Simulation of nonlinear physical processes with the generalized phenomenological equation

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***Abstract*—**Using a phenomenologically constructed Lagrangian function, a nonlinear partial differential equation is obtained, which describes the space – time distribution of a physical scalar parameter $n(\vec{r},t)$, which can be density, concentration, temperature, etc. For the one-dimensional case phase trajectories in three dimensions are obtained depending on the numerical parameter of the problem, designated as x_3.

Keywords—Dissipative processes, Lagrangian function, phase trajectory, phase portrait, nonlinear dynamics.

I. INTRODUCTION

THEORETICAL description of any physical process is associated first of all with the set up of the corresponding physical problem. However, in those cases when it comes to equations of mathematical physics [1 - 15], the main tool is a method for building a particular Lagrange's function, which is based on the well-known rule of finding the difference between the densities of the kinetic and potential energies. Further, by using the action functional, almost all the basic equations of mathematical physics are obtained, from the Maxwell equations to the Poisson and Laplace equations. The exception here are the equations of parabolic type, which include only the dissipative equations as heat conduction and diffusion equations, and in the special case also the Navier - Stokes equation. To obtain parabolic equations class, artificial method is often used in theoretical physics. It is based on building certain functional S, and them using the phenomenological approach

$$\frac{\partial n}{\partial t}=\gamma\frac{\delta S}{\delta n}, \qquad (1)$$

(where γ is a certain constant value providing correct equation dimension (1), and symbol $\frac{\delta}{\delta}$ means drawing of the functional derivative from functional S) the unknown equations are found.

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II. PROBLEM FORMULATIONN

That is why here the main aim is always to build quite specific invariant functional $S\{n(\vec{r},t)\}$. The problem, which is solved in this paper, is general in nature and reduced to building just the most general form of functional $S\{n(\vec{r},t)\}$. The only requirement which is applied to subintegral function $L\left(t,\vec{r},\{n\},\{\dot{n}\},\{\ddot{n}\},...,\frac{\partial\{n\}}{\partial x_i},\frac{\partial^2\{n\}}{\partial x_i\partial x_k},...\right)$ (where $\{n\}=\{n_\alpha\}=\left(n_1(\vec{r},t),n_2(\vec{r},t),...n_p(\vec{r},t)\right)-$, a certain generalized physical parameter (concentration, temperature, density, hydrodynamic flow velocity components etc.), $t-$ time, $p-$ quantity of unknown functions, $\alpha=1,2,...,p$, indices $i,k,...=1,2,3$ refer to three Cartesian coordinates x,y,z) is its invariance with respect to the replacement operations $t\to -t,\ \vec{r}\to-\vec{r}$.

III. PROBLEM SOLUTION

As a result for the case of two independent functions n_0,n_1 it can be written that

$$\begin{aligned}&L\left(t,\vec{r},\{n\},\{\dot{n}\},\{\ddot{n}\},...,\frac{\partial\{n\}}{\partial x_i},\frac{\partial^2\{n\}}{\partial x_i\partial x_k},...\right)=\\&=L(n_0,n_1)=\frac{\alpha}{2}\left(\dot{n}_0^2+\dot{n}_1^2\right)-K_0n_0n_1-\\&[\frac{K_1}{3}\left(n_0^3+n_1^3\right)++K_2\left(n_0n_1^2+n_1n_0^2\right)]+\\&+\frac{K_3}{2}\left(n_0^2+n_1^2\right)+\frac{\gamma}{2}\left[(\nabla n_0)^2+(\nabla n_1)^2\right]- \qquad (2)\\&-\gamma_{12}\nabla n_0\nabla n_1-\frac{\vec{b}}{2}\left(n_1^2\nabla n_0+n_0^2\nabla n_1\right)+\\&+\frac{\vec{A}}{3}\left[(\nabla n_0)^3+(\nabla n_1)^3\right]-...\end{aligned}$$

where all the presented here coefficients $\alpha,\beta,\gamma,\gamma_{12},l,\vec{b},\vec{A},K_{1,2,3}$ have the relevant dimension, which for the right part of expression (2) by its nature should define the energy density, simply to say the pressure, and $K_1 = \frac{\beta}{D}$, D - the diffusion coefficient, $K_3 = 3K_1$. It also should be noticed that constant values K_0, K_1, K_2 и K_3 will have the meaning of chemical reaction speeds if n_0, n_1 mean, for example, concentration of the reactive substances in the chemical reaction. In an abstract way, they are just a few functions. Since the action may be specified as (see. references [16-18] as example)

$$S = \int_V \int_{t_0}^{t_1} L(n_0, n_1) dt d^3 x, \qquad (3)$$

then by insertion of (2) into (3) and them into(1), and by making simple calculations of the variation derivatives by the relevant concentrations n_α, the following system of two nonlinear differential equations in partial derivatives can be obtained

$$\dot{n}_0 = \lambda[K_0 n_1 - K_3 n_0 - \alpha \ddot{n}_0 - \vec{b}(n_1 - n_0)\nabla n_1 + \\ +\gamma \Delta n_0 + \vec{A}\nabla\Delta n_0 + K_1 n_0^2 + 2K_2 n_0 n_1 + K_2 n_1^2] \qquad (4)$$

$$\dot{n}_1 = \lambda[K_0 n_0 - K_3 n_1 - \alpha \ddot{n}_1 + \vec{b}(n_1 - n_0)\nabla n_0 + \\ +\gamma \Delta n_1 + \vec{A}\nabla\Delta n_1 + K_1 n_1^2 + 2K_2 n_0 n_1 + K_2 n_0^2] \qquad (5)$$

For the purpose of specificity, we will understand n_0, n_1 as concentrations of the reacting substances in the chemical reaction. However, it should be emphasized that all the following consideration is general.

