Numerical Solution of Boundary Inverse Problems for Plane Orthotropic Elastic Solids

Igor Brilla, František Janíček

***Abstract*—**We deal with numerical analysis of inverse problems for orthotropic solids when measured data are given only on the boundary of the domain. In this paper we have elaborated an iterative procedure to the solution of inverse problems for orthotropic solids when input data measured from suitable states are sufficient for determination of unknown material parameters. We deal with numerical experiments.

Keywords—Inverse problem, orthotropic solid, finite difference method.

I. Introduction

INVERSE problems are very important from a practical point of view and interesting from a theoretical point of view as they are improperly posed problems. An important class of inverse problems is a class of identification problems. These problems are important, for example, in the non-destructive testing of materials, the identification of material parameters, the study of aquifer problems as well as for electrical impedance tomography, etc.

We deal with analysis of inverse problems for orthotropic solids when measured data are given only on the boundary of the domain. The inverse problems for orthotropic solids have special features in comparison with those for isotropic solids. In order to solve orthotropic problems, more unknown material parameters of governing differential equations than the total number of equations must be determined and therefore, in order to determine them, we need input data measured from more than one field state. These input states as we show cannot be chosen arbitrarily. This fact leads to new theoretical problems in the analysis of inverse problems for orthotropic solids and also complicates numerical analysis.

For numerical analysis of such problems we apply discrete methods. These are very convenient because in the case of practical problems we have to measure input states in discrete points. In [1] and [2] we elaborated an iterative procedure to the numerical solution of plane orthotropic and plane anisotropic boundary inverse problems using governing equations and equations of Hooke`s law. In this paper, we have elaborated an iterative procedure to the numerical solution of boundary inverse problems for orthotropic plates when the input data measured from suitable states are sufficient to determine the unknown material parameters using generalization of so called Sophie Germain`s equation for orthotropic plates. We derive the number of measured input states and conditions for these measured input states which secure determinability of the numerical solution. We also deal with numerical experiments from mathematical point of view. Another approach is derived in [3] and [4].

II. Formulation of the Problem

At first we derive convenient form of the problem. We consider Hooke's law

$$\tau_{ij} = c_{ijkl}\,\varepsilon_{k,l} \quad , \quad i,j,k,l = 1,2 \quad , \tag{1}$$

where τ is stress tensor, c_{ijkl} are elastic coefficients and ε is strain tensor. We apply the summation and differentiation rule with respect to indices. The elastic coefficients are symmetric. It holds $c_{ijkl} = c_{klij} = c_{jikl} = c_{ijlk}$. Hooke's law can be written for plane non-homogeneous anisotropic problem in the following forms

$$\begin{aligned} \tau_{xx} &= c_{1111}(x,y)\,u_{x,x} + c_{1112}(x,y) \\ &\quad .\left(u_{x,y} + u_{y,x}\right) + c_{1122}(x,y)\,u_{y,y} \quad , \\ \tau_{yy} &= c_{2211}(x,y)\,u_{x,x} + c_{2212}(x,y) \\ &\quad .\left(u_{x,y} + u_{y,x}\right) + c_{2222}(x,y)\,u_{y,y} \quad , \\ \tau_{xy} &= c_{1211}(x,y)\,u_{x,x} + c_{1212}(x,y) \\ &\quad .\left(u_{x,y} + u_{y,x}\right) + c_{1222}(x,y)\,u_{y,y} \quad , \end{aligned} \tag{2}$$

where u_x and u_y are displacements in x and y-direction. Using Kirchhoff assumption we can obtain

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$$u_{x,x} = -z\, w,_{xx} \quad , \quad u_{y,y} = -z\, w,_{yy} \quad ,$$

$$u_{x,y} + u_{y,x} = -2\, z\, w,_{xy} \quad ,$$

where w is displacement in z-direction. Now we can rewrite equations (2) of Hooke's law in the following forms

$$\tau_{xx} = -z\left[c_{1111}\, w,_{xx} + 2\, c_{1112}\, w,_{xy} + c_{1122}\, w,_{yy}\right] ,$$

$$\tau_{yy} = -z\left[c_{2211}\, w,_{xx} + 2\, c_{2212}\, w,_{xy} + c_{2222}\, w,_{yy}\right], \qquad (3)$$

$$\tau_{xy} = -z\left[c_{1211}\, w,_{xx} + 2\, c_{1212}\, w,_{xy} + c_{1222}\, w,_{yy}\right] .$$

