equation is partially discretized in the class of generalized functions (Dirac delta function). An analytical solution characterizing the behavior of a small perturbation $\delta f$ in time has been obtained. The efficiency of the proposed approach is based on the simplicity of the solution and visualization of its results. This is illustrated by the example of stability analysis of resonance oscillations at the basic frequency of physically and geometrically nonlinear systems. The results obtained in this paper are in good agreement with well-known results obtained by other methods.

***Keywords* –** Dynamics, nonlinear oscillations, stability zone, partial discretization.

## I. INTRODUCTION

THE problem of stability of motion is one of the main problems of modern dynamical systems having a wide range of practical applications in engineering. Nonlinear dynamical systems are of special interest.

Nonlinearity of dynamical systems may be caused by different factors. In practice of machine design and operation, nonlinearity of systems may be due to the influence of large inertial forces and technological loads. They may cause elastic [1] or plastic deformation of individual elements or of the system as a whole. Another source of nonlinearity is initial stresses, physically nonlinear properties of elements of the system, nonlinear viscous friction and other factors [2]-[3]. They can lead to the emergence of complex oscillatory processes with modulation frequencies and resonance phenomena at sub- and ultra frequencies. Therefore, in order to provide stable movement of nonlinear systems it is necessary to identify resonance frequencies and to exclude them from the operating modes.

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L. Khajiyeva is with the Department of Mechanics and Mathematics al-Farabi Kazakh National University, Almaty 050040, Kazakhstan (corresponding author e-mail: khadle@mail.ru).

Askat Kudaibergenov is with the Department of Mechanics and Mathematics al-Farabi Kazakh National University, Almaty 050040, Kazakhstan (e-mail: ask7hat@mail.ru).

Askar Kudaibergenov is with the Department of Mechanics and Mathematics al-Farabi Kazakh National University, Almaty 050040, Kazakhstan (e-mail: as5kar@mail.ru).

A. Kydyrbekuly is with the Institute of Mechanics and Theoretical Ingineering, Almaty 050010, Kazakhstan (e-mail: almatbek@list.ru).

Investigation of stability of motion of mechanisms and machines depends on the choice of the dynamic model. The most widely studied model is a nonlinear dynamical model with one degree of freedom. This model not only describes movements of machine elements as absolutely rigid units, but it can also describe oscillations of systems with discrete masses.

On the other hand, such models can be used for rough description of oscillations of systems with distributed parameters. In this case, dependences on spatial variables are eliminated using the Bubnov-Galerkin, Rayleigh-Ritz methods, finite element method and others. This approach is widely used by the authors studying stability of motion of elements of machines and mechanisms taking into account their nonlinear deformations.

Among the early studies of resonant oscillations in nonlinear systems with one degree of freedom, it is necessary to mention the work of W. Szemplinska-Stupnicka [4]. The author was the first to study resonant oscillations at high frequencies and problems of their stability.

In some papers the stability of periodic oscillations was studied by asymptotic methods and methods of a small parameter. They refer to quasi-linear and quasi-Lyapunov systems [5]-[7], etc. In these works the conditions of asymptotic stability were obtained using Lyapunov's function with rather rigid restrictions on the degree of nonlinearity. In [8] the stability of periodic oscillations of a nonlinear system was considered without restrictions on its nonlinearity and non-autonomous terms.

The first Lyapunov's method is also used to study the problem of robust stability and stabilization of linear time-invariant systems with delayed perturbation [9]-[11], etc, as well as stability of nonlinear systems including delayed perturbations [12]. This problem is especially difficult for systems with time-varying delay. In this case the analysis of stability becomes a much more difficult job. In [13] the stability of the trivial equilibrium position of mechanical systems with time-varying delay was studied.

The tool most widely used to analyze the stability of linear and nonlinear dynamic systems is the second (main) A.M. Lyapunov's method [14]. Most authors base their study of stability of motion on the application of the second Lyapunov's method. In this case, Lyapunov stability implies the existence of initial disturbances or conditions under which the motion of the system remains within certain limits. These limits depend on the strength and geometrical parameters of the element. The stability boundary evaluation problem is important in many engineering disciplines. The methods developed for this purpose avoid the need of carrying out extensive experiments and simulations of the systems design process [15].