Well in view of the just said we introduce the following designation for diffusion coefficient $D = \gamma\lambda$, and consider the following special cases

1. If $\alpha = \vec{b} = \vec{A} = K_0 = K_1 = K_2 = K_3 = 0$

as a result we will obtain the regular equation of diffusion

$$\frac{\partial n_0}{\partial t} = D\Delta n_0 . \qquad (6)$$

2.And if $\frac{1}{\lambda} = \vec{b} = \gamma = \vec{A} = K_0 = K_1 = K_2 = 0$

the equation describing the Belousov-Zhabotinskii reaction is obtained.

$$\ddot{n}_0 + \omega_0^2 n_0 = 0, \qquad (7)$$

where frequency is $\omega_0 = \sqrt{\frac{K_3}{\alpha}}$.

3. $\frac{1}{\lambda} = \vec{b} = \vec{A} = K_0 = K_1 = K_2 = K_3 = 0$.

As a result the equation of acoustic vibrations (concentration)

$$\frac{\partial^2 n_0}{\partial t^2} = c_s^2 \Delta n_0, \qquad (8)$$

where sound velocity is $c_s = \sqrt{\frac{\gamma}{\alpha}}$.

4. $\alpha = \gamma = K_0 = K_1 = K_2 = K_3 = 0$.

From equation (4) it appears

$$\dot{n}_0 = -\lambda\vec{b}(n_1 - n_0)\nabla n_1 + \lambda\vec{A}\nabla\Delta n_0 . \qquad (9)$$

Shall the number of particles in the solution maintains, i.e. $n_0 + n_1 = \bar{n} = const$, and the constant vectors $\vec{b}$ and $\vec{A}$ have one non-zero component $\vec{b} = (b,0,0)$, $\vec{A} = (A,0,0)$, each we can find from here $\dot{n}_0 = \lambda b(\bar{n} - 2n_0)n_0' + \lambda A n_0'''$, where the dashes mean differentiation by coordinate x. On the assumption that $b < 0$ and $A > 0$, we obtain $\dot{n}_0 + \lambda|b|(\bar{n} - 2n_0)n_0' - \lambda A n_0''' = 0$. If it is necessary to find solution of this equation in the form of a solitary wave by accepting that $n_0(x,t) = n_0(x - V_0 t)$, where V_0 is a certain velocity, then we will obtain $(\lambda|b|\bar{n} - V_0)n_0' - 2\lambda|b|n_0 n_0' - \lambda A n_0''' = 0$, where dash here means already differentiation by argument $x - V_0 t$. Finally, after selecting constant λ in form of $\lambda = \frac{V_0}{3|b|\bar{n}}$ in this equation, we immediately get well-known Korteweg-de Vries equation

$$n_0' + \frac{n_0}{\bar{n}} n_0' + \beta n_0''' = 0, \qquad (10)$$

where constant is $\beta = \frac{V_0 A}{2|b|\bar{n}}$.

So, as we have seen that the system of equations (4) allows us in the relevant special cases obtaining any known differential equations describing a particular physical phenomenon. Our challenge now will be to find possible solutions of the equation system (4-5) by numerical methods in the one-dimensional case, provided that the solution is self-similar form and depends on difference $x - V_0 t$. We will assume that $n_0 + n_1 = \bar{n} = const$. The considering (5), equation (4) can be traced to the following form:

$$\dot{n}_0 = \frac{V_0}{3|b|\bar{n}}[K_0\bar{n} + 3K_2\bar{n}^2 - (K_0 + K_3)n_0 + \\ +(K_1 - K_2)n_0^2 - \alpha(\ddot{n}_0 - c_s^2\Delta n_0) - \\ -\vec{b}(\bar{n} - 2n_0)\nabla n_0 + \vec{A}\nabla\Delta n_0 ...]$$

and in the one-dimensional case it comes from here immediately that

$$\dot{n}_0 = \frac{V_0}{3|b|\overline{n}}[K_0\overline{n} + 3K_2\overline{n}^2 - (K_0 + K_3)n_0 +$$
$$+(K_1 - K_2)n_0^2 - \alpha(\ddot{n}_0 - c_s^2 n_0'') - \quad (11)$$
$$-b(\overline{n} - 2n_0)n_0' + An_0'''...]$$

We will try solution of equation (11) in form of $n_0(x,t) = n_0(x - V_0 t)$. After insertion and dimensionalizationless of variables we will have the following nonlinear and nonhomogeneous equation

$$a_5 y''' - a_3 y'' + (1 - a_4) y' - y + \\ +a_2 y^2 + 2a_4 y y' = a_1 \quad (12)$$

where dimensionless function is $y = \frac{n_0}{\overline{n}}$, new dimensionless argument is $\xi = \frac{(x - V_0 t)(K_0 + K_3)}{3|b|\overline{n}}$, and coefficients are

$$a_1 = \frac{K_0 + 3K_2\overline{n}}{(K_0 + K_3)\overline{n}}, a_2 = \frac{K_1 - K_2}{K_0 + K_3}\overline{n},$$
$$a_3 = \frac{\alpha(V_0^2 - c_s^2)(K_0 + K_3)}{9b^2\overline{n}^2}, a_4 = \frac{1}{3}, \quad (13)$$
$$a_5 = \frac{A(K_0 + K_3)^2}{(3|b|\overline{n})^3}.$$

Differentiation in (12) is made by the argument ξ. If the constant b is turned to zero then those terms in the equation, which are proportional to the first derivative ξ, disappear and the following nonlinear equation of the third order $a_0 y''' - a_3 y'' - y + a_2 y^2 = -a_1$ can be obtained. The above equation, unfortunately, cannot be solved analytically, and the asymptotic cases (for particular values of the parameters, as well as for small values of the argument or function) have very little relevance to the reality. Therefore, in order to study the possible phase trajectories we analyzed a much more complex equation. In some particular cases below its solution is illustrated by various phase portraits and with different values of the parameters. When considering all the other nonlinear terms, which appear in functional (2), equation (12) can be presented in the following rather general form

$$a_5 y''' + \alpha y'' = \alpha(b_0 + b_1 y + b_2 y^2 + b_3 y^3 + ...) \quad (14)$$

By assuming here that $a_5 = 0$, we have

$$y'' = b_0 + b_1 y + b_2 y^2 + b_3 y^3 + ... \quad (15)$$

To build phase portraits of this equation we should be written in the form of the following system of differential equations

$$\begin{cases} x_1' = b_0 + b_1 x_2 + b_2 x_2^2 + b_3 x_2^3 + ..., \\ x_2' = x_1. \end{cases} \quad (16)$$

In Fig. 1-22 some possible phase portraits in coordinates $x_1 - x_2$. Graphs of the solutions and the phase portraits in the form of functions from the numerical parameter for the systems of 1-7 are of illustrated. It is necessary to note that the phase portraits describe the connection of relative concentrations with the speed of their change, therefore not it goes about what acoustic waves here of speech. If we are interested in the dependence of concentration on the time, these fluctuations will be identical to the acoustic waves, ,which is evident from figures 2,5,11,14,17,20, which describe purely oscillating processes, in particular, acoustic waves.

