Extension the matrices of one dimensional beam elements for solution of rectangular plates resting on elastic foundation problems

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***Abstract*—**Complex medium of foundations is a frequently recurring problem for many engineering structures in case of transmission of rational, vertical or horizontal forces. In general it is often difficult to find suitable analytical models for plates on elastic foundation problems. In this study, it is proposed to extend analytical solutions of the discrete one-dimensional beam elements resting on one- or two-parameter elastic foundation for solution of plate problems. Firstly, the derivations of the governing differential equations and exact shape functions are obtained. In order to observe the influences of foundation parameters, some graphical comparisons have been done on stiffness terms and the shape functions for solving general plate problems.

***Keywords*—**Finite element, Shape functions, Stiffness matrices, elastic foundation.

I. Introduction

In many engineering structures assessment of stress conditions created by vertical or horizontal forces to the foundation is a frequent problem of design. In order to include behaviour of foundation properly into the mathematically simple representation it is necessary to make some assumptions. One of the most useful simplified models known as the Winkler model assumes the foundation behaves elastically, and that the vertical displacement and pressure underneath it are linearly related to each other. That is, it is assumed that the supporting medium is isotropic, homogeneous and linearly elastic, provided that the displacements are “small”. This simplest simulation of an elastic foundation is considered to provide vertical reaction by a composition of closely spaced independent vertical linearly elastic springs. There are several more realistic foundation models as well as their proper mathematical formulations given in several references [1-6]. Owing to its convenience in solution of plate problems as a numerical method the finite strip method have attracted much attention from many authors as [7-9]. Among them Huang and Thambiratnam [9] suggested a procedure incorporating the finite strip method together with spring systems is proposed for treating plates on elastic supports. In order to simplify the problem it is possible to use a grid of beam elements to model .

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II. Problem Definition

The governing equation for transverse displacement w(x,y) of plates subjected to lateral loads given in Eqn 1.a can be rearranged for plates resting on two-parameter elastic foundation by using two-dimensional Laplacian operator in Eqn 1.b as follows:

$$D\left(\frac{\partial^4 w(x,y)}{\partial x^4}+2\frac{\partial^4 w(x,y)}{\partial x^2\partial y^2}+\frac{\partial^4 w(x,y)}{\partial y^4}\right)=q(x,y) \qquad (1.a)$$

$$D\nabla^2\nabla^2 w+k_1 w+k_2\nabla^2 w=q(x,y) \qquad (1.b)$$

where k_1 is the Winkler parameter with the unit of force per unit area/per unit length (force/length3), k_2 is the second foundation parameter (force/length) and D is the flexural rigidity of the plate element.

This equation is applicable to all types of rectangular plates including two-parameter elastic foundation problems. Classical methods that provide mathematically exact solutions of plate problems are available for a limited number of limited cases. There are a few load and boundary conditions that permit Eq. (1) to be solved analytically for all load and boundary conditions. Currently, there exist approximate and numerical methods to solve the governing differential equations of plates resting on one-parameter and two-parameter elastic foundation for transverse displacement w. A broad range of the beam or plates as engineering problems has been solved by computer-based numerical methods such as finite element and boundary element methods [10-17]. However closed form solutions for plates have been published for a limited number of cases.

In this study gridwork model of plates for general applications suggested solving a wide range of plate problems. A differential part of a plate supported by a generalized foundation as shown in Fig. 1 can be represented by two parallel sets of beam elements for rectangular plates [18]. On the other hand the similar elements can be formed in radial and tangential directions for circular plates [19].

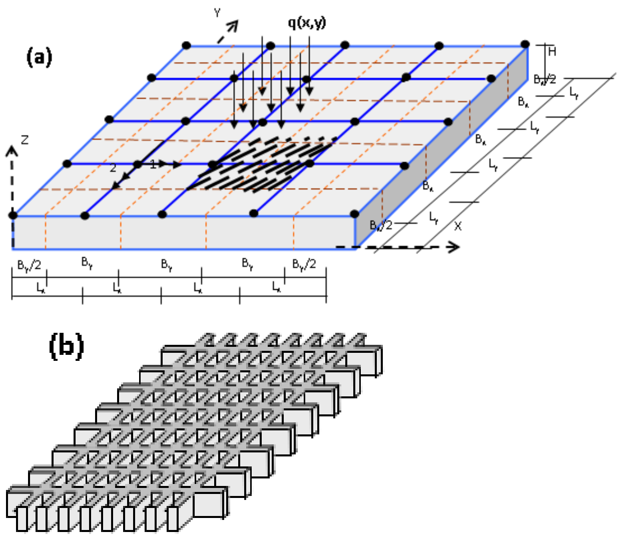

Fig. 1 The discrete system a) the elements are connected at finite nodal points of a rectangular thin plate in flexure, b) Parallel sets of one-dimensional elements replaced by the continuous surface

Many solution methods have been proposed by researchers for the problem of beams resting on elastic foundation have inserted Hermitian polynomials into strain energy functions that has been derived in this study. In order to converge to the better solution, the beam needs to be divided into smaller segments. A representation of the foundation with closely linear translational and rotational springs underlying a beam element is illustrated in Fig. 2. The generalized foundation as a representation of two-parameter model implies that at the end of each translational spring element there must be also a rotational spring to produce a reaction moment (k_θ) proportional to the local slope at that point.

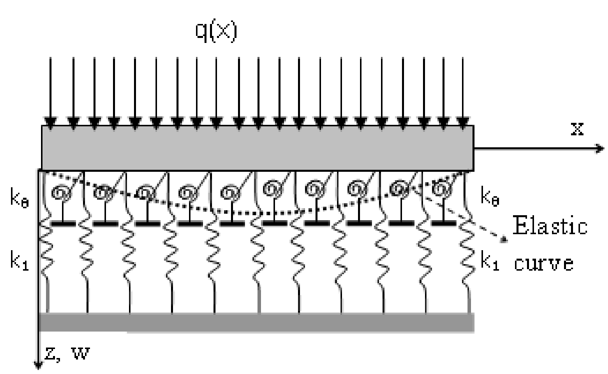

Fig. 2 Representation of the beam element resting on generalized foundation

For generalized foundations the model assumes that at the point of contact between plate and foundation there is not only pressure but also distributed moments caused by the interaction between linear springs. These moments are assumed to be proportional to the slope of the elastic curve by a second parameter for foundation.