To determine the areas of instability of motion the authors often use the Floquet theory [4], [16]-[18], which enables them to construct a characteristic determinant based on the perturbation method and known properties of equations of Mathieu or Hill type [4], [17]-[18]. It sets the boundaries of such areas. The fundamental works of T. Hayashi [17], A. Tondl [18] and V.V. Bolotin [19] on studies of parametric instability of nonlinear mechanical systems are also well known. They consider in details stability of vibrations of mechanical systems for sub- and ultra-harmonics and methods of determining their boundaries.

The other, not less well-known approach, is identification of areas of instability in accordance with the Routh-Hurwitz criterion. In this method, the rank of the determinant significantly affects the calculations. For example, in case of resonance at high frequencies it is difficult to determine the areas of instability due to the high rank of the characteristic determinant [4]. In some papers the stability of solutions of dynamic models is studied based on the geometric representation of stability (geometric method) [20], discretization of models in time, application of the iteration methods [16], [21] and other methods.

A great variety of methods is used to solve stability problems. Nevertheless, the development of tools of dynamic analysis of mechanical systems is of great practical interest.

In this paper stability of nonlinear dynamical systems is analyzed by the method of partial discretization. A steady motion of a mechanical system is defined here as its motion without resonance oscillations. In order to exclude them from the operating modes of the mechanical system, a quasianalytic evaluation of the solution to the perturbed equation of state is made. For this purpose the equation is partially discretized in the class of generalized functions. Discretization of the equation enables us to obtain an analytical solution characterizing the behavior of a small perturbation $\delta f$ in time.

The paper describes the state of the problem and the principle of partial discretization of the equation in the class of generalized functions. In this work the Hill-type equation was partially discretized and its analytic solution was obtained. As an example, a resonance at the basic frequency in physically and geometrically nonlinear systems with one degree of freedom was considered. A numerical analysis of the behavior of solutions of Hill equations was made. The results obtained in this work are in good agreement with the results of other authors, which shows the efficiency of the proposed approach based on the simplicity of solution and visualization of the obtained results.

## II. STATEMENT OF THE PROBLEM

Let us consider a nonlinear dynamic model with one degree of freedom:

$$\ddot{f} + \Phi(\dot{f}, f) + \omega_0^2 f = F(\Omega t). \qquad (1)$$

In case of motion of bodies with distributed parameters, when spatial variables are excluded from the model of elastic body motion, the unknown $f(t)$ can represent a function of generalized displacements. This approach is widely used in practice of dynamic analysis and solution of problems of stability of elastic body motion, in particular, motion of rod elements. It is known that in many technical areas, structural elements, reducible to the design of the rod, are widely used.

The degree of nonlinearity of $\Phi(\dot{f}, f)$ with respect to the function of generalized displacements $f(t)$ corresponds to the assumptions of the model. In addition, it characterizes nonlinearity of elastic characteristics (geometrical and physical nonlinearity) and dissipative forces.

Let us consider a periodic solution of (1). To study stability of the periodic solution $f_0(t)$ we will set a small deviation $\delta f$ from its equilibrium state:

$$f(t) = f_0(t) + \delta f\ . \qquad (2)$$

Stability of the periodic solution $f_0(t)$ depends on the nature of behavior of its small deviation $\delta f$ in time, i.e. solution of the equation for the perturbed state of the system:

$$\delta \ddot{f} + \left(\frac{\partial \Phi}{\partial \dot{f}}\right)_0 \delta \dot{f} + \left(\frac{\partial \Phi}{\partial f}\right)_0 \delta f = 0\,, \qquad (3)$$

where the symbol $(\ )_0$ means that the solution $f_0(t)$ is taken as an argument of functions.

If the solution $\delta f$ of (3) is limited for $t \to \infty$, the motion of the system is considered to be stable. If $\delta f \to \infty$ for $t \to \infty$, by definition, the motion is unstable, which is identical to the criterion of Lyapunov stability.

The possibility of transition from (1) to (3) is presented in [10], where the author refers to Trefftz's research of properties of periodic solutions of equations in the form (1). The limitation of the solution (1) and its asymptotic stability lead to its periodicity with a very small period equal or multiple to the period of external perturbing force.