1. $\begin{cases} x_1' = x_2 \\ x_2' = x_1 + 6x_1^2 - 7x_1^3 + x_3 \sin x_2 \\ x_3' = 0 \end{cases}$

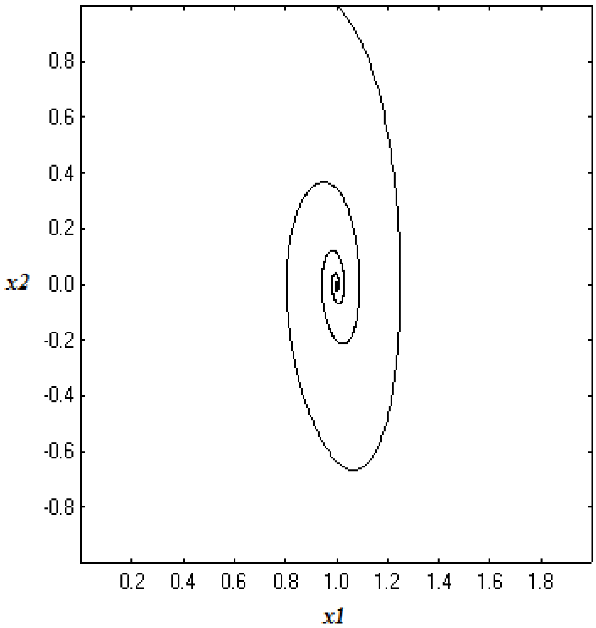

Fig.1. The phase portrait of the system is obtained with $x_3 = -1$ and the initial conditions $x_1(0) = x_2(0) = 1$. (Temporary) Time dependence $x_1(t)$ is given in the following figure.

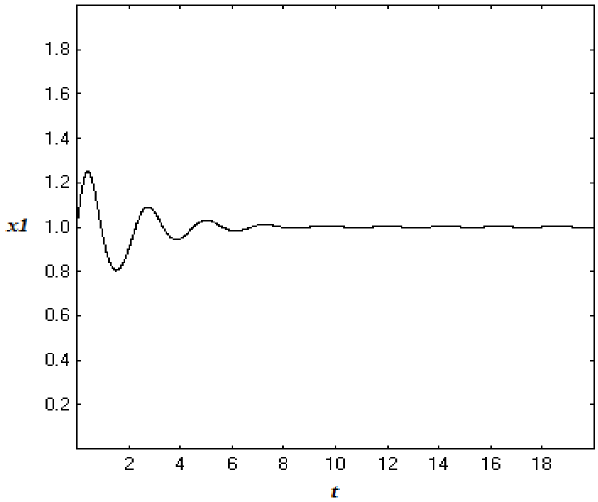

Fig.2. Solution of the system $x_1(t)$ with $x_3 = -1$ and the initial conditions $x_1(0) = x_2(0) = 1$.

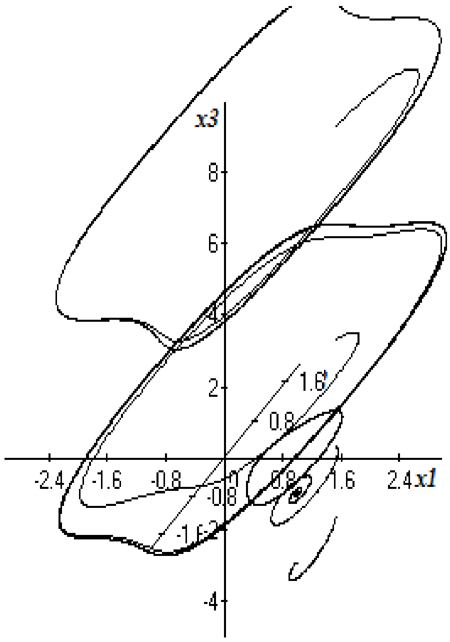

Fig.3. It is shown that the phase portraits of the system qualitatively change depending on the value of the coefficient with $\sin x_2$, whose values are shown along the vertical axis. With $x_3 < 0$ the equilibrium point is of the focus type, with $x_3 = 0$ it is of the center type, and with $x_3 < 0$ - the phase trajectories become limiting cycles. The initial conditions everywhere $x_1(0) = x_2(0) = 1$.

2. $\begin{cases} x_1' = x_2 \\ x_2' = -100 \sin x_1 + x_3 \sin x_2 \\ x_3' = 0 \end{cases}$

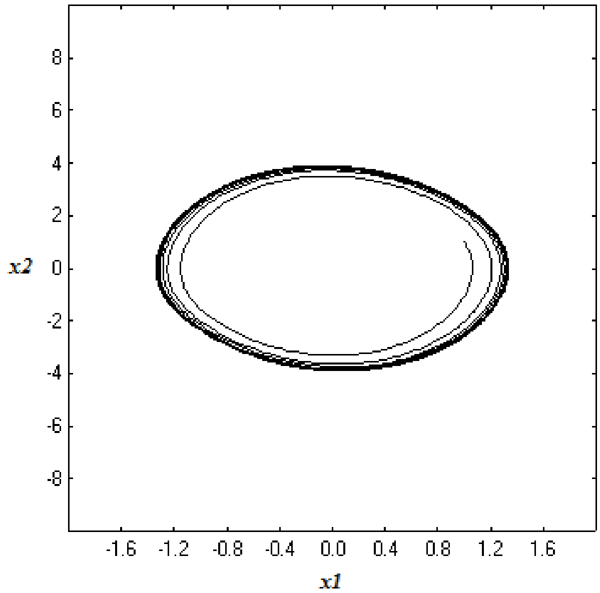

Fig.4. The phase portrait of the system is obtained with $x_3 = -1$ and the initial conditions $x_1(0) = x_2(0) = 1$. Time dependence $x_1(t)$ is given in the following figure.

