Analysis of the impact-induced two-to-one internal resonance in nonlinear doubly curved shallow panels with rectangular platform*

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Abstract – The problem of the low-velocity impact of an elastic sphere upon a nonlinear doubly curved shallow panel with rectangular platform is investigated. The approach utilized in the present paper is based on the fact that during impact only the modes strongly coupled by the two-to-one internal resonance condition are initiated. Such an approach differs from the Galerkin method, wherein resonance phenomena are not involved. Since is it assumed that shell's displacements are finite, then the local bearing of the shell and impactor's materials is neglected with respect to the shell deflection in the contact region. In other words, the Hertz's theory, which is traditionally in hand for solving impact problems, is not used in the present study; instead, the method of multiple time scales is adopted, which is used with much success for investigating vibrations of nonlinear systems subjected to the conditions of the internal resonance.

Keywords – Impact-induced internal resonance, impact response, doubly curved shallow panels with rectangular platform, method of multiple time scales.

I. Introduction

Doubly curved shallow panels under impact load are encountered frequently in practice [1]. Nonlinear behavior of some types of thin-walled doubly curved panels under large deformation is very sensitive to their parameters involving their curvatures. Thus, it has been revealed in [2, 3] that such an important nonlinear phenomenon as the occurrence of internal resonances, which are governed by the shell's parameters, is of fundamental importance in the study of large-amplitude vibrations of doubly curved shallow shells with rectangular base, simply supported at the four edges and subjected to various dynamic loads.

In spite of the fact that the impact theory is substantially developed, there is a limited number of papers devoted to the problem of impact over geometrically nonlinear shells. Literature review on this subject could be found in Kistler and Wass [4].

An analysis to predict the transient response of a thin, curved laminated plate subjected to low velocity transverse impact by a rigid object was carried out by Ramkumar and Thakar [5], in so doing the contact force history due to the impact phenomenon was assumed to be a known linear-dependent input to the analysis. The coupled governing equations, in terms of the Airy stress function and shell deformation, were solved using Fourier series expansions for the variables.

A methodology for the stability analysis of doubly curved orthotropic shells with simply supported boundary condition and under impact load from the viewpoint of nonlinear dynamics was suggested in [1]. The nonlinear governing differential equations were derived based on a Donnell-type shallow shell theory, and the displacement was expanded in terms of the eigenfunctions of the linear operator of the motion equation using the Galerkin procedure. To analyze the influence of each single mode on the response to impact loading, only one term composed of two half-waves was used in developing the governing equation, whereas the contact force was proposed a priori to be a sine function during the contact duration.

The review of papers dealing with the impact response of curved panels and shells shows that a finite element method and such commercial finite element software as ABAQUS and its modifications are the main numerical tools adopted by many researchers [6]–[20], in so doing the load due to low velocity impact was treated as an equivalent quasi-static load and Hertzian law of contact was used for finding the peak contact force.

Recently a new approach has been proposed for the analysis of the impact interactions of nonlinear doubly curved shallow shells with rectangular base under the low-velocity impact by an elastic sphere [21]. It has been as-

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sumed that the shell is simply supported and partial differential equations have been obtained in terms of shell's transverse displacement and Airy's stress function. The local bearing of the shell and impactor's materials has been neglected with respect to the shell deflection in the contact region, and therefore, the contact force is found analytically without utilizing the Hertzian contact law. The equations of motion have been reduced to a set of infinite nonlinear ordinary differential equations of the second order in time and with cubic and quadratic nonlinearities in terms of the generalized displacements. Assuming that only two natural modes of vibrations dominate and interact during the process of impact and applying the method of multiple time scales, the set of equations has been obtained, which allows one to find the time dependence of the contact force and to determine the contact duration and the maximal contact force.

Further the approach proposed by Rossikhin et al. [21] has been generalized by the authors [22] for studying the influence of the impact-induced three-to-one internal resonances on the low velocity impact response of a nonlinear doubly curved shallow shell with rectangular platform.