A new variable $\eta$ is introduced:

$$\delta f = \eta \exp\left(-0{,}5\left(\frac{\partial \Phi}{\partial \dot{f}}\right)_0\right). \qquad (4)$$

Then (3) is reduced to the Hill parametrical equation with respect to the variable $\eta$.

For the case of basic resonance the Hill equation is written as:

$$\frac{d^2\eta}{dt^2} + \eta\left[\theta_0 + \theta_{1s}\sin\Omega t + \theta_{1c}\cos\Omega t + \theta_{2s}\sin 2\Omega t + \theta_{2c}\cos 2\Omega t\right] = 0, \qquad (5)$$

where $\theta_0$, $\theta_{1s}$, $\theta_{1c}$, $\theta_{2s}$, $\theta_{2c}$, are functions of frequencies, amplitudes, and phases of oscillations of harmonic solutions of Eq. (1) $\Omega$, $r_1$, $\phi_1$), respectively.

Solutions of Hill equations will determine the behavior of small perturbations $\delta f$ in time t.

To exclude secular terms, growing with time, from the solution, various methods are used, for example, Newton's iteration [16].

Among the methods of dynamic analysis of vibrations of mechanical systems there are methods based on the construction of characteristic determinants defining the boundaries of instability regions of resonant modes. For this purpose, the Floquet theory is used [17].

The boundaries of instability regions can be determined directly from the amplitude-frequency characteristics, that is, from the resonance curves using the Routh-Hurwitz criterion. In this case it is not necessary to solve the Hill-type equation.

One of the methods used to solve such equations is partial discretization of equations in space or in time (the straight line method) [22]. A similar approach was used in [23] for partial discretization of the equation for oscillations of a curved beam. The discretization was made in the class of generalized functions.

At each step of partial discretization the variable coefficient of the equation was represented as a constant. This enabled the author to determine the magnitude of flexure of the beam analytically.

Here the analysis of stability of the solution (1) is based on discretization of the second term of the parametric Hill-type equation in the class of generalized functions.

In this paper the method of partial discretization is applied to the Hill-type equation in order to avoid a time-consuming process of determination of characteristic determinants and boundaries of instability regions of resonant modes. The Hill-type equation is simplified, which enables us to obtain analytical solutions of Hill equations characterizing the behavior of small perturbations $\delta f$ in time t.

## III. PARTIAL DISCRETIZATION OF THE HILL EQUATION

According to the method of partial discretization, the second term in Eq. (5) is represented discretely in the class of the generalized functions:

$$\frac{d^2\eta}{dt^2}+\frac{1}{2}\sum_{k=1}^{n}(t_k+t_{k+1})[(\theta_0+\theta_{1s}\sin\Omega t_k+\theta_{1c}\cos\Omega t_k
+\theta_{2s}\sin 2\Omega t_k+\theta_{2c}\cos 2\Omega t_k)\cdot\eta(t_k)\delta(t-t_k)
-(\theta_0+\theta_{1s}\sin\Omega t_{k+1}+\theta_{1c}\cos\Omega t_{k+1}+\theta_{2s}\sin 2\Omega t_{k+1}
+\theta_{2c}\cos 2\Omega t_{k+1})\cdot\eta(t_{k+1})\delta(t-t_{k+1})]=0, \tag{6}$$

where

$\eta(t_k)$ is discrete representation of function $\eta(t)$ for the value of the argument $t=t_k$;

$k=\overline{1,n}$ is number of splitting of the argument $t$;

$\delta(t-t_k)$ is Dirac's delta function.

It is not difficult to solve (16). For the initial conditions:

$$t=0 \quad \eta(0)=\eta_0,\quad \dot\eta(0)=\dot\eta_0,$$

it is written as:

$$\eta(t)=-\frac{1}{2}\sum_{k=1}^{n}(t_k+t_{k+1})[(\theta_0+\theta_{1s}\sin\Omega t_k+\theta_{1c}\cos\Omega t_k
+\theta_{2s}\sin 2\Omega t_k+\theta_{2c}\cos 2\Omega t_k)\cdot\eta(t_k)H(t-t_k)
-(\theta_0+\theta_{1s}\sin\Omega t_{k+1}+\theta_{1c}\cos\Omega t_{k+1}+\theta_{2s}\sin 2\Omega t_{k+1}
+\theta_{2c}\cos 2\Omega t_{k+1})\cdot\eta(t_{k+1})H(t-t_{k+1})]+\dot\eta_0 t+\eta_0, \tag{7}$$