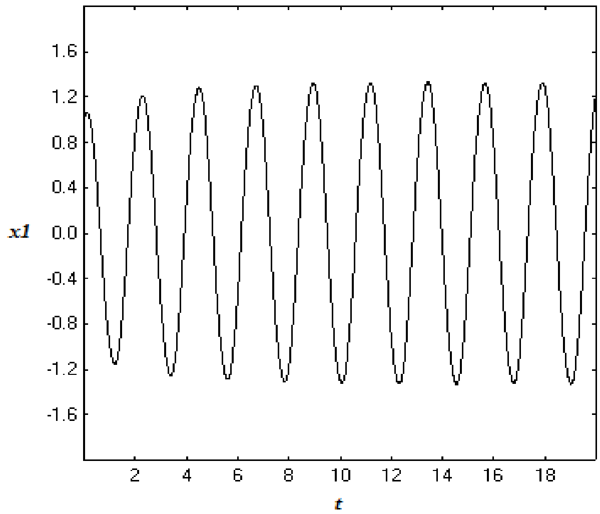

Fig.5. Solution of the system $x_1(t)$ with $x_3 = -1$ and the initial conditions $x_1(0) = x_2(0) = 1$.

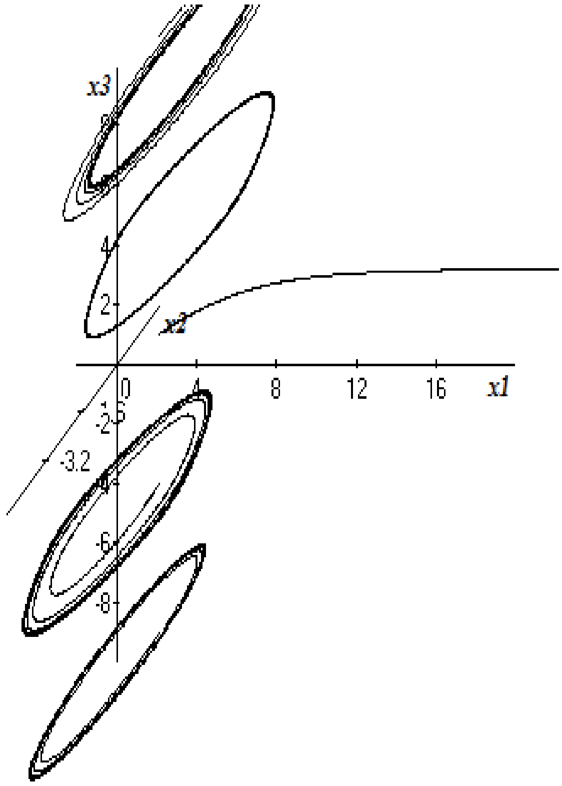

Fig.6. It is shown how the phase portrait of the system changes depending on the parameter of the system - coefficient at $\sin x_2$ (its values are plotted along the vertical axis). The critical value $x_3 = 0$ leads to “the disruption” when the phase trajectory goes to infinity. The initial conditions everywhere $x_1(0) = x_2(0) = 1$.

3. $$\begin{cases} x_1' = x_2 \\ x_2' = x_1 + 6x_1^2 - 7x_1^3 + x_3 \exp x_2 \\ x_3' = 0 \end{cases}$$

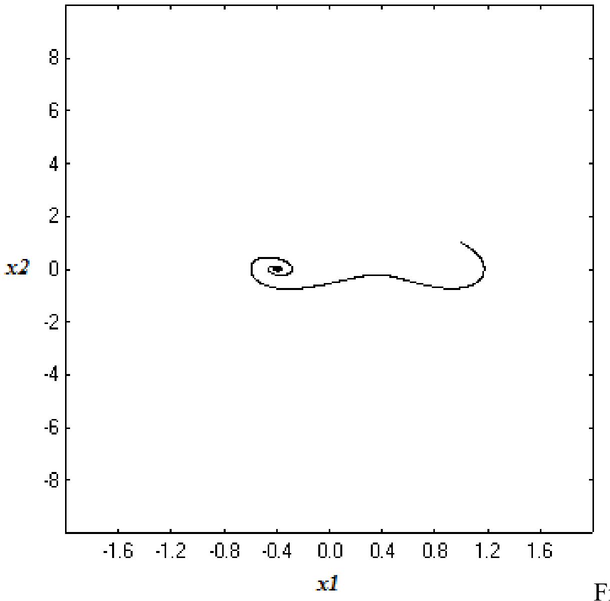

Fig.7. The phase portrait of the system is obtained with $x_3 = -1$ and the initial conditions $x_1(0) = x_2(0) = 1$. Time dependence $x_1(t)$ is given in the following figure.

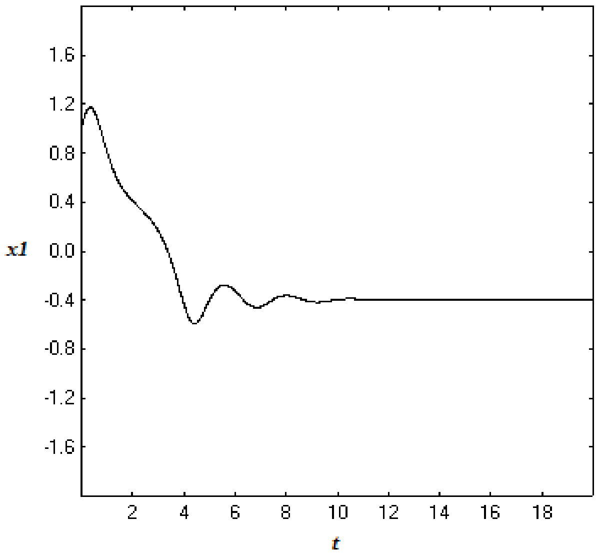

Fig.8. Solution of the system $x_1(t)$ with $x_3 = -1$ and the initial conditions $x_1(0) = x_2(0) = 1$.