There are many researches concerning analysis of beam element resting on elastic foundation [20–25]. Among the references, Eisenberger and Clastornik [25] developed the formulations based on interpolation (shape) functions for of solution beams by finite element method with the exact stiffness matrices. This derivations extended to an analytical solution for the shape functions of a beam segment on a generalized two-parameter elastic foundation given in [18], leading to exact element-level matrices. This study let plates to be represented by a discrete number of intersecting beams. Thus, it is possible to use mechanical properties of one-dimensional beam for solution of plate problems of different types of loading and boundary conditions.

III. Theory and Formulations

For particular plate problems, closed form solutions have been obtained for Eq. (1). However, even for conventional plate analysis these solutions can usually be applied to the problems with simple geometry, load and boundary conditions. With most elements developed to date, there exists no rigorous solution for plates except in the form of infinite Fourier series for a Levy-type solution. The series solutions are valid for very limited cases such as when the second parameter has been eliminated, and simple loading and boundary conditions exist. Networks of beam elements that have no such limitations can represent the plates. The properties of beam elements resting on elastic foundations will be a very useful tool to solve such complicate problems. However, the equation of the elastic curve derived for a beam element resting on a two-parameter elastic foundation from the equilibrium equations of an infinitesimal segment of the structural member is:

$$EI\frac{d^4w(x)}{dx^4}+k_1w(x)-k_\theta\frac{d^2w(x)}{dx^2}=q(x) \qquad (2)$$

For different types of loading and boundary conditions it is possible to extend the exact solution of Eq. (2) for a beam element supported on a two-parameter elastic foundation to plates on generalized foundations when the plate is represented by a discrete number of intersecting beams. Then finite element based matrix methods will be used to determine the exact shape, fixed end forces and stiffness matrices of beam elements resting on elastic foundations. These individual element matrices will be used to form the system load and stiffness matrices for plates.

3.1 Derivation of Exact Shape Functions

For a beam element resting on two-parameter elastic foundation, the homogeneous form of Eq. (2) is obtained by using q(x)=0

$$\frac{d^4w(x)}{dx^4}-A\frac{d^2w(x)}{dx^2}+Bw(x)=0 \quad A=\frac{k_\theta}{EI},\ B=\frac{k_1}{EI} \qquad (3)$$

By the operator method, let then the characteristic Eq. (3) can be written as:

$$(D^4-AD^2+B)w(x)=0 \quad \frac{d^n}{dx^n}=D^n \qquad (4)$$

Then the roots of the characteristic equation are

$$D_1 = \frac{\sqrt{A+\sqrt{(A^2-4B)}}}{\sqrt{2}}, D_2 = -\frac{\sqrt{A+\sqrt{(A^2-4B)}}}{\sqrt{2}}$$
$$D_3 = \frac{\sqrt{A-\sqrt{(A^2-4B)}}}{\sqrt{2}}, D_4 = -\frac{\sqrt{A-\sqrt{(A^2-4B)}}}{\sqrt{2}} \quad (5)$$

There are three possible combinations of parameters A and B that must be considered to define Eq. (5). The cases are;

$$A < 2\sqrt{B},\ A = 2\sqrt{B}\ \text{and}\ A > 2\sqrt{B} \quad (6)$$

Since the case $A = 2\sqrt{B}$ (or $k_\theta = \sqrt{4k_1 EI}$) is a v ery special one it is not necessary to obtain solution of the equation for this case. It is possible to get an accurate solution by increasing kθ a very small amount that let to use the solution for $A > 2\sqrt{B}$ case. Therefore, solution of the differential equation would be obtained for the other possible cases.

For the case $A < 2\sqrt{B}$ Eq. (5) yields as;

$$D_1 = \frac{\sqrt{A+i\sqrt{(4B-A^2)}}}{\sqrt{2}}, D_2 = \frac{\sqrt{-A-i\sqrt{(4B-A^2)}}}{\sqrt{2}}$$
$$D_3 = \frac{\sqrt{A-i\sqrt{(4B-A^2)}}}{\sqrt{2}}, D_4 = \frac{\sqrt{-A+i\sqrt{(4B-A^2)}}}{\sqrt{2}} \quad (7)$$

by utilizing auxiliary quantities for the first and second parameter as;

$$\lambda = \sqrt[4]{\frac{B}{4}} = \sqrt[4]{\frac{k_1}{4EI}} \quad \text{and} \quad \delta = \frac{A}{4} = \frac{k_\theta}{4EI} \quad (8)$$

then the first root can be expressed in the following way;

$$D_1 = \frac{\sqrt{4\delta + i\sqrt{(16\lambda^4 - 16\delta^4)}}}{\sqrt{2}} = \frac{2\sqrt{\delta + i\sqrt{(\lambda^4-\delta^4)}}}{\sqrt{2}}$$
$$= \sqrt{2}\sqrt{\delta + i(\sqrt{\lambda^2-\delta^2})(\sqrt{\lambda^2+\delta^2})} \quad (9)$$

By defining new quantities to simplify the terms, both α and β have dimension of 1/L. Then substitute the new quantities into Eq. (9), the first root can be written in simplified form as;

$$D_1 = \sqrt{(\alpha^2-\beta^2)+2i\alpha\beta} = \sqrt{(\alpha+i\beta)^2} = \alpha + i\beta \quad (10)$$

where
$$\alpha = \sqrt{\lambda^2+\delta}$$
$$\beta = \sqrt{\lambda^2-\delta} \quad \rightarrow \quad \delta = \frac{\alpha^2-\beta^2}{2}$$