Next we integrate with respect to z equations (3) and introduce moment intensity functions

$$M_x = \int_{-h/2}^{h/2} z\, \tau_{xx}\, dz =$$

$$= -\int_{-h/2}^{h/2} z^2 \left[c_{1111}\, w,_{xx} + 2\, c_{1112}\, w,_{xy} + c_{1122}\, w,_{yy}\right] dz = \qquad (4)$$

$$= -\left(h^3/12\right)\left[c_{1111}\, w,_{xx} + 2\, c_{1112}\, w,_{xy} + c_{1122}\, w,_{yy}\right] ,$$

where h is a thickness of the plate. Similarly we have

$$M_y = \int_{-h/2}^{h/2} z\, \tau_{yy}\, dz =$$

$$= -\left(h^3/12\right)\left[c_{2211}\, w,_{xx} + 2\, c_{2212}\, w,_{xy} + c_{2222}\, w,_{yy}\right] ,$$

$$(5)$$

$$M_{xy} = \int_{-h/2}^{h/2} z\, \tau_{xy}\, dz =$$

$$= -\left(h^3/12\right)\left[c_{1211}\, w,_{xx} + 2\, c_{1212}\, w,_{xy} + c_{1222}\, w,_{yy}\right] = M_{yx} .$$

The equation of equilibrium between moments and load applied normal to the midplane of the plane p is

$$M_{x,xx} + 2\, M_{xy,xy} + M_{y,yy} + p(x,y) = 0 \quad \text{in} \quad \Omega , \qquad (6)$$

where Ω is a two dimensional Lipschitz domain. Then replacing M_x, M_{xy} and M_y using equations (4) and (5) the equation (6) can be written in the following form

$$c_{1111}\, w,_{xxxx} + 2\, c_{1111,x}\, w,_{xxx} + c_{1111,xx}\, w,_{xx} + 2\, c_{1122}\, w,_{xxyy}$$

$$+ 2\, c_{1122,x}\, w,_{xyy} + 2\, c_{1122,y}\, w,_{xxy} + c_{1122,xx}\, w,_{yy} +$$

$$+ c_{1122,yy}\, w,_{xx} + 4\, c_{1112}\, w,_{xxxy} + 6\, c_{1112,x}\, w,_{xxy} +$$

$$+ 2\, c_{1112,y}\, w,_{xxx} + 2\, c_{1112,xx}\, w,_{xy} + 2\, c_{1112,xy}\, w,_{xx} +$$

$$+ 4\, c_{1222}\, w,_{xyyy} + 2\, c_{1222,x}\, w,_{yyy} + 6\, c_{1222,y}\, w,_{xyy} + \qquad (7)$$

$$+ 2\, c_{1222,yy}\, w,_{xy} + 2\, c_{1222,xy}\, w,_{yy} + 4\, c_{1212}\, w,_{xxyy} +$$

$$+ 4\, c_{1212,x}\, w,_{xyy} + 4\, c_{1212,y}\, w,_{xxy} + 4\, c_{1212,xy}\, w,_{xy} +$$

$$c_{2222}\, w,_{yyyy} + 2\, c_{2222,y}\, w,_{yyy} + c_{2222,yy}\, w,_{yy} = 12\, p/h^3$$

in Ω .

For simplicity we are going to deal with orthotropic solids. In the plane orthotropic case c has only 4 nonzero components and (7) can be written in the following form

$$c_{11}\, w,_{xxxx} + 2\, c_{11},_{x}\, w,_{xxx} + c_{11},_{xx}\, w,_{xx} + 2\, c_{12}\, w,_{xxyy} +$$

$$+ 2\, c_{12},_{x}\, w,_{xyy} + 2\, c_{12},_{y}\, w,_{xxy} + c_{12},_{xx}\, w,_{yy} + c_{12},_{yy}\, w,_{xx} + \qquad (8)$$

$$+ 4\, c_{66}\, w,_{xxyy} + 4\, c_{66},_{x}\, w,_{xyy} + 4\, c_{66},_{y}\, w,_{xxy} + 4\, c_{66},_{xy}\, w,_{xy}$$

$$+ c_{22}\, w,_{yyyy} + 2\, c_{22},_{y}\, w,_{yyy} + c_{22},_{yy}\, w,_{yy} = 12\, p/h^3$$

in Ω, where we use following notations $c_{1111} = c_{11}$, $c_{2222} = c_{22}$, $c_{1122} = c_{12}$, $c_{1212} = c_{66}$.