Such an additional nonlinear phenomenon as the internal resonance could be examined only via analytical treatment, since any of existing numerical procedures could not catch this subtle phenomenon. Moreover, impact-induced internal resonance phenomena should be studied in detail as their initiation during impact interaction may lead to the fact that the impacted shell could occur under extreme loading conditions resulting in its invisible and/or visible damage and even failure.

Because of this and to investigate the mechanism of impact-induced internal resonance, in the present paper, a semi-analytical method proposed previously [21, 22] is applied to investigate doubly curved shallow panels with a rectangular boundary and simply supported boundary condition subjected to impact load, resulting in the impact-induced two-to-one internal resonance.

II. Problem Formulation and Governing Equations

Assume that an elastic or rigid sphere of mass M moves along the z-axis towards a thin-walled doubly curved shell with thickness h, curvilinear lengths a and b, principle curvatures k_x and k_y and rectangular base, as shown in Fig. 1. Impact occurs at the moment $t = 0$ with the low velocity εV_0 at the point N with Cartesian coordinates x_0, y_0, where ε is a small dimensionless parameter.

According to the Donnell-Mushtari nonlinear shallow shell theory [23], the equations of motion in terms of lateral deflection w and Airy's stress function ϕ have the form

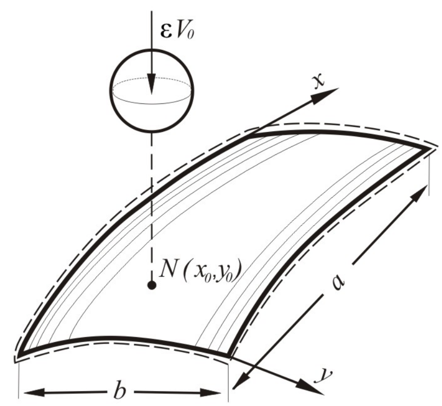

Figure 1: Geometry of the doubly curved shallow shell

$$\frac{D}{h}\left(\frac{\partial^4 w}{\partial x^4} + 2\,\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4}\right) = \frac{\partial^2 w}{\partial x^2}\,\frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 w}{\partial y^2}\,\frac{\partial^2 \phi}{\partial x^2} - 2\,\frac{\partial^2 w}{\partial x \partial y}\,\frac{\partial^2 \phi}{\partial x \partial y} + k_y\,\frac{\partial^2 \phi}{\partial x^2} + k_x\,\frac{\partial^2 \phi}{\partial y^2} + \frac{F}{h} - \rho \ddot{w}, \qquad (1)$$

$$\frac{1}{E}\left(\frac{\partial^4 \phi}{\partial x^4} + 2\,\frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial y^4}\right) = -\frac{\partial^2 w}{\partial x^2}\,\frac{\partial^2 w}{\partial y^2} + \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 - k_y\,\frac{\partial^2 w}{\partial x^2} - k_x\,\frac{\partial^2 w}{\partial y^2}, \qquad (2)$$

where $D = \frac{Eh^3}{12(1-\nu^2)}$ is the cylindrical rigidity, ρ is the density, E and ν are the elastic modulus and Poisson's ratio, respectively, t is time, $F = P(t)\delta(x - x_0)\delta(y - y_0)$ is the contact force, $P(t)$ is yet unknown function, δ is the Dirac delta function, x and y are Cartesian coordinates, overdots denote time-derivatives, $\phi(x, y)$ is the stress function which is the potential of the in-plane force resultants

$$N_x = h\,\frac{\partial^2 \phi}{\partial y^2}, \quad N_y = h\,\frac{\partial^2 \phi}{\partial x^2}, \quad N_{xy} = -h\,\frac{\partial^2 \phi}{\partial x \partial y}. \qquad (3)$$