The other roots also can be found by the same procedures. Then the roots are:

$$D_1 = \alpha + i\beta,\ D_2 = -\alpha - i\beta,$$
$$D_3 = \alpha - i\beta,\ D_4 = -\alpha + i\beta \quad (11)$$

Considering the above roots for solution of Eq. (3) is:

$$w(x) = \begin{array}{l} a_1 e^{\alpha x}(cos[\beta x] + sin[\beta x]) + \\ a_2 e^{-\alpha x}(cos[\beta x] - sin[\beta x]) + \\ a_3 e^{\alpha x}(cos[\beta x] - sin[\beta x]) + \\ a_4 e^{-\alpha x}(cos[\beta x] + sin[\beta x]) \end{array} \quad (12)$$

Using hyperbolic functions,

$$e^{\alpha x} = Cosh[\alpha x] + Sinh[\alpha x]$$
$$e^{-\alpha x} = Cosh[\alpha x] - Sinh[\alpha x] \quad (13)$$

Substituting the above hyperbolic functions and rearrange Eq. (12) with defining the new constants, the closed form of the solution in terms of hyperbolic and trigonometric functions is obtained as:

$$w(x) = \begin{array}{l} c_1\, cos[\beta x] cosh[\alpha x] + \\ c_2\, cos[\beta x] sinh[\alpha x] + \\ c_3\, sin[\beta x] cosh[\alpha x] + \\ c_4\, sin[\beta x] sinh[\alpha x] + \end{array} \quad (7) \quad (14)$$

By neglecting foundation effects for torsional degree of freedoms, a linear description of the angular displacement at any point along the element can be expressed as Ø(x)= a1+a2x. Inserting the angular displacements due to torsional effects, Eq. (14) can be rearranged as follows:

$$w(x) = \begin{array}{l} c_1 + \\ c_2\, cos[\beta x] cosh[\alpha x] + \\ c_3\, cos[\beta x] sinh[\alpha x] + \\ c_4 + \\ c_5\, sin[\beta x] cosh[\alpha x] + \\ c_6\, sin[\beta x] sinh[\alpha x] + \end{array} \quad (9) \quad (15)$$

then, the closed form equation can be expressed in matrix form as:

$$w = \underline{B}^T \underline{C} \quad (16)$$

The generalized displacement vector which forms boundary conditions shown in Fig. 3 is obtained with x= 0 and x= L. From the figure:

$$\{\underline{d}\}^T = \{\phi_1, \theta_1, w_1, \phi_2, \theta_2, w_2\}$$
$$\{\underline{F}\}^T = \{T_1, M_1, V_1, T_2, M_2, V_2\} \quad (17)$$

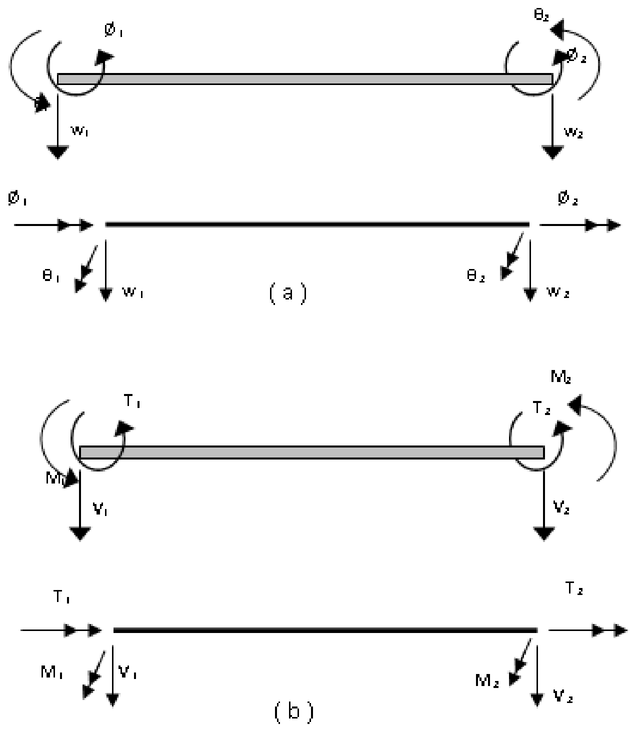

Fig. 3 a finite element of a beam (a) generalized displacements (b) loads applied to nodes

Then the arbitrary constant elements of the vector C can be related to the end displacements in matrix form as follows;

$$[\underline{d}]=[\underline{H}]\cdot[\underline{C}] \text{ or } [\underline{C}]=[\underline{H}]^{-1}\cdot[\underline{d}] \tag{18}$$

where [H] is a 6x6 ,Substitute Eq. (18) into Eq. (16) then the closed form solution of the differential equation can be written in matrix form as:

$$[\underline{w}]=[\underline{B}]^{T}\cdot[\underline{H}]^{-1}\cdot[\underline{d}] \tag{19}$$

Eq. (19) can be redefined by introducing vector N that includes six shape functions. Then the closed form of the solution in terms of shape functions and the generalized displacements defined in Fig. 3 is:

$$[\underline{w}]=[\underline{N}]\cdot\begin{Bmatrix}\phi(x=0)\\ \dfrac{dw}{dx}(x=0)\\ w(x=0)\\ \phi(x=L)\\ \dfrac{dw}{dx}(x=L)\\ w(x=L)\end{Bmatrix} \quad where\ [\underline{N}]=[\underline{B}]^{T}\cdot[\underline{H}]^{-1} \tag{20}$$

Finally for $A<2\sqrt{B}$ after the necessary evaluations the shape functions determined and can be represented as follows;