where $H(t-t_k)$ denotes the Heaviside step function. Specifying $t$ discretely, we obtain a recurrent formula for calculation of unknown $\eta(t)$ on $k$-th step of splitting of the argument $t$:

$$\eta(t_k)=\Big[-(t_1+t_2)(\theta_0+\theta_{1s}\sin\Omega t_1+\theta_{1c}\cos\Omega t_1+\theta_{2s}\sin 2\Omega t_1
+\theta_{2c}\cos 2\Omega t_1)\eta(t_1)\left(\frac{t_k+t_{k+1}}{2}-t_1\right)\Big]\Big/\Big[1+\frac{1}{2}(t_{k+1}-t_k)
\cdot(t_{k+1}-t_{k-1})(\theta_0+\theta_{1s}\sin\Omega t_k+\theta_{1c}\cos\Omega t_k+\theta_{2s}\sin 2\Omega t_k
+\theta_{2c}\cos 2\Omega t_k)\Big]-\Big[\sum_{j=2}^{k-1}(t_{j+1}-t_{j-1})(\theta_0+\theta_{1s}\sin\Omega t_j
+\theta_{1c}\cos\Omega t_j+\theta_{2s}\sin 2\Omega t_j+\theta_{2c}\cos 2\Omega t_j)\eta(t_j)
\cdot\left(\frac{t_k+t_{k+1}}{2}-t_j\right)\Big]\Big/\Big[1+\frac{1}{2}(t_{k+1}-t_k)(t_{k+1}-t_{k-1})
\cdot(\theta_0+\theta_{1s}\sin\Omega t_k+\theta_{1c}\cos\Omega t_k+\theta_{2c}\sin 2\Omega t_k
+\theta_{2c}\cos 2\Omega t_k)\Big]+\Big[\dot\eta_0\frac{t_k+t_{k+1}}{2}+\eta_0\Big]\Big/\Big[1+\frac{1}{2}(t_{k+1}-t_k)
\cdot(t_{k+1}-t_{k-1})(\theta_0+\theta_{1s}\sin\Omega t_k+\theta_{1c}\cos\Omega t_k
+\theta_{2s}\sin 2\Omega t_k+\theta_{2c}\cos 2\Omega t_k)\Big]. \tag{8}$$

Unlike [14]-[15], where the method of partial discretization is applied to studying of parametric system oscillations, in this paper it is used directly for the solution of the perturbation equation in terms of $\delta(t)$. It is possible to determine stability of the state analyzing the nature of $\delta(t)$ behavior, according to the Lyapunov stability criterion. If the magnitude $\delta(t)$ decreases with time $t$ (damping process) then $\delta f \to 0$, i.e. the state is stable. If the oscillatory process is growing, then we have an unstable state.

The efficiency of the proposed method will be shown below on the example of stability analysis of resonant oscillations of physically nonlinear systems.

## IV. ANALYTICAL SOLUTION OF THE HILL EQUATION

### *A. Physically nonlinear systems*

As an example, let us consider the motion of physically nonlinear systems. The equations of motion for these systems are taken in the form:

$$\frac{d^2 f}{dt^2} + k_1 \frac{df}{dt} + k_2 \left(\frac{df}{dt}\right)^2 + \alpha_1 f + \alpha_2 f^2 = F_0 + F_1 \cos\Omega t. \tag{9}$$

In (9) dissipative forces, which are supposed to be nonlinear and viscous due to damping properties of physically nonlinear media (rubber and similar materials are used as oscillation dampers), are taken into account.

Physical nonlinearity of the system (soft-type nonlinearity) is characterized by an arbitrary angle of rotation of cross elements, which corresponds to quadratic nonlinearity of the restoring force.