In the case of the inverse problems we have to determine the elastic coefficients we need for their determination following boundary conditions

$$c_{11}(s) = a_{11}(s) , \quad c_{22}(s) = a_{22}(s) ,$$

$$c_{12}(s) = a_{12}(s) , \quad c_{66}(s) = a_{66}(s) \quad , \quad s \in \partial\Omega \quad , \qquad (9)$$

In the case of the boundary inverse problems we have also to determine the displacement w using measured values of the displacement w on the boundary $\partial\Omega$. We consider for the displacement w following boundary conditions

$$w(s) = g_1(s) \quad , \quad w_{,n}(s) = g_2(s) \quad , \quad s \in \partial\Omega \quad , \qquad (10)$$

where $(.)_{,n}$ denotes the differentiation in direction of the outer normal.

In the case of plane orthotropic problem moments have the following forms

$$^{w}M_x = -\left(h^3/12\right)\left[c_{11}\, w,_{xx} + c_{12}\, w,_{yy}\right] ,$$

$$^{w}M_y = -\left(h^3/12\right)\left[c_{12}\, w,_{xx} + c_{22}\, w,_{yy}\right] , \qquad (11)$$

$$ {}^{w}M_{xy} \;=\; -\left(h^3/6\right)\, c_{66}\, w,_{xy} \;=\; M_{yx} $$

and equation (6)

$$ {}^{w}M_{x,xx} + 2\,{}^{w}M_{xy,xy} + {}^{w}M_{y,yy} + p \;=\; 0 \;\text{ in } \Omega \qquad (12) $$

with following boundary conditions

$$ {}^{w}M_{x}(s) = {}^{w}j_1(s)\,, \quad {}^{w}M_{y}(s) = {}^{w}j_2(s)\,, $$
$$ {}^{w}M_{xy}(s) = {}^{w}j_3(s)\,, \quad s \in \partial\Omega \qquad (13) $$

and

$$ {}^{w}M_{x,x}(s) = {}^{w}j_4(s)\,, \quad {}^{w}M_{y,y}(s) = {}^{w}j_5(s)\,, $$
$$ {}^{w}M_{xy,x}(s) = {}^{w}j_6(s)\,, \quad {}^{w}M_{xy,y}(s) = {}^{w}j_7(s) \qquad (14) $$

on corresponding parts of the boundary $\partial\Omega$.

The convenient formulation of our boundary inverse problems for orthotropic plates represents equation for displacements (8) with corresponding boundary conditions (9), (10), relations for moments (11), equation of equilibrium (12) with corresponding boundary conditions (13), (14).

However, in the case of boundary inverse problems for orthotropic plates, the system (8) – (14) does not form a complete system of equations and is not sufficient for determination of the unknown elastic coefficients. We show that for determination of the unknown elastic coefficients, it is necessary to add input data measured from next state of the displacement v. For this next state of input data we consider the equations and boundary conditions analogical to (8) and (10)

$$ c_{11}\, v,_{xxxx} + 2\, c_{11},_{x}\, v,_{xxx} + c_{11},_{xx}\, v,_{xx} + 2\, c_{12}\, v,_{xxyy} + $$
$$ + 2\, c_{12},_{x}\, v,_{xyy} + 2\, c_{12},_{y}\, v,_{xxy} + c_{12},_{xx}\, v,_{yy} + c_{12},_{yy}\, v,_{xx} + \qquad (15) $$
$$ + 4\, c_{66}\, v,_{xxyy} + 4\, c_{66},_{x}\, v,_{xyy} + 4\, c_{66},_{y}\, v,_{xxy} + 4\, c_{66},_{xy}\, v,_{xy} $$
$$ + c_{22}\, v,_{yyyy} + 2\, c_{22},_{y}\, v,_{yyy} + c_{22},_{yy}\, v,_{yy} \;=\; 12\, q/h^3 $$