The equation of motion of the sphere is written as

$$M\ddot{z} = -P(t) \qquad (4)$$

subjected to the initial conditions

$$z(0) = 0, \quad \dot{z}(0) = \varepsilon V_0, \qquad (5)$$

where $z(t)$ is the displacement of the sphere, in so doing

$$z(t) = w(x_0, y_0, t). \qquad (6)$$

Considering a simply supported shell with movable edges, the following conditions should be imposed at each edge:
at $x = 0,\ a$

$$w = 0,\ \ \int_0^b N_{xy} dy = 0,\ \ N_x = 0,\ \ M_x = 0, \quad (7)$$

and at $y = 0,\ b$

$$w = 0,\ \ \int_0^a N_{xy} dx = 0,\ \ N_y = 0,\ \ M_y = 0, \quad (8)$$

where M_x and M_y are the moment resultants.

The suitable trial function that satisfies the geometric boundary conditions is

$$w(x, y, t) = \sum_{p=1}^{\tilde{p}} \sum_{q=1}^{\tilde{q}} \xi_{pq}(t)\ \sin\left(\frac{p\pi x}{a}\right) \sin\left(\frac{q\pi y}{b}\right), \quad (9)$$

where p and q are the number of half-waves in x and y directions, respectively, and $\xi_{pq}(t)$ are the generalized coordinates. Moreover, $\tilde{p}$ and $\tilde{q}$ are integers indicating the number of terms in the expansion.

Substituting (9) in (6) and using (4), we obtain

$$P(t) = -M \sum_{p=1}^{\tilde{p}} \sum_{q=1}^{\tilde{q}} \ddot{\xi}_{pq}(t) \sin\left(\frac{p\pi x_0}{a}\right) \sin\left(\frac{q\pi y_0}{b}\right). \quad (10)$$

In order to find the solution of the set of equations (1) and (2), it is necessary first to obtain the solution of (2). For this purpose, let us substitute (9) in the right-hand side of (2) and seek the solution of the equation obtained in the form

$$\phi(x, y, t) = \sum_{m=1}^{\tilde{m}} \sum_{n=1}^{\tilde{n}} A_{mn}(t) \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right), \quad (11)$$

where $A_{mn}(t)$ are yet unknown functions.

Substituting (9) and (11) in (2) and using the orthogonality conditions of sines within the segments $0 \le x \le a$ and $0 \le y \le b$, we have

$$A_{mn}(t) = \frac{E}{\pi^2} K_{mn} \xi_{mn}(t) + \frac{4E}{a^3 b^3} \left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)^{-2} \times \sum_k \sum_l \sum_p \sum_q B_{pqklmn} \xi_{pq}(t) \xi_{kl}(t), \quad (12)$$

where

$$B_{pqklmn} = pqkl B^{(2)}_{pqklmn} - p^2 l^2 B^{(1)}_{pqklmn},$$

$$B^{(1)}_{pqklmn} = \int_0^a \int_0^b \sin\left(\frac{p\pi x}{a}\right) \sin\left(\frac{q\pi y}{b}\right) \sin\left(\frac{k\pi x}{a}\right) \times \sin\left(\frac{l\pi y}{b}\right) \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) dx dy,$$

$$B^{(2)}_{pqklmn} = \int_0^a \int_0^b \cos\left(\frac{p\pi x}{a}\right) \cos\left(\frac{q\pi y}{b}\right) \cos\left(\frac{k\pi x}{a}\right) \times \cos\left(\frac{l\pi y}{b}\right) \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) dx dy,$$

$$K_{mn} = \left(k_y \frac{m^2}{a^2} + k_x \frac{n^2}{b^2}\right)^2 \left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)^{-2}.$$