The stability of the main resonance is studied. The solution of (9) is expressed as

$$f(t) = r_0 + r_1 \cos(\Omega t - \varphi_1). \tag{10}$$

In this case the Hill equation is written as (5):

$$\frac{d^2\eta}{dt^2} + \eta\left[\theta_0 + \theta_{1s}\sin\Omega t + \theta_{1c}\cos\Omega t + \theta_{2s}\sin 2\Omega t + \theta_{2c}\cos 2\Omega t\right] = 0, \tag{11}$$

where

$$\theta_0 = \alpha_1 + 2\,\alpha_2 r_0 - 0.25\,k_1^2 - 0.5\,k_2^2 r_1^2\Omega^2,$$

$$\theta_{1s} = (2\,\alpha_2 r_1 + k_2 r_1\Omega^2)\sin\varphi_1 + k_1 k_2 r_1\Omega\cos\varphi_1,$$

$$\theta_{1c} = (2\,\alpha_2 r_1 + k_2 r_1\Omega^2)\cos\varphi_1 - k_1 k_2 r_1\Omega\sin\varphi_1, \tag{12}$$

$$\theta_{2s} = 0.5\,k_2^2 r_1^2\Omega^2 \sin 2\varphi_1,$$

$$\theta_{2c} = 0.5\,k_2^2 r_1^2\Omega^2 \cos 2\varphi_1.$$

According to the above technique, using the method of partial discretization for the initial conditions $\eta(0) = \eta_0$ and $\eta(0) = \eta_0$, we obtain an analytical solution to physically nonlinear systems (11):

$$\begin{aligned}\eta(t_k) = \Big[&-(t_1 + t_2)(\theta_0 + \theta_{1s}\sin\Omega t_1 + \theta_{1c}\cos\Omega t_1 \\ &+ \theta_{1c}\cos\Omega t_k + \theta_{2s}\sin 2\Omega t_k \\ &+ \theta_{2c}\cos 2\Omega t_k)\Big] - \Bigg[\sum_{j=2}^{k-1}(t_{j+1} - t_{j-1})(\theta_0 + \theta_{1s}\sin\Omega t_1 \\ &+ \theta_{1c}\cos\Omega t_j + \theta_{2s}\sin 2\Omega t_j + \theta_{2c}\cos 2\Omega t_{j1})\eta(t_j) \\ &\cdot\left(\frac{t_k + t_{k+1}}{2} - t_j\right)\Bigg]\Bigg/\Bigg[1 + \frac{1}{2}(t_{k+1} - t_k)(t_{k+1} - t_{k-1}) \\ &\cdot(\theta_0 + \theta_{1s}\sin\Omega t_k + \theta_{1c}\cos\Omega t_k + \theta_{2c}\sin 2\Omega t_k \\ &+ \theta_{2c}\cos 2\Omega t_k)\Bigg] + \left[\dot\eta_0\,\frac{t_k + t_{k+1}}{2} + \eta_0\right]\Bigg/\Bigg[1 + \frac{1}{2} \\ &\cdot(t_{k+1} - t_k)(t_{k+1} - t_{k-1})(\theta_0 + \theta_{1s}\sin\Omega t_k + \theta_{1c}\cos\Omega t_k \\ &+ \theta_{2s}\sin 2\Omega t_k + \theta_{2c}\cos 2\Omega t_k)\Bigg]\end{aligned} \tag{13}$$

with coefficients (12).

### *B. Geometrically nonlinear systems*

Another type of nonlinearity of mechanical systems is geometric nonlinearity (nonlinearity of rigid type). It may be caused by finite deformations, initial stresses and other factors.

For a nonlinear system of the type

$$\frac{d^2 f}{dt^2} + k_1 \frac{df}{dt} + k_2 \left(\frac{df}{dt}\right)^2 + \alpha_1 f + \alpha_3 f^3 = F\cos\Omega t, \tag{14}$$

we studied stability of the basic frequency of the resonance:

$$f(t) = r_1\cos(\Omega t - \varphi_1). \tag{15}$$

The perturbed equation of state (3) is reduced to the parametric Hill-type equation with respect to the variable $\eta$ (5). For this equation, the functions $\theta_0$, $\theta_{1s}$, $\theta_{1c}$, $\theta_{2s}$, $\theta_{2c}$ are defined as:

$$\theta_0 = \alpha_1 + 1,5\,\alpha_3 r_1^2 - 0,25\,k_1^2 - 0,5\,k_2^2 r_1^2 \Omega^2,$$