Shape functions for $A<2\sqrt{B}$

$$\psi_1=\left(1-\frac{x}{L}\right)$$

$$\psi_2=\left(\frac{\beta\cosh[\alpha x]\sin[\beta x]-\beta\sin[\beta x]\cosh[\alpha(2L-x)]-\alpha\sinh[\alpha x]\cos[\beta(2L-x)]+\alpha\sinh[\alpha x]\cos[\beta x]}{\left(\alpha^2+\beta^2-\alpha^2\cos[2\beta L]-\beta^2\cosh[2\alpha L]\right)}\right)$$

$$\psi_3=\left(\frac{\alpha^2\cos[\beta x]\cosh[\alpha x]+\beta^2\cos[\beta x]\cosh[\alpha x]-\beta^2\cos[\beta x]\cosh[\alpha(2L-x)]-\alpha^2\cosh[\alpha x]\cos[\beta(2L-x)]-\alpha\beta\sin[\beta x]\sinh[\alpha(2L-x)]+\alpha\beta\sinh[\alpha x]\sin[\beta(2L-x)]}{\left(\alpha^2+\beta^2-\alpha^2\cos[2\beta L]-\beta^2\cosh[2\alpha L]\right)}\right)$$

$$\psi_4=\left(\frac{x}{L}\right)$$

$$\psi_5=\left(\frac{\beta\cosh[\alpha(L+x)]\sin[\beta(L-x)]-\beta\cosh[\alpha(L-x)]\sin[\beta(L-x)]-\alpha\sinh[\alpha(L-x)]\cos[\beta(L-x)]+\alpha\sinh[\alpha(L-x)]\cos[\beta(L+x)]}{\left(\alpha^2+\beta^2-\alpha^2\cos[2\beta L]-\beta^2\cosh[2\alpha L]\right)}\right)$$

$$\psi_6=\left(\frac{\alpha^2\cosh[\alpha(L-x)]\cos[\beta(L-x)]+\beta^2\cos[\beta(L-x)]\cosh[\alpha(L-x)]-\alpha^2\cosh[\alpha(L-x)]\cos[\beta(L+x)]-\beta^2\cos[\beta(L-x)]\cosh[\alpha(L+x)]+\alpha\beta\sin[\beta(L+x)]\sinh[\alpha(L-x)]-\alpha\beta\sinh[\alpha(L+x)]\sin[\beta(L-x)]}{\left(\alpha^2+\beta^2-\alpha^2\cos[2\beta L]-\beta^2\cosh[2\alpha L]\right)}\right)$$

On the other hand for the case $A>2\sqrt{B}$ it is noted that the roots of Eq. (5) are definite. Therefore, by substituting the auxiliary parameters defined in Eq. (8) into Eq. (5) the first root can be expressed. Similarly using the same procedures and after the necessary evaluations as previously done the shape functions for this case be found and illustrated as follows;

Shape functions for $A>2\sqrt{B}$

$$\psi_1=\left(1-\frac{x}{L}\right)$$

$$\psi_2 = \left(\frac{\begin{gathered}\alpha \sinh[(\alpha-\beta)x] + \beta \sinh[(\alpha-\beta)x] + \alpha \sinh[(\alpha+\beta)x] - \\ \beta \sinh[(\alpha+\beta)x] - \beta \sinh[2\alpha L - \alpha x - \beta x] + \alpha \sinh[2\beta L - \alpha x - \beta x] - \\ \alpha \sinh[2\beta L + \alpha x - \beta x] + \beta \sinh[2\alpha L - \alpha x + \beta x]\end{gathered}}{2\left(\alpha^2 - \beta^2 + \beta^2 \cosh[2\alpha L] - \alpha^2 \cosh[2\beta L]\right)} \right)$$

$$\psi_3 = \left(-\cosh[\alpha x]\cosh[\beta x] - \frac{\begin{gathered}(\alpha \sinh[2\beta L] + \beta \sinh[2\alpha L]) \\ (\alpha \cosh[\alpha x]\sinh[\beta x] - \beta \cosh[\beta x]\sinh[\alpha x]) + \\ \alpha\beta \sinh[\alpha x]\sinh[\beta x](\cosh[2\beta L] - \cosh[2\alpha L])\end{gathered}}{\left(\alpha^2 - \beta^2 + \beta^2 \cosh[2\alpha L] - \alpha^2 \cosh[2\beta L]\right)} \right)$$

$$\psi_4 = \frac{x}{L}$$

$$\psi_5 = \left(\frac{\begin{gathered}-\alpha \sinh[(\alpha+\beta)(L-x)] + \beta \sinh[(\alpha+\beta)(L-x)] + \\ \alpha \sinh[\alpha L - \beta L - \alpha x - \beta x] - \beta \sinh[\alpha L + \beta L + \alpha x - \beta x] - \\ \alpha \sinh[\alpha L - \beta L - \alpha x + \beta x] - \beta \sinh[\alpha L - \beta L - \alpha x + \beta x] + \\ \alpha \sinh[\alpha L + \beta L - \alpha x + \beta x] + \beta \sinh[\alpha L - \beta L + \alpha x + \beta x]\end{gathered}}{2\left(\alpha^2 - \beta^2 + \beta^2 \cosh[2\alpha L] - \alpha^2 \cosh[2\beta L]\right)} \right)$$

$$\psi_6 = \left(\frac{\begin{gathered}2\cosh[\beta x]\left(\beta^2 \cosh[\beta L]\sinh[\alpha L] + \alpha\beta \cosh[\alpha L]\sinh[\beta L]\sinh[\alpha x]\right) - \\ 2\cosh[\alpha x]\left(\alpha\beta \cosh[\beta L]\sinh[\alpha L] + \alpha^2 \cosh[\alpha L]\sinh[\beta L]\sinh[\beta x]\right) + \\ 2(\alpha^2 - \beta^2)(\cosh[\alpha L]\sinh[\beta L]\sinh[\alpha x]\sinh[\beta x])\end{gathered}}{\left(\alpha^2 - \beta^2 + \beta^2 \cosh[2\alpha L] - \alpha^2 \cosh[2\beta L]\right)} \right)$$

For both $A < 2\sqrt{B}$ and $A > 2\sqrt{B}$ cases, when foundation parameter k1 and kθ tends to zero (dependently λ→0, δ→0, α→0, β→0), the terms of shape functions reduces to Hermitian functions as expected.