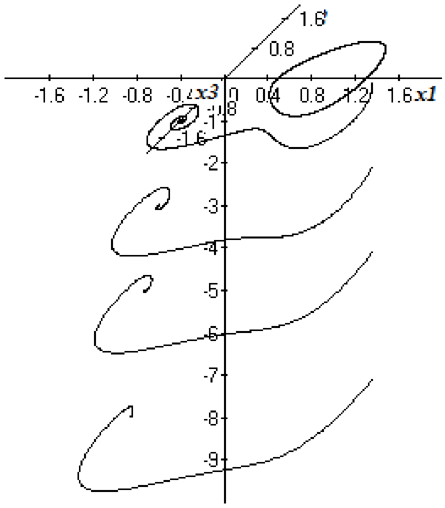

Fig.9. Simulation of the dependence of the phase portraits of the system on its parameter – the coefficient at $\exp x_2$ (plotted along the vertical axis). It is evident that the phase portrait changes radically: at $x_3 < 0$ the phase trajectories are «twisted» in a spiral towards the equilibrium point, and at $x_3 = 0$ - the phase portrait is a center. The initial conditions everywhere $x_1(0) = x_2(0) = 1$.

4. $$\begin{cases} x_1' = x_2 \\ x_2{}' = x_1 + x_3 x_1{}^2 - 7x_1 \\ x_3' = 0 \end{cases}$$

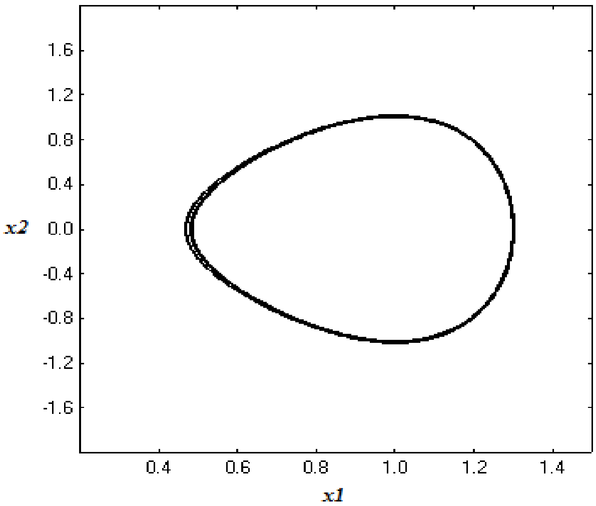

Fig.10. The phase portrait of the system is obtained with $x_3 = 6$ and the initial conditions $x_1(0) = x_2(0) = 1$.

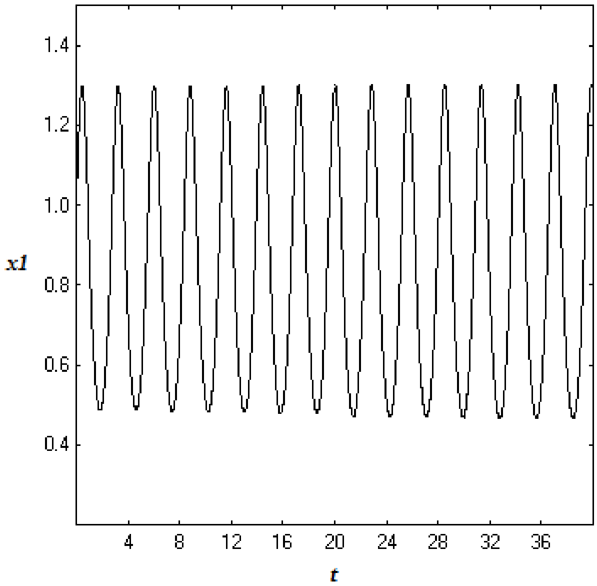

Fig.11. Solution of the system $x_1(t)$ with $x_3 = 6$ and the initial conditions $x_1(0) = x_2(0) = 1$.

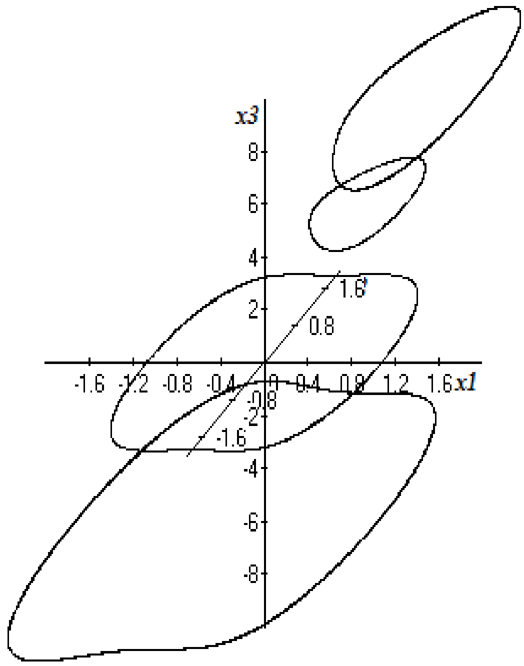

Fig.12. Changes in the parameter of the system - coefficient at x_1^2 - from “- 10” to “10” does not lead to a change in the nature of the dynamics of the system. The only change is the shift in the equilibrium point. At $x_3 = 0$ the equilibrium point is $(0;0)$, at $x_3 \neq 0$ this point smoothly moves away from the origin of coordinates. The initial conditions everywhere $x_1(0) = x_2(0) = 1$.