$$\theta_{1s} = -k_2 r_1 \Omega^2 \sin\varphi_1 + k_1 k_1 r_1 \Omega \cos\varphi_1,$$

$$\theta_{1c} = -k_2 r_1 \Omega^2 \cos\varphi_1 - k_1 k_2 r_1 \Omega \sin\varphi_1, \qquad (16)$$

$$\theta_{2s} = 1,5\,\alpha_3 r_1^2 \sin 2\varphi_1 + 0,5\,k_2^2 r_1^2 \Omega^2 \sin 2\varphi_1,$$

$$\theta_{2c} = 1,5\,\alpha_3 r_1^2 \cos 2\varphi_1 + 0,5\,k_2^2 r_1^2 \Omega^2 \cos 2\varphi_1.$$

As in the previous case, the analytical solution of Hill-type equations is determined by formula (13). In this case, the coefficients $\theta_0$, $\theta_{1s}$, $\theta_{1c}$, $\theta_{2s}$, $\theta_{2c}$ are defined by relations (16).

The solution (13) is a recurrent formula for discrete representation of solution $\eta(t)$ in time $t$ on the $k$-th step of partition of the argument $t$. By analyzing the nature of behavior of $\eta(t)$, we can judge about the stability of the studied state.

## V. NUMERICAL RESULTS

### *A. Physically nonlinear system*

In this work the methods of numerical analysis were used to analyze the behavior of $\eta(t)$ representing the behavior of a small variation $\delta f$ with time.

Calculations were made for parameters of the system $k_1 = 0.2;\ k_2 = 0.1;\ \alpha_1 = 5;\ \alpha_2 = 0.5;\ F_0 = 5;\ F_1 = 50.$ The step of discretization was $\Delta t$ =0.05.

Stability of the solution (13) was studied by subdividing the amplitude-frequency characteristics of the main resonance (Fig.1) into three frequency areas – subresonant, resonant and postresonant modes of oscillations.

It has been established that both subresonant and postresonant modes of oscillations are damping (Fig. 2, Fig. 3), which does not contradict the physical sense of the studied phenomenon.

In the zone of resonant frequencies an increase in the oscillation amplitude is observed, which shows that the process is instable (Fig.4).

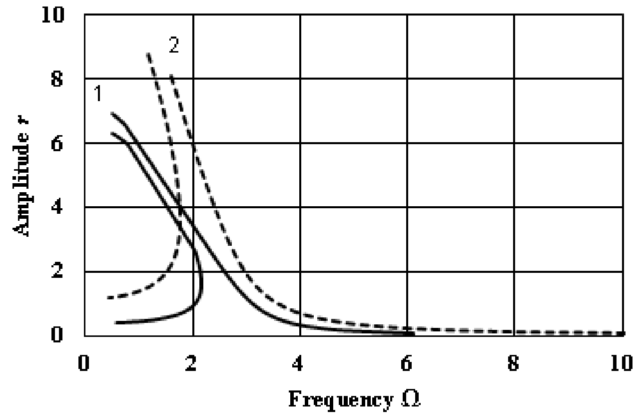


Fig. 1 amplitude-frequency characteristics of a main resonance at $\alpha_2 = 1;\ F_1 = 10$ (curve 1), $\alpha_2 = 0.5;\ F_1 = 50$ (curve 2)

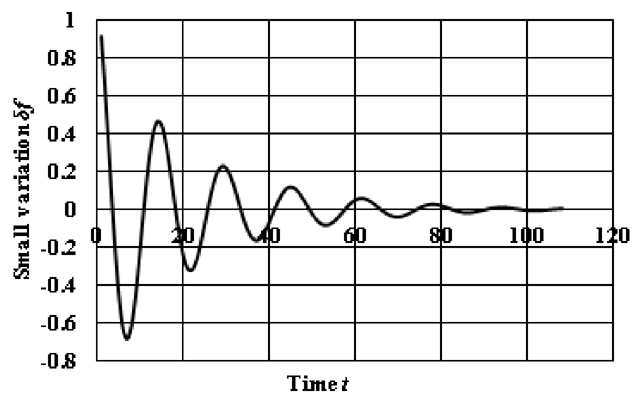


Fig. 2 behavior of the physically nonlinear system in the subresonant zone of oscillations at $\Omega = 0.5, r = 1.5$