in Ω,

$$ v(s) = g_3(s) \;,\quad v_{,n}(s) = g_4(s) \;,\quad s \in \partial\Omega \qquad (16) $$

corresponding equations for moments

$$ {}^{v}M_{x} \;=\; -h^3\left[c_{11}\, v,_{xx} + c_{12}\, v,_{yy}\right]/\,12 \;, $$

$$ {}^{v}M_{y} \;=\; -h^3\left[c_{12}\, v,_{xx} + c_{22}\, v,_{yy}\right]/\,12 \;, \qquad (17) $$

$$ {}^{v}M_{xy} \;=\; -h^3\, c_{66}\, v,_{xy}/\,6 \;=\; {}^{v}M_{yx} $$

corresponding equation to (12)

$$ {}^{v}M_{x,xx} + 2\,{}^{v}M_{xy,xy} + {}^{v}M_{y,yy} + q \;=\; 0 \;\text{ in } \Omega \qquad (18) $$

with corresponding boundary conditions

$$ {}^{v}M_{x}(s) = {}^{v}j_1(s)\,, \quad {}^{v}M_{y}(s) = {}^{v}j_2(s)\,, $$
$$ {}^{v}M_{xy}(s) = {}^{v}j_3(s)\,, \quad s \in \partial\Omega \qquad (19) $$

and

$$ {}^{v}M_{x,x}(s) = {}^{v}j_4(s)\,, \quad {}^{v}M_{y,y}(s) = {}^{v}j_5(s)\,, $$
$$ {}^{v}M_{xy,x}(s) = {}^{v}j_6(s)\,, \quad {}^{v}M_{xy,y}(s) = {}^{v}j_7(s) \qquad (20) $$

on corresponding parts of the boundary $\partial\Omega$.

Now the question of whether the states of displacements w and v can be chosen arbitrarily arises. We show that these states of displacements cannot be chosen arbitrarily.

III. Solution of the Problem

For solving boundary inverse problems for orthotropic plates (8) – (20) we have elaborated the following iterative procedure:

- determination of an initial approximation of the elastic coefficients c^0_{11}, c^0_{12}, c^0_{22}, c^0_{66} as the linear interpolation of the boundary conditions (9);
- determination of the displacements w^0 from the equation (8) and v^0 from the equation (15);
- determination of the moments ${}^{w}M^0_x$, ${}^{w}M^0_y$, ${}^{w}M^0_{xy}$, from the equation (12) rewritten in the following forms

$$ {}^{w}M^0_{x,xx} \;=\; -\,p + h^3\left[c^0_{12}\, w^0,_{xx} + c^0_{22}\, w^0,_{yy}\right],_{yy}/\,12 + $$
$$ +\; h^3\left(c^0_{66}\, w^0,_{xy}\right),_{xy}/\,3 \;, \qquad (21) $$

$$ {}^{w}M^0_{y,yy} \;=\; -\,p + h^3\left[c^0_{11}\, w^0,_{xx} + c^0_{12}\, w^0,_{yy}\right],_{xx}/\,12 + $$
$$ +\; h^3\left(c^0_{66}\, w^0,_{xy}\right),_{xy}/\,3 \;, \qquad (22) $$

$$ {}^{w}M^0_{xy,xy} = p^0_1 \;=\; +\; h^3\left[c^0_{11}\, w^0,_{xx} + c^0_{12}\, w^0,_{yy}\right],_{xx}/\,24 $$
$$ -\,p/2 + h^3\left[c^0_{12}\, w^0,_{xx} + c^0_{22}\, w^0,_{yy}\right],_{yy}/\,24 \;. \qquad (23) $$

We can consider equations (21), (22) as ordinary differential equations of the second order. The equation (23) can be written for example in the following form

$$\left({}^{w}M^{0}_{xy,y} \right)_{,x} = p^{0}_{1} \quad . \tag{24}$$

The equation (24) we can solve at first as ordinary differential equation of the first order according variable x and after it as ordinary differential equation of the first order according variable y. We also determine ${}^{v}M^{0}_{x}$, ${}^{v}M^{0}_{y}$, ${}^{v}M^{0}_{xy}$ using similar approach;