$$\underset{\alpha \to 0}{Lim}\,\underset{\beta \to 0}{Lim}\,\psi_2 \to x\left(1 - \frac{x}{L}\right)^2$$

$$\underset{\alpha \to 0}{Lim}\,\underset{\beta \to 0}{Lim}(\psi_3) \to 3\left(\frac{x}{L}\right)^2 - 2\left(\frac{x}{L}\right)^3 - 1$$

$$\underset{\alpha \to 0}{Lim}\,\underset{\beta \to 0}{Lim}(\psi_5) \to x\left[\left(\frac{x}{L}\right)^2 - \left(\frac{x}{L}\right)\right]$$

$$\underset{\alpha \to 0}{Lim}\,\underset{\beta \to 0}{Lim}(\psi_6) \to 2\left(\frac{x}{L}\right)^3 - 3\left(\frac{x}{L}\right)^2 \tag{21a}$$

the non-dimensional forms of the shape functions as Hermitian polynomials for $\xi = \dfrac{x}{L}$ can be formed

$$\frac{\psi_2}{L} = \xi - 2\xi^2 + \xi^3$$

$$\psi_3 = 3\xi^2 - 2\xi^3 - 1$$

$$\frac{\psi_5}{L} = \xi^3 + \xi^2$$

$$\psi_6 = 2\xi^3 - 3\xi^2 \tag{21b}$$

For observing the influence of the foundation parameters, it is necessary to compare the expressions in Equations For both $A < 2\sqrt{B}$ and $A > 2\sqrt{B}$ cases of shape functions with the Hermitian polynomials in Eq. (21). For clarifying the comparison let the parameters be;

For $A < 2\sqrt{B}$

$$\alpha = \sqrt{\lambda^2 + \delta} = \lambda\sqrt{1+t}$$

$$\beta = \sqrt{\lambda^2 - \delta} = \lambda\sqrt{1-t}$$

and For $A > 2\sqrt{B}$

$$\alpha = \sqrt{\lambda^2 + \delta} = \lambda\sqrt{1+t}$$

$$\beta = \sqrt{\delta - \lambda^2} = \lambda\sqrt{t-1}$$

where t is dimensionless as;

$$t = \frac{\delta}{\lambda^2} = \frac{\dfrac{k_\theta}{4EI}}{\sqrt{\dfrac{k_1}{4EI}}}$$

Then the effect of the foundation parameters k1 and kθ on the shape function terms given for both $A < 2\sqrt{B}$ and $A > 2\sqrt{B}$ cases with corresponding terms of Hermitian polynomials are shown in Figs. 4 – 7. From the figures it can be noted that the shape functions related to beams on elastic foundations are very sensitive to variation of foundation parameters after some limits. On the other hand it is obvious that shape functions converge towards Hermitian polynomials when the parameter λL becomes smaller.

3.2 Derivation of Element Stiffness Matrix

The element stiffness matrix of a b eam element, which relates the nodal forces to the nodal displacements resting on two-parameter elastic foundation, can be obtained by the same procedures. As a s ummary, the stiffness matrix [Ke], for the prismatic beam element shown in Fig. 2. can be obtained from the minimization of strain energy functional U as follows:

$$\left[\underline{K_e}\right] = \frac{\partial U}{\partial\{\underline{d}\}} \tag{22}$$

$$\text{where} \quad U = \frac{EI}{2}\int_0^L \frac{d^2w(x)}{dx^2}\frac{d^2w(x)}{dx^2}dx + \frac{k_1}{2}\int_0^L w(x)w(x)dx - \frac{k_\theta}{2}\int_0^L \frac{dw(x)}{dx}\frac{dw(x)}{dx}dx$$

Substituting w(x) and its derivatives from the shape functions into Eq. (22), the stiffness matrix can be written in the following form;

$$\left[\underline{K_e}\right] = EI\int_0^L \left\{\frac{d^2\{N\}}{dx^2}\right\}^T \left\{\frac{d^2\{N\}}{dx^2}\right\}dx + k_1\int_0^L \{N\}^T\{N\}dx - k_\theta\int_0^L \left\{\frac{d\{N\}}{dx}\right\}^T \left\{\frac{d\{N\})}{dx}\right\}dx \tag{23}$$

where N is, a 6x1 matrix of the exact shape functions, given in Appendix. Their first and second derivatives in matrix forms are;

$$\left\{\frac{d\{N\}}{dx}\right\} = \left[\underline{H}^{-1}\right]^T \left\{\frac{d\{B\}}{dx}\right\}$$
$$\left\{\frac{d^2\{N\}}{dx^2}\right\} = \left[\underline{H}^{-1}\right]^T \left\{\frac{d^2\{B\}}{dx^2}\right\} \tag{24}$$

Substituting N from the shape functions and its derivatives, using Eq. (24), into Eq. (23) and carrying out the necessary integrals and procedures the stiffness terms are obtained. For the cases $A < 2\sqrt{B}$ and $A > 2\sqrt{B}$ the terms are defined as;