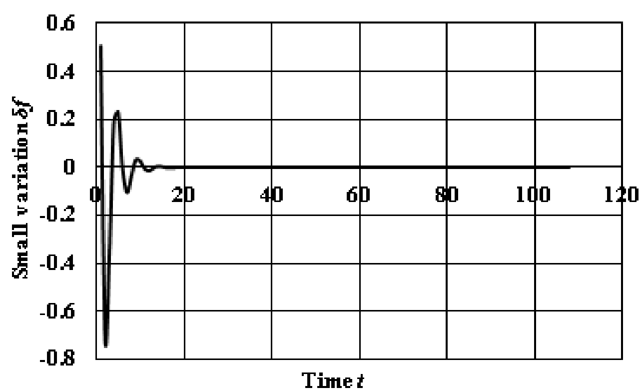


Fig. 3 behavior of the physically nonlinear system in the postresonant zone of oscillations at $\Omega = 7.26, r = 0.15$

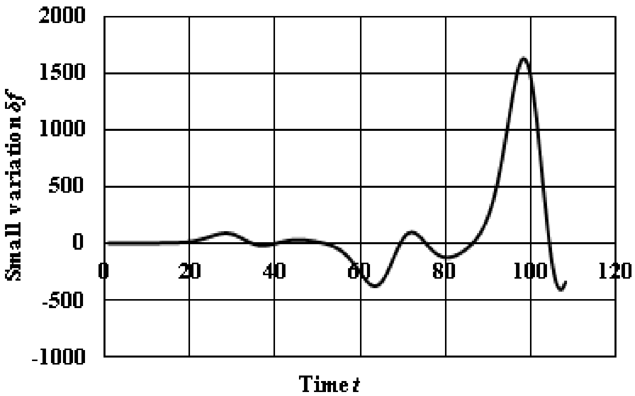


Fig. 4 behavior of the physically nonlinear system in the resonant zone of oscillations at $\Omega = 1.8, r = 8$

*B. Geometrically nonlinear system*

In this case numerical analysis of behavior of $\eta(t)$ was made for parameters $k_1 = 0,2;\ k_3 = 0;\ \alpha_1 = 1;\ \alpha_3 = 1.$ The step of discretization in time was taken equal to $\Delta t$ =0.05.

For this purpose, three frequency ranges – the areas outside the resonance zone (to the left and to the right) and the area inside the resonance zone were defined on the graph of instability zones of the main resonance (Fig. 5).

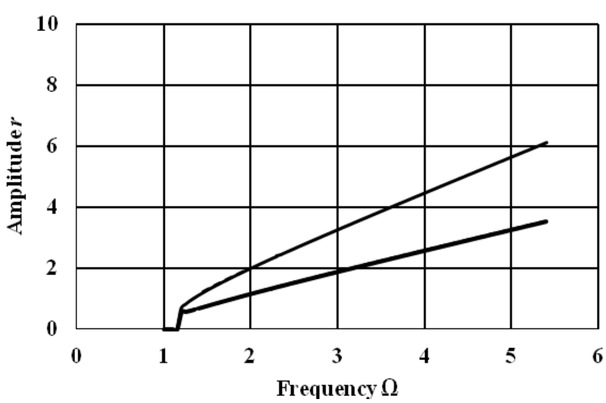


Fig. 5 instability zone of the main resonance of the geometrically nonlinear system

The instability zone of the main resonance (Fig. 5) was obtained using the Floquet theory for testing results.

It was found that for frequencies outside the resonance zone vibrations damped (Fig. 6, Fig. 7) as it was expected. In the zone of instability of the main resonance, as in the case of physically nonlinear systems, growing oscillation amplitude, indicating instability of the process, was observed (Fig. 8).

The application of the method of partial discretization to the problem of stability of oscillations gives an analytical solution. It enables us to detect zones of stable and unstable oscillations of the system. Selecting appropriate geometrical and physical parameters of the system by varying them, we can choose such operating modes of the system that enable us to avoid unwanted resonances.

The approach used in this paper is universal. It is successfully applied to nonlinear systems with one degree of freedom. It is also possible to apply it to the analysis of stability of nonlinear deformable systems. To do this, it is necessary to transform the model of elastic motion of a body by excluding spatial variables and transforming the model to the form (1).