- determination of new state of the elastic coefficients $c^{1}_{11}, c^{1}_{12}, c^{1}_{22}, c^{1}_{66}$ from the system of six equations of (11) and (17) using following formulas

$$\begin{aligned} c^{1}_{11} &= k \left[a^{0}_{1} \left(w^{0},_{yy} \right)^{2} - a^{0}_{2} \left(v^{0},_{yy} \right)^{2} \right] / \Delta^{0} , \\ c^{1}_{22} &= k \left[a^{0}_{1} \left(w^{0},_{xx} \right)^{2} - a^{0}_{2} \left(v^{0},_{xx} \right)^{2} \right] / \Delta^{0} , \\ c^{1}_{12} &= - \left[k \ {}^{w}M^{0}_{x} + c^{1}_{11} \, w^{0},_{xx} \right] / \ w^{0},_{yy} , \\ c^{1}_{66} &= -k \ {}^{v}M^{0}_{xy} \ / \ \left(2 \, v^{0},_{xy} \right) , \end{aligned} \tag{25}$$

where

$$a^{0}_{1} = {}^{v}M^{0}_{y} \, v^{0},_{yy} - {}^{v}M^{0}_{x} \, v^{0},_{xx} ,$$

$$a^{0}_{2} = {}^{w}M^{0}_{y} \, w^{0},_{yy} - {}^{w}M^{0}_{x} \, w^{0},_{xx} ,$$

$$k = 12 / h^{3} ,$$

$$\Delta^{0} = \left(w^{0},_{yy} \, v^{0},_{xx} \right)^{2} - \left(w^{0},_{xx} \, v^{0},_{yy} \right)^{2} ;$$

- we can continue with determination of w^{1}, v^{1}, ${}^{w}M^{1}_{x}$, ${}^{w}M^{1}_{y}$, ${}^{w}M^{1}_{xy}$, ${}^{v}M^{1}_{x}$, ${}^{v}M^{1}_{y}$, ${}^{v}M^{1}_{xy}$, $c^{2}_{11}, c^{2}_{12}, c^{2}_{22}, c^{2}_{66}$ and etc.

From Eqs. (25) we can see that the iterative procedure can be used only if

$$\left(w,_{yy} \, v,_{xx} \right)^{2} - \left(w,_{xx} \, v,_{yy} \right)^{2} \neq 0 ,$$

$$w,_{yy} \neq 0 , \quad v,_{xy} \neq 0 \quad \text{in } \Omega \ .$$

It means that the input states of displacements cannot be chosen arbitrarily.

IV. Numerical Experiments

For numerical analysis we can apply discrete methods. They are very convenient because in the case of practical problems we have to measure input states in discrete points. We assume that the domain Ω is rectangular. We consider on this domain uniform grid. Using central differences we can rewrite the iterative procedure from previous part to discrete form.

We deal with numerical experiments from mathematical point of view. It means that we construct the problem with the exact solution, afterwards we compute the numerical solution of this problem using discrete form of the iterative procedure and in the end we compare the computed solution with the exact one.

We use discrete form of the iterative procedure with stopping condition that the difference of two computed consecutive states of the material parameters is less than 10^{-7}. We consider the following plate $\Omega = \langle 0,2 \rangle \times \langle 0,1 \rangle$, $h = 0.06$. For example for the following constant elastic coefficients

$$c_{11} = 7 , \quad c_{22} = 5 , \quad c_{12} = 1 , \quad c_{66} = 2 \tag{26}$$

the displacements

$$w = 0.05 \, y \, (1 - y) , \quad v = 0.01 \, x^{2} \, y \tag{27}$$

and corresponding moments and loads

$$\begin{aligned} & {}^{w}M_{x} = 1.8 \ 10^{-6} , \quad {}^{w}M_{y} = 9 \ 10^{-6} , \quad {}^{w}M_{xy} = 0 , \\ & {}^{v}M_{x} = -2.52 \ 10^{-6} \ y , \quad {}^{v}M_{y} = -3.6 \ 10^{-7} \ y , \\ & {}^{v}M_{xy} = -1.44 \ 10^{-6} \ x ; \end{aligned} \tag{28}$$

$$p = 0, \quad q = 0 \tag{29}$$

using loads given by the equation (29) and the boundary conditions constructed from the equations (26) - (28), using discrete form of the iterative procedure we obtain on all meshes results at once with the error of computation about 10^{-9} %. Similar situation is also for the linear elastic coefficients. For example for the following elastic coefficients