The stiffness terms for $A < 2\sqrt{B}$

$$k_{22} = \frac{2EI\alpha\beta(\beta\,sinh[2\alpha L] - \alpha\,sin[2\beta L])}{\left(-\alpha^2 - \beta^2 + \alpha^2 cos[2\beta L] + \beta^2 cosh[2\alpha L]\right)}$$
$$k_{23} = \frac{EI(\alpha^2+\beta^2)\left(\beta^2 - \alpha^2 + \alpha^2 cos[2\beta L] - \beta^2 cosh[2\alpha L]\right)}{\left(-\alpha^2 - \beta^2 + \alpha^2 cos[2\beta L] + \beta^2 cosh[2\alpha L]\right)}$$
$$k_{25} = \frac{-4EI\alpha\beta^2(-\alpha\,cosh[\alpha L]sin[\beta L] + \beta\,cos[\beta L]sinh[\alpha L])}{\beta\left(-\alpha^2 - \beta^2 + \alpha^2 cos[2\beta L] + \beta^2 cosh[2\alpha L]\right)}$$
$$k_{26} = \frac{4EI\alpha\beta\left(\alpha^2+\beta^2\right)\left(sin[\beta L]sinh[\alpha L]\right)}{\left(-\alpha^2 - \beta^2 + \alpha^2 cos[2\beta L] + \beta^2 cosh[2\alpha L]\right)}$$
$$k_{33} = \frac{2EI\alpha\beta\left(\alpha^2+\beta^2\right)\left(\alpha\,sin[2\beta L] + \beta\,sinh[2\alpha L]\right)}{\left(-\alpha^2 - \beta^2 + \alpha^2 cos[2\beta L] + \beta^2 cosh[2\alpha L]\right)}$$
$$k_{36} = -\frac{4EI\beta\alpha\left(\alpha^2+\beta^2\right)\left(\alpha\,cosh[\alpha L]sin[\beta L] + \beta\,cos[\beta L]sinh[\alpha L]\right)}{\left(-\alpha^2 - \beta^2 + \alpha^2 cos[2\beta L] + \beta^2 cosh[2\alpha L]\right)}$$

The stiffness terms for $A > 2\sqrt{B}$

$$k_{22} = \frac{2EI\beta\alpha\left(\beta\,sinh[\alpha L]cosh[\alpha L] - \alpha\,sinh[\beta L]cosh[\beta L]\right)}{\left(-\alpha^2 sinh\left[(\beta L)^2\right] + \beta^2 sinh\left[(\alpha L)^2\right]\right)}$$
$$k_{23} = EI\left[\left(\alpha^2+\beta^2\right) + \frac{\left(\alpha^2\beta^2\left(cosh(\alpha L)^2 sinh(\beta L)^2 - cosh(\beta L)^2 sinh(\alpha L)^2\right)\right)}{\left(-\alpha^2 sinh\left[(\beta L)^2\right] + \beta^2 sinh\left[(\alpha L)^2\right]\right)}\right]$$
$$k_{25} = \frac{2EI\beta\alpha\left(\alpha\,sinh[\beta L]cosh[\alpha L] - \beta\,sinh[\alpha L]cosh[\beta L]\right)}{\left(-\alpha^2 sinh\left[(\beta L)^2\right] + \beta^2 sinh\left[(\alpha L)^2\right]\right)}$$
$$k_{26} = \frac{2EI\beta\alpha\left(\alpha^2-\beta^2\right)sinh[\alpha L]sinh[\beta L]}{\left(-\alpha^2 sinh\left[(\beta L)^2\right] + \beta^2 sinh\left[(\alpha L)^2\right]\right)}$$
$$k_{33} = \frac{EI\beta\alpha\left(\alpha^2-\beta^2\right)\left(\alpha\,sinh[2\beta L] + \beta\,sinh[2\alpha L]\right)}{\left(-\alpha^2 sinh\left[(\beta L)^2\right] + \beta^2 sinh\left[(\alpha L)^2\right]\right)}$$
$$k_{36} = \frac{-2EI\beta\alpha\left(\alpha^2-\beta^2\right)\left(\alpha\,cosh[\alpha L]sinh[\beta L] + \beta\,cosh[\beta L]sinh[\alpha L]\right]}{\left(-\alpha^2 sinh\left[(\beta L)^2\right] + \beta^2 sinh\left[(\alpha L)^2\right]\right)}$$

The other terms of the stiffness matrix are;

$$k_{11} = k_{44} = -k_{14} = -k_{41} = \frac{GJ}{L}$$
$$k_{12} = k_{13} = k_{15} = k_{16} = k_{42} = k_{43} = k_{45} = k_{46} = 0$$
$$k_{32} = k_{35} = k_{53} = k_{23}, \quad k_{56} = k_{65} = k_{62} = k_{26}$$
$$k_{52} = k_{25}, \quad k_{22} = k_{55}, \quad k_{63} = k_{36}, \quad k_{66} = k_{33}$$

These terms tends to be the conventional stiffness terms when foundation parameter k1 and kθ tends to zero (λ→0 and δ→0 or α→0 and β→0). They are verified as expected for the both cases as follows;

$$\lim_{\alpha\to 0}\lim_{\beta\to 0} k_{22} \to \frac{4EI}{L}$$
$$\lim_{\alpha\to 0}\lim_{\beta\to 0} k_{23} \to -\frac{6EI}{L^2}$$
$$\lim_{\alpha\to 0}\lim_{\beta\to 0} k_{25} \to \frac{2EI}{L}$$
$$\lim_{\alpha\to 0}\lim_{\beta\to 0} k_{26} \to \frac{6EI}{L^2}$$
$$\lim_{\alpha\to 0}\lim_{\beta\to 0} k_{33} \to \frac{12EI}{L^3}$$
$$\lim_{\alpha\to 0}\lim_{\beta\to 0} k_{36} \to -\frac{12EI}{L^3} \tag{25}$$

IV. REPRESENTATION OF CONTINUOUS SURFACE BY GRILLAGES OF BEAMS

As Wilson [26] has indicated the structural behaviour of a beam resembles that of a s trip in a p late, so replacing a continuous surface by an idealized discrete system can represent a t wo-dimensional plate. The differential equation requires that the bond between the foundation and the plate be accounted for the soffit of the "equivalent" strip is not affected by the foundation in twisting. The torsion constant for the rectangular beam strips is adopted by Bowles [27]. The representation of a plate through the grid work (or lattice) analogy at which the discrete elements are connected at finite nodal points is shown in Fig. 1. The plate through the lattice analogy at which the discrete elements are connected at finite nodal points can be represented by one dimensional beam elements. The plate is modelled as an assemblage of individual beam elements interconnected at their neighbouring joints. In gridwork systems at edge nodes two or three, at interior nodes four of the typical discrete individual beam elements shown in Fig. 4 are intersected. The beam element has 3 Degree of Freedom at each node. That is, element node DOF's at i are two rotations, 1 and 2, and one translation, 3, at j they are similarly 4 and 5 for rotations and 6 for translation. The replacement implies that there are rigid intersection joints between all sets of beam elements, ensuring slope and rotation continuity. Because of plane rigid intersection, the elements can resist torsion as well as bending moment and shear.