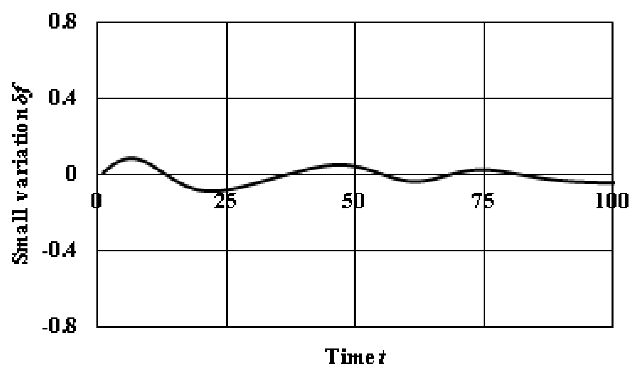


Fig. 6 behavior of the geometrically nonlinear system in the stability zone of oscillations at $\Omega = 1, r = 2$

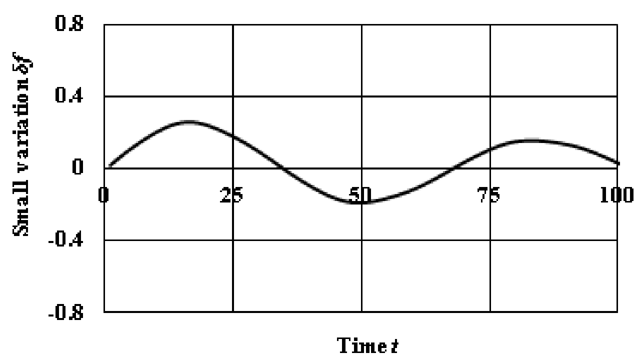


Fig. 7 behavior of the geometrically nonlinear system in the stability zone of oscillations at $\Omega = 4, r = 0,7$

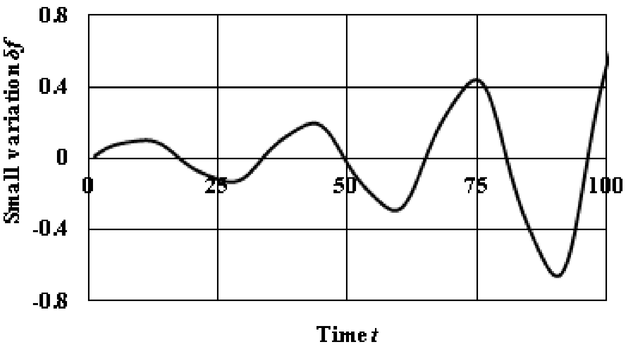


Fig. 8 behavior of the geometrically nonlinear system in the instability zone of oscillations at $\Omega = 4; \quad r = 2,5$

## CONCLUSION

The paper considers the problems of stability of nonlinear mechanical systems and methods of their analysis. As a stable state of the dynamic system the authors define the movement of the system in the absence of resonance oscillations. These requirements are identical to the definition of Lyapunov stability. Therefore, we used the method of analysis of stability of nonlinear systems based on the assessment of behavior of solutions to perturbed equations of state.

Application of the method of partial discretization to the perturbed equations of state enabled us to conduct quasi-analytic evaluation of the behavior of its solutions. For this purposes, the Hill-type equation was partially discretized in the class of generalized functions. This considerably simplified the solution of the Hill-type equation, as its variable coefficients were presented as constants at each step of discretization in time.

The analytical solution of the perturbed equation of state is a recurrent formula for calculating the oscillation amplitude. It enables us to predict the parametric instability of resonant modes of motion of nonlinear systems. The efficiency of the proposed method is shown on the example of physically and geometrically nonlinear systems. The results are in good agreement with the results obtained by other methods.

The proposed method of stability analysis of nonlinear systems is applicable not only to the systems with one degree of freedom but also to the systems with distributed parameters (mainly, to one-dimensional elements, which are widely used in engineering). In this paper the authors considered the case of a basic resonance.

The results of this research can be applied to the analysis of stability of oscillations at high frequencies. The other subject of research can be oscillations of elastic systems with more complex topology of deformation.

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