$$\begin{aligned} & c_{11} = x + 1 , \quad c_{22} = y + 1 , \\ & c_{12} = (x+1) + (y+1) , \quad c_{66} = (x+1)(y+1) \end{aligned} \tag{30}$$

displacements given by (27) and corresponding moments and loads

$${}^{w}M_{x} = 1.8 \ 10^{-6} \left[(x+1) + (y+1) \right], \ {}^{w}M_{y} = 1.8 \ 10^{-6} \, (y+1),$$

$$ {}^{w}M_{xy}=0\,,\quad {}^{v}M_{x}=-3.6\;\;10^{-7}\;y\left(x+1\right)\,, $$
$$ {}^{v}M_{y}=-3.6\;\;10^{-7}\;y\left[\left(x+1\right)+\left(y+1\right)\right]\,, \tag{31} $$
$$ {}^{v}M_{xy}=-7.2\;\;10^{-7}\;x\left(x+1\right)\left(y+1\right)\;; $$

$$ p=0\,,\quad q=7.2\;\;10^{-7}\left(4\,x+3\right) \tag{32} $$

using discrete form of the iterative procedure we also obtain on all meshes results at once with the error of computation about 10^{-9} % .

Different situation is for the nonlinear elastic coefficients. For example for

$$ c_{11}\;=\;\left(x+1\right)^{2}\,,\;c_{22}\;=\;\left(y+1\right)^{2}\,, $$
$$ c_{12}\;=\;\left(x+1\right)+\left(y+1\right),\;c_{66}\;=\;\left(x+1\right)\left(y+1\right) \tag{33} $$

the displacements given by (27) and corresponding moments and loads

$$ {}^{w}M_{x}=1.8\;\;10^{-6}\left[\left(x+1\right)+\left(y+1\right)\right],{}^{w}M_{y}=1.8\;\;10^{-6}\left(y+1\right)^{2}, $$
$$ {}^{w}M_{xy}=0\,,\quad {}^{v}M_{x}=-3.6\;\;10^{-7}\;y\left(x+1\right)^{2}\,, $$
$$ {}^{v}M_{y}=-3.6\;\;10^{-7}\;y\left[\left(x+1\right)+\left(y+1\right)\right]\,, \tag{34} $$
$$ {}^{v}M_{xy}=-7.2\;\;10^{-7}\;x\left(x+1\right)\left(y+1\right)\;; $$

$$ p=-3.6\;\;10^{-6}\,,\quad q=7.2\;\;10^{-7}\left(4\,x+y+3\right) \tag{35} $$

in the Table 1 we are able to see the percentage of errors in the computed solutions in the second column with respect to the exact solutions of the meshes given in the first column. In the third column we report the numbers of iterations after which we obtain the numerical solution with the specific stopping condition on the given mesh. We can see from the results that we obtain small errors for a course mesh and when the number of grid points increases, errors also increase slightly but are still small.

Table 1 numerical results for the problem (27), (33) – (35)

Mesh	Percent error	Number of iterations
8 x 4	$3.7\ 10^{-5}$	157
12 x 6	$8.1\ 10^{-5}$	330
16 x 8	$1.5\ 10^{-4}$	573

For the following elastic coefficients

$$ c_{11}=\left(x+1\right)^{2}\left(y+1\right)\,,\quad c_{22}=\left(x+1\right)\left(y+1\right)^{2}, $$
$$ c_{12}=\left(x+1\right)+\left(y+1\right)\,,\quad c_{66}=\left(x+1\right)\left(y+1\right) \tag{36} $$

the displacements given by (27) and corresponding moments and loads

$$ {}^{w}M_{x}=1.8\;\;10^{-6}\left[\left(x+1\right)+\left(y+1\right)\right]\,, $$
$$ {}^{w}M_{y}=1.8\;\;10^{-6}\left(x+1\right)\left(y+1\right)^{2}, $$
$$ {}^{w}M_{xy}=0\,,\quad {}^{v}M_{x}=-3.6\;\;10^{-7}\;y\left(x+1\right)^{2}\left(y+1\right)\,, \tag{37} $$
$$ {}^{v}M_{y}=-3.6\;\;10^{-7}\;y\left[\left(x+1\right)+\left(y+1\right)\right]\,, $$
$$ {}^{v}M_{xy}=-7.2\;\;10^{-7}\;x\left(x+1\right)\left(y+1\right)\;; $$

$$ p=-3.6\;\;10^{-6}\left(x+1\right),\quad q=7.2\;\;10^{-7}\left[4\,x+y\left(\,y+1\right)+3\right] \tag{38} $$

we obtain similar results as it is shown in the Table 2.