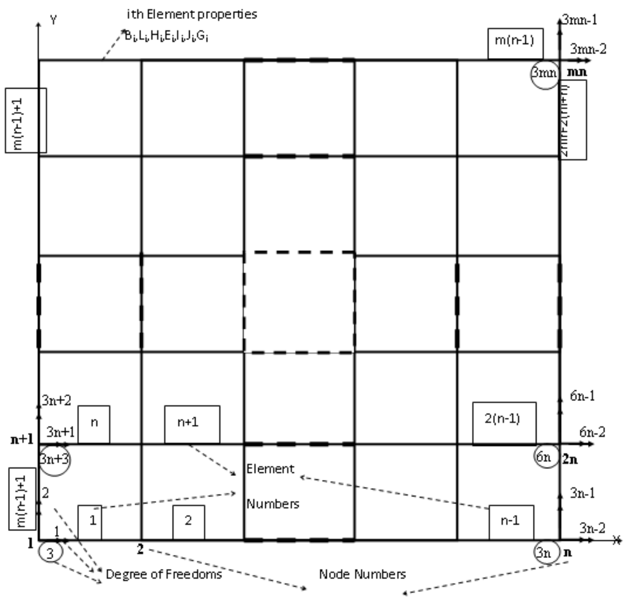

Fig. 4. Typical numbering of nodes, DOF's and elements of a rectangular plate gridding for free edges boundary conditions

Matrix displacement method based on stiffness-matrix approach is a useful tool to solve gridworks with arbitrary load and boundary conditions. The solution can be obtained by using a proper numbering shame to collect all displacements for each nodal point in a co nvenient sequence the stiffness matrix of the system.

The SI unit for magnetic field strength *H* is A/m. However, if you wish to use units of T, either refer to magnetic flux density *B* or magnetic field strength symbolized as $\mu_0 H$. Use the center dot to separate compound units, e.g., "A·m^2."

V. Results and Discussions

Some examples of plates on elastic foundation solved to check the validity of the solution technique by the finite grid solution (FGM). Comparison with known analytical and other numerical solutions yields accurate results as an approximate method.

The first example is to analyze the plane-grid system solved by Wang [28] shown in Fig. 5. The system is a monolithic reinforced-concrete simple supported on four columns at A, B, C and D. the values of flexural and torsional rigidities for all elements are EI=288000 kip-ft2 and GJ=79142.4 kip-ft2 respectively. Two loading conditions are to be investigated: firstly a 10-kip concentrated load applied at H and (LC1) and than a uniform load of 3 klf on the element BF (LC2).

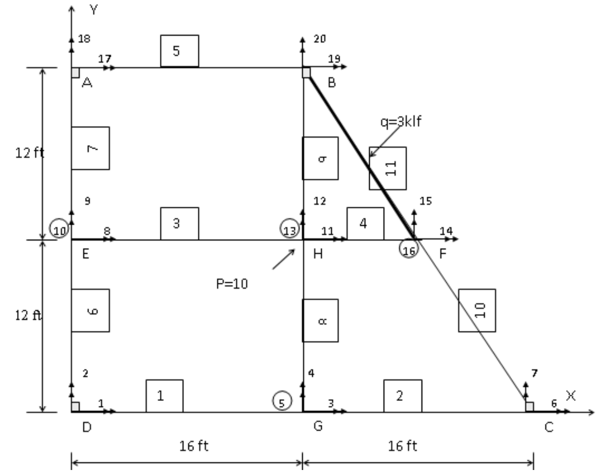

Fig. 5. The reference grid system [28].

There are 8 nodes, 11 elements and 20 degrees of freedoms in the Figure. The element internal bending and torsional forces and the displacements values of the reference for both loading conditions are compared with the Finite Grid Solution and they are tabulated in Tables I and II respectively.

Table I The Comparison of the End Forces with the Reference, [28].

COMPARISON OF INTERNAL FORCES

El. No		LC1			LC2		
		L. End M	R. End M	Tors. M	L. End M	R. End M	Tors. M
1	Ref.	0.2245	20.1033	0.4221	0.6877	26.6887	3.5376
	FGM	0.22432	20.1	0.42159	0.68671	26.67	3.5349
2	Ref.	-19.7902	0.753	-0.1456	-33.3826	-0.4439	6.6647
	FGM	-19.785	0.75496	-0.14548	-33.357	-0.43532	6.6642
3	Ref.	2.761	23.1228	-1.2437	6.0058	-8.286	-0.8249
	FGM	2.7607	23.128	-1.2435	6.0043	-8.2632	-0.82376
4	Ref.	-23.4002	0.3811	0.2419	16.4602	24.026	10.2151
	FGM	-23.408	0.38659	0.24008	16.424	24	10.206
5	Ref.	-2.9855	1.2746	-2.528	-6.6936	-49.4419	-9.9166
	FGM	-2.985	1.2754	-2.522	-6.691	-49.436	-9.9123
6	Ref.	-0.4221	11.8008	0.22432	-3.5376	6.2845	0.68671
	FGM	-0.42159	11.802	0.22432	-3.5349	6.2884	0.68671
7	Ref.	-10.557	2.5228	2.985	-5.4596	9.9166	6.691
	FGM	-10.558	2.522	2.985	-5.4646	9.9123	6.691
8	Ref.	0.5677	28.9562	0.31442	-3.1271	49.0296	-6.6871
	FGM	0.56707	28.948	0.31442	-3.1293	48.992	-6.6871
9	Ref.	-30.4418	-4.9497	0.034322	-60.0696	43.5324	1.4741
	FGM	-30.432	-4.9445	0.034322	-60.022	43.554	1.4741
10	Ref.	0.2966	22.6675	0.70886	5.2992	92.2964	-4.0588
	FGM	0.29773	22.663	0.70886	5.3034	92.266	-4.0588
11	Ref.	-22.6776	2.7461	0.25402	-174.4697	32.9234	10.249
	FGM	-22.678	2.7422	0.25402	-174.46	32.909	10.249

Table II The Comparison of the Displacements with the Reference [28].