5. $\begin{cases} x_1' = x_2 \\ x_2' = x_3 \sin x_1 \\ x_3' = 0 \end{cases}$

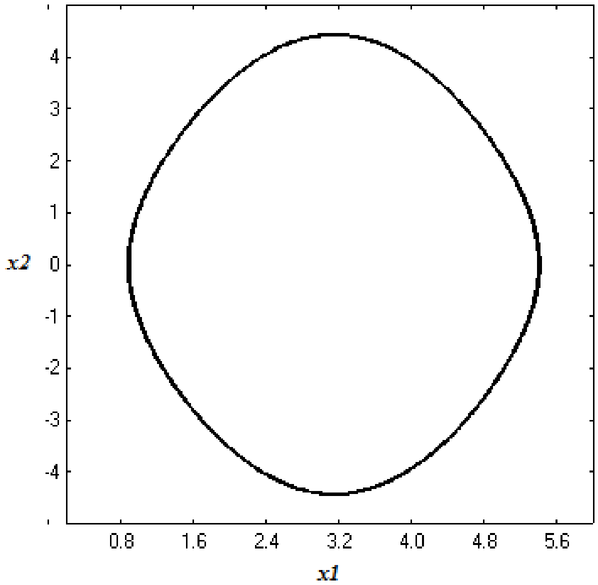

Fig.13. The phase portrait of the system is obtained with $x_3 = 6$ and the initial conditions $x_1(0) = x_2(0) = 1$. Time dependence $x_1(t)$ is given in the following figure.

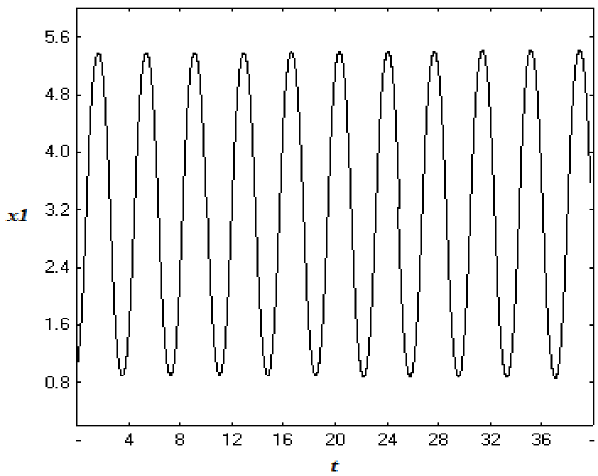

Fig.14. Solution of the system $x_1(t)$ with $x_3 = 6$ and the initial conditions $x_1(0) = x_2(0) = 1$.

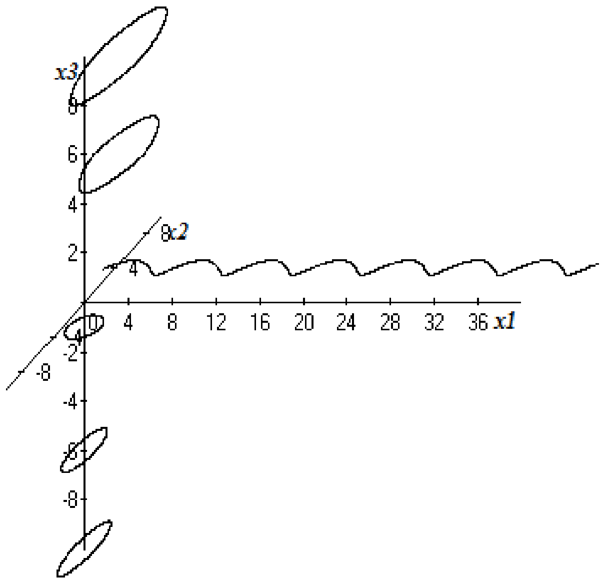

Fig.15. If the parameter of the system - coefficient at $\sin x_1$ (shown along the vertical axis) - falls into the region $(0;1)$, then the phase portrait changes: a trajectory of the type "center" transforms into a curve similar to a sine. The initial conditions everywhere $x_1(0) = x_2(0) = 1$.

6. $\begin{cases} x_1' = x_2 \\ x_2' = x_3 \cos x_1 \\ x_3' = 0 \end{cases}$

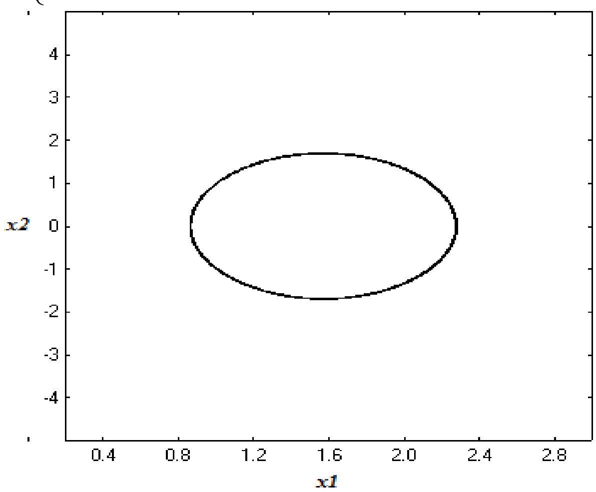

Fig.16. The phase portrait of the system is obtained with $x_3 = 6$ and the initial conditions $x_1(0) = x_2(0) = 1$.

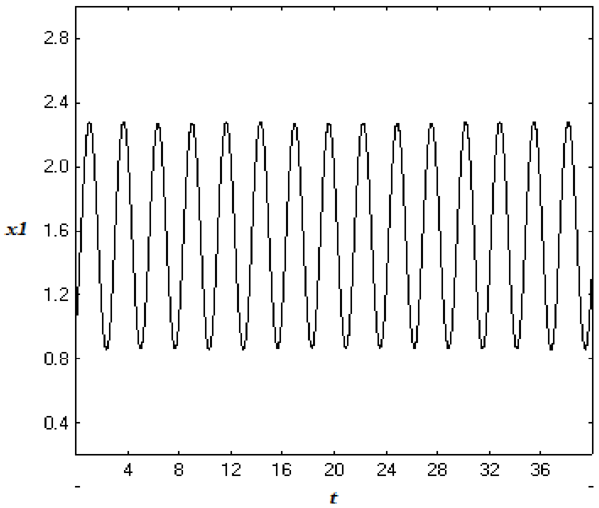

Fig.17. Time dependence $x_1(t)$, obtained on the assumption that $x_3 = 6$ and the initial conditions $x_1(0) = x_2(0) = 1$.