Table 2 numerical results for the problem (27), (36) – (38)

Mesh	Percent error	Number of iterations
8 x 4	$4.0\ 10^{-5}$	259
12 x 6	$7.4\ 10^{-5}$	578
16 x 8	$1.3\ 10^{-4}$	1032

If we change the displacements

$$ w\;=\;0.05\,y^{2}\quad,\qquad v\;=\;0.01\,x^{2}\,y \tag{39} $$

for corresponding moments and loads

$$ {}^{w}M_{x}=-1.8\;\;10^{-6}\left[\left(x+1\right)+\left(y+1\right)\right]\,, $$
$$ {}^{w}M_{y}=-1.8\;\;10^{-6}\left(x+1\right)\left(y+1\right)^{2}, $$
$$ {}^{w}M_{xy}=0\,,\quad {}^{v}M_{x}=-3.6\;\;10^{-7}\;y\left(x+1\right)^{2}\left(y+1\right), \tag{40} $$
$$ {}^{v}M_{y}=-3.6\;\;10^{-7}\;y\left[\left(x+1\right)+\left(y+1\right)\right]\,, $$
$$ {}^{v}M_{xy}=-7.2\;\;10^{-7}\;x\left(x+1\right)\left(y+1\right)\;; $$

$$p = 3.6\ 10^{-6}\left(x+1\right),\quad q = 7.2\ 10^{-7}\left[4x + y\left(y+1\right)+3\right] \tag{41}$$

we obtain the same results for the same elastic coefficients as it is shown in the Table 3.

Table 3 numerical results for the problem (36), (39), (40), (41)

Mesh	Percent error	Number of iterations
8 x 4	$4.0\ 10^{-5}$	259
12 x 6	$7.4\ 10^{-5}$	578
16 x 8	$1.3\ 10^{-4}$	1032

For the following elastic coefficients

$$c_{11} = \exp x,\ c_{22} = \exp y,$$
$$c_{12} = 1,\ c_{66} = \left(x+1\right)\left(y+1\right) \tag{42}$$

the displacements given by (27) and corresponding moments and loads

$${}^{w}M_{x} = 1.8\ 10^{-6},\quad {}^{w}M_{y} = 1.8\ 10^{-6}\exp y,\quad {}^{w}M_{xy} = 0,$$
$${}^{v}M_{x} = -3.6\ 10^{-7}\ y\ \exp x,\quad {}^{v}M_{y} = -3.6\ 10^{-7}\ y, \tag{43}$$
$${}^{v}M_{xy} = -7.2\ 10^{-7}\ x\left(x+1\right)\left(y+1\right);$$

$$p = -1.8\ 10^{-6}\exp y,$$
$$q = 1.44\ 10^{-6}\left(2x+1\right) + 3.6\ 10^{-7}\ y\ \exp x \tag{44}$$

as it is obvious from the Table 4 that the accuracy of computation is not so good as in the previous cases. This fact is caused by the discretization error, which is in this case rather greater than in the previous cases. If we want to obtain better results we have to use better discretization scheme.

Table 4 numerical results for the problem (27), (42) – (44)

Mesh	Percent error	Number of iterations
8 x 4	1.5	118
12 x 6	2.4	238
16 x 8	4.2	978

V. Conclusion

In this paper we have elaborated iterative procedure to the numerical solution of plane orthotropic boundary inverse problems when the input data measured from two suitable states are sufficient for determination of four unknown elastic coefficients despite of the fact that we have only two governing differential equations for their determination.

We derive conditions for measured input states which secure determinability of the numerical solution.

From computed examples we can see that the errors of computed solutions depend on the discretization errors. If we want to obtain better results we have to use better discretization scheme.

This approach is possible to generalize also to plane anisotropic boundary inverse problems.

This approach is possible to use for identification of unknown elastic properties for new materials.

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