COMPARISON OF THE DISPLACEMENTS				
DOF	LC1		LC2	
NO:	Reference	FGM	Reference	FGM
1	-2.5330E-04	-2.5331E-04	-2.2105E-04	-2.2107E-04
2	5.5381E-04	5.5371E-04	7.8536E-04	7.8485E-04
3	-3.3864E-04	-3.3854E-04	-9.3624E-04	-9.3573E-04
4	1.6270E-06	1.6224E-06	6.3109E-05	6.3081E-05
5	5.9493E-03	5.9481E-03	8.8156E-03	8.8099E-03
6	-3.0922E-04	-3.0913E-04	-2.2836E-03	-2.2830E-03
7	-5.6902E-04	-5.6894E-04	-8.5186E-04	-8.5142E-04
8	1.3430E-06	1.3421E-06	-1.6420E-05	-1.6423E-05
9	5.1977E-04	5.1970E-04	6.8108E-04	6.8072E-04
10	1.9859E-03	1.9860E-03	1.5393E-03	1.5397E-03
11	2.5279E-04	2.5274E-04	1.5035E-04	1.5012E-04
12	-4.5839E-05	-4.6053E-05	1.0781E-03	1.0770E-03
13	7.6946E-03	7.6928E-03	1.5444E-02	1.5435E-02
14	2.2834E-04	2.2847E-04	-8.8223E-04	-8.8162E-04
15	-3.6555E-04	-3.6579E-04	9.7300E-04	9.7183E-04
16	5.6087E-03	5.6048E-03	2.4398E-02	2.4379E-02
17	2.7384E-04	2.7385E-04	3.0392E-04	3.0393E-04
18	6.7089E-05	6.7087E-05	-3.3384E-04	-3.3383E-04
19	7.8387E-04	7.8373E-04	2.3087E-03	2.3079E-03
20	-5.1258E-05	-5.1258E-05	8.5352E-04	8.5352E-04

From the Tables I and II apart from errors associated with rounding the input numbers, which are less than 0.05%, the results obtained are almost the same as the reference values. The results those can be accepted as exact are valuable for checking the correctness of the method.

However in order to check the validity of the method for plates resting elastic foundation, an example of a simply supported square plate subjected to a uniformly distributed load considered [29]. In the reference the side length a, the flexural rigidity D and Poisson ration υ were chosen as 8 m, 1000 Nm and 0.3 respectively. The uniformly distributed load q was taken as 1N/mm2. Firstly for the simple supported case, Winkler and two-parameter foundations considered. The comparison of the FGM results with the Local Boundary Integral Equation method (LBIE) on the centerline of the plate for three different Winkler coefficients is given in Table III. From the table one can see that the maximum relative error for deflections of points located on the axis passing through the centre of the plate is about less than 1%. This reflects a high degree of accuracy.

Table III The comparison of the deflections at the centerline for a simply supported plate resting on a Winkler foundation with the LBIE [29].

Distance (m)	k1=100 N/m3 LBIE (mm)	FGM (mm)	Relative Error %	k1=300 N/m3 LBIE (mm)	FGM (mm)	Relative Error %	k1=500 N/m3 LBIE (mm)	FGM (mm)	Relative Error %
0	7.925	7.933	0.25	3.751	3.719	0.84	2.399	2.373	1.08
0.8	7.596	7.614	0.12	3.622	3.596	0.73	2.331	2.309	0.94
1.6	6.604	6.62	0.28	3.211	3.202	0.30	2.103	2.092	0.50
2.4	4.95	4.99	0.80	2.472	2.481	0.36	1.657	1.66	0.19
3.2	2.683	2.727	1.64	1.376	1.392	1.14	0.944	0.952	0.89
4	0	0	-	0	0	-	0	0	-

On the other hand for two parameter foundation case, the numerical results of the maximum deflection w_{max} are given in Table IV. The relative error is also less than 1 % as for the one-parameter foundation. Then it can be concluded that the accuracy is high and comparable with the one valid for Winkler model.

Table 4. The comparison of the maximum deflections for a simply supported plate resting on a two-parameter foundation with the LBIE,[29].

coefficients k1 (N/m3)	k2 (N/m)	LBIE (mm)	FGM (mm)	Relative Error %
100	100	6.8147	6.7913	0.34
300	300	3.0276	3.0034	0.80
500	500	1.911	1.8941	0.88

For all of the three cases, comparison with the other solution methods, with less than 1% error, yields accurate results as an approximate numerical method.

VI. Conclusion

For particular plate problems, closed form solutions have been obtained. However, even for conventional plate analysis these solutions can only be applied to the problems with simple geometry, load and boundary conditions. For plates supported by the two-parameter elastic foundations the solution is usually much too complex and there is apparently no analytical solution other than for simple cases. A grid work analogy called the Finite Grid Method involving discretized plate properties mapped onto equivalent beams with adjusted parameters and matrix displacement analysis are used to develop a more general simplified numerical approach for plates on elastic foundations. In this solution the plate is modeled as an assemblage of individual beam elements interconnected at joints. The solution method of this technique is acceptable as a correct treatment from the point of view of use the strain energy functions.

In this method the plate is modeled as an assemblage of individual beam elements interconnected at their neighboring joints. By this representation, also the plate problems which have non-uniform thickness and foundation properties, arbitrary boundary and loading conditions and discontinuous surfaces, can be solved in a general form. It has been verified the validity of the solution with a broad range of applications. of plates on either one or two parameter elastic foundation.

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