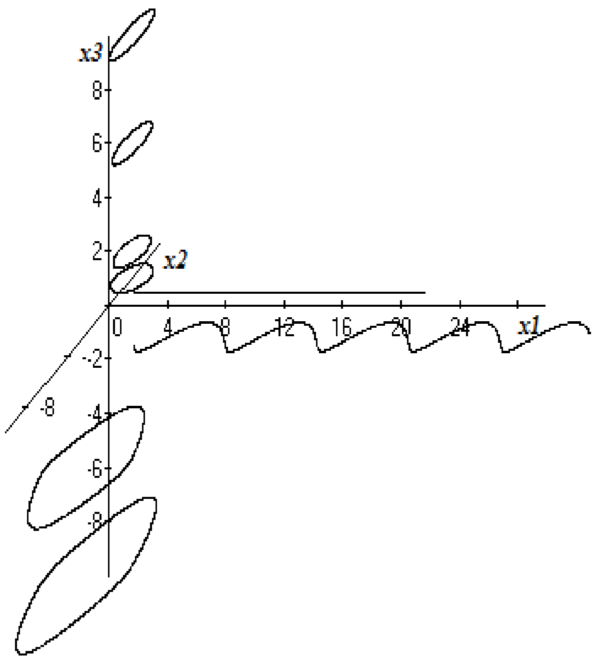

Fig.18. The parameter of the system is the coefficient at $\cos x_1$ (shown along the vertical axis). If $x_3 \in (-3;0)$, then a trajectory of the center type again transforms into a sine-type curve, and at $x_3 = 0$ it becomes a horizontal line. The initial conditions everywhere $x_1(0) = x_2(0) = 1$

7. $\begin{cases} x_1' = x_2 \\ x_2{}' = 6\cos x_1 + x_3 \sin x_2 \\ x_3' = 0 \end{cases}$

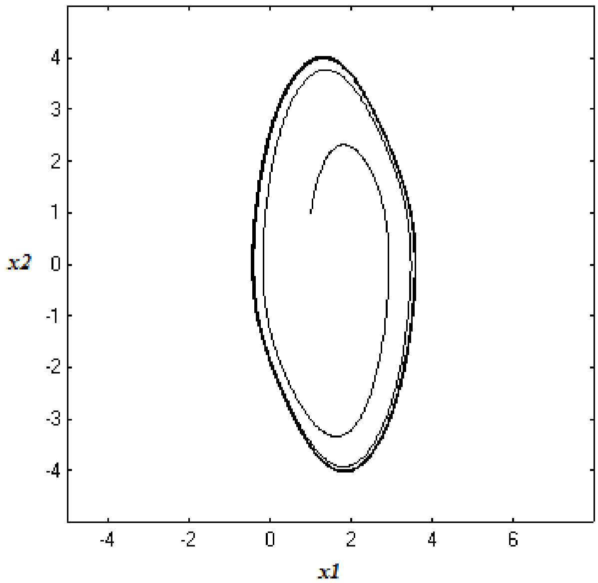

Fig.19. The phase portrait of the system is obtained with $x_3 = -7$ and the initial conditions $x_1(0) = x_2(0) = 1$.

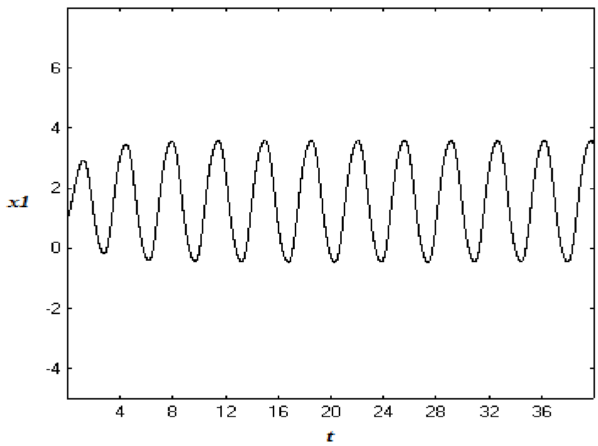

Fig.20. Solution $x_1(t)$ obtained on the assumption that $x_3 = 6$ and the initial conditions $x_1(0) = x_2(0) = 1$.

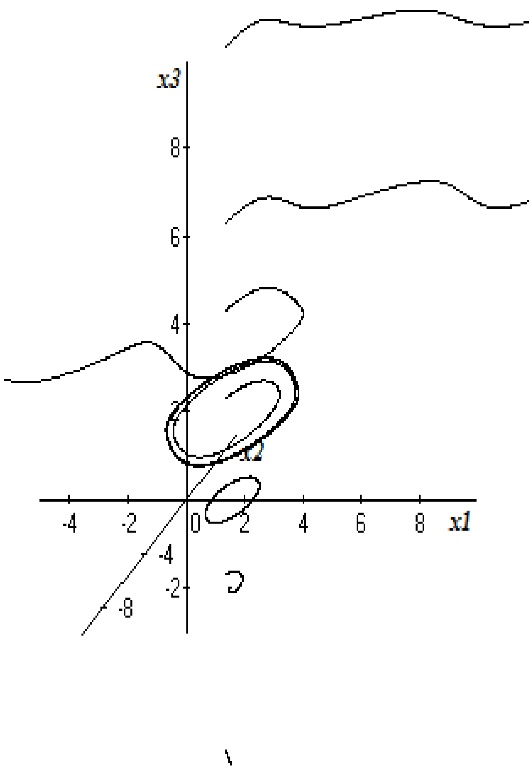

Fig.21. A change in the coefficient at $\sin x_2$ leads to a change in the phase portraits of the system. At $x_3 < 0$ the phase portraits can be seen well in Fig.19, at $x_3 = 0$ it is already a center, at $0 < x_3 < 3,5$ the phase portraits are limiting cycles. Beginning from the critical value $x_3 = 3,5$ we observe the disruption of the trajectory to infinity according to the sinusoidal law. The initial conditions everywhere $x_1(0) = x_2(0) = 1$.

The influence of the value of the coefficient at $\cos x_1$ on the nature of phase portraits is shown in the following figure.

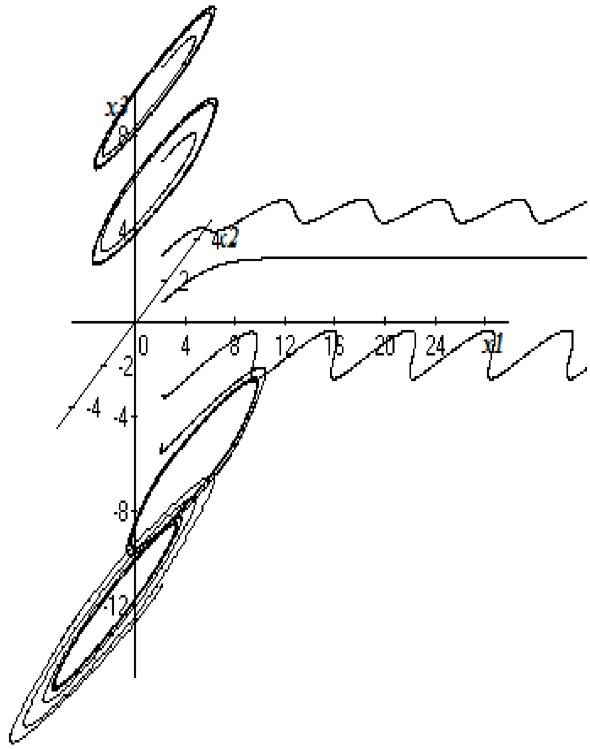

Fig.22. The parameter of the system is the coefficient at $\cos x_1$. The phase trajectory at $x_3 = 0$ is "almost a straight line", at $-4 < x_3 < 4, x_3 \neq 0$ it is bent in the form of a wave, and if $|x_3| \geq 4$, it turns into a cycle. The initial conditions everywhere $x_1(0) = x_2(0) = 1$.

IV. Conclusion

In conclusion of the paper we should note two main points.
1. A general method for deriving any nonlinear differential equations describing the real physical processes, is suggested;
2. Numerical calculation of the phase portraits for some types of nonlinear differential equations and their graphical illustration are provide

References

[1] N.S. Koshliakov, E.B. Gleaner, M.M. Smirnov. *Equations in the partial derivatives of mathematical physics.* M.: Higher school. 1970. 710 pp.
[2] L.D. Landau, E.M. Lifshitz. *Mechanics.* V. 1. M.: Nauka, 1973. 207 pp.
[3] S.L.Muller, T.Plesser, B.Hess. Two dimensional spectrophotometry of spiral wave propagation in the Belousov-Zhabotinskii reaction: I. Experiment and digital data representation. *Physica D,* 1987. V.24. p.71-86.
[4] S.L.Muller, T.Plesser, B.Hess. Two dimensional spectrophotometry of spiral wave propagation in the Belousov-Zhabotinskii reaction: II. Geometric and kinematic parpmeters. *Physica D,* 1987. V.24. p.87-96
[5] R.Feeney, S.L.Schmidt, P.Ortoleva. Experiments on electric field-BZ chemical wave interactions: Annihilation and the crescent wave. *Physica D, 1981. V.2. p.536-547.*
[6] P. Glendinning. Travelling wave solutions near isolated double-pulse solitary waves of nerve axon equations. *Physics Letters A, 1987.*V.121. p.411-413.
[7] E.N.Lorenz. Deterministic Nonperiodic Flows. *J.Atm.Sci., 1963.* V.20. p.130-141.

[8] A.M. Zhabotinskii. Periodic motion of oxidizing the malonic acid in the solution. *Biophysics, 1964*. V.9. v.3. p.306-311.
[9] M.D.Korzukhin, A.M. Zhabotinskii. *The mathematical simulation of the chemical and ecological self-oscillating systems.* M.: Nauka, 1965. 231 pp.
[10] G.B.Whitham. The linear and nonlinear waves. M.: Mir, 1977. 622 pp.
[11] L.D. Landau, E.M. Lifshitz. *Theory of field.* V. 2. M.: Nauka, 1975. 502 pp.
[12] L.D. Landau, E.M. Lifshitz. *Theory of elasticity*. V. 7. M.: Nauka, 1987. 244 pp.
[13] Nguyen Huu Khanh. Stability Analysis in Competitin Population Model. *International journal of mathematical models and methods in applied sciences.* **2012. V. 6. №1. p. 11-19.**
[14] **Vagif R. Ibrahimov, Mehriban N. Imanova. On a Research of Symmetric Equations of Volterra Type.** *International journal of mathematical models and methods in applied sciences.* 2014. V.8. p. 434-440.
[15] L. A. Fil'Shtinskii, T. A. Kirichok, D. V. Kushnir. One-Dimensional Fractional Quasi-Static Thermoelasticity Problem for a Half-Space. ***WSEAS Transactions on Heat And Mass Transfer.*** **2013. V 8. №2. p.31-36.**
[16] S.O.Gladkov, I.V. Gladyshev. On fluctuations in liquids and gases. *Technical Physics Journal,* 2001. V. 71. V. 3. p. 1 – 8.
[17] S.O.Gladkov, I.V. Gladyshev. Nonuniform fluctuations in liquids and gases. *Technical Physics Journal,* 2001. V. 71. v. 4. p. 1-5.
[18] S.O.Gladkov. Model description crystal growth of heterogeneous substances. *Reports of the Russian Academy of the Sciences.* 2004. V. 394. v. 4. p. 469 – 472.