Substituting then (9)–(12) in (1) and using the orthogonality condition of sines within the segments $0 \le x \le a$ and $0 \le y \le b$, we obtain an infinite set of coupled nonlinear ordinary differential equations of the second order in time for defining the generalized coordinates

$$\begin{aligned}\ddot{\xi}_{mn}(t) &+ \Omega^2_{mn} \xi_{mn}(t) + \frac{8\pi^2 E}{a^3 b^3 \rho} \\ &\times \sum_p \sum_q \sum_k \sum_l B_{pqklmn} \left(K_{kl} - \frac{1}{2} K_{mn}\right) \\ &\times \xi_{pq}(t) \xi_{kl}(t) + \frac{32\pi^4 E}{a^6 b^6 \rho} \\ &\times \sum_r \sum_s \sum_i \sum_j \sum_k \sum_l \sum_p \sum_q B_{rsijmn} \\ &\times B_{pqklij} \xi_{rs}(t) \xi_{pq}(t) \xi_{kl}(t) \\ &+ \frac{4M}{ab\rho h} \sin\left(\frac{m\pi x_0}{a}\right) \sin\left(\frac{n\pi y_0}{b}\right) \\ &\times \sum_p \sum_q \ddot{\xi}_{pq}(t) \sin\left(\frac{p\pi x_0}{a}\right) \sin\left(\frac{q\pi y_0}{b}\right) = 0, \end{aligned} \quad (13)$$

where Ω_{mn} is the natural frequency of the mnth mode of the shell vibration defined as

$$\Omega^2_{mn} = \frac{E}{\rho} \left[\frac{\pi^4 h^2}{12(1-\nu^2)} \left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)^2 + K_{mn}\right]. \quad (14)$$

The last term in each equation from (13) describes the influence of the coupled impact interaction of the target with the impactor of the mass M applied at the point with the coordinates x_0, y_0.

It is known [24, 25] that during nonstationary excitation of thin bodies not all possible modes of vibration would be excited. Moreover, the modes which are strongly coupled by any of the so-called internal resonance conditions are initiated and dominate in the process of vibration, in so doing the types of modes to be excited are dependent on the character of the external excitation.

Thus, in order to study the additional nonlinear phenomenon induced by the coupled impact interaction due to (13), we suppose that only two natural modes of vibrations are excited during the process of impact, namely, $\Omega_{\alpha\beta}$ and $\Omega_{\gamma\delta}$.

Then the set of equations (13) is reduced to the following two nonlinear differential equations:

$$p_{11} \ddot{\xi}_{\alpha\beta} + p_{12} \ddot{\xi}_{\gamma\delta} + \Omega^2_{\alpha\beta} \xi_{\alpha\beta} + p_{13} \xi^2_{\alpha\beta} + p_{14} \xi^2_{\gamma\delta} + p_{15} \xi_{\alpha\beta} \xi_{\gamma\delta} + p_{16} \xi^3_{\alpha\beta} + p_{17} \xi_{\alpha\beta} \xi^2_{\gamma\delta} = 0, \quad (15)$$

$$p_{21}\ddot{\xi}_{\alpha\beta} + p_{22}\ddot{\xi}_{\gamma\delta} + \Omega^2_{\gamma\delta}\xi_{\gamma\delta} + p_{23}\xi^2_{\gamma\delta} + p_{24}\xi^2_{\alpha\beta} + p_{25}\xi_{\alpha\beta}\xi_{\gamma\delta} + p_{26}\xi^3_{\gamma\delta} + p_{27}\xi^2_{\alpha\beta}\xi_{\gamma\delta} = 0, \quad (16)$$

where coefficients p_{ij} $(i, j = 1, 2)$ depending on impactor's mass, the coordinates of the impact point, as well as on the parameters of the target and the numbers of the induced modes have the following form:

$$p_{11} = 1 + \frac{4M}{\rho hab}\, s_1^2, \quad p_{22} = 1 + \frac{4M}{\rho hab}\, s_2^2,$$

$$p_{12} = p_{21} = \frac{4M}{\rho hab}\, s_1 s_2,$$

$$s_1 = \sin\left(\frac{\alpha\pi x_0}{a}\right)\sin\left(\frac{\beta\pi y_0}{b}\right),$$

$$s_2 = \sin\left(\frac{\gamma\pi x_0}{a}\right)\sin\left(\frac{\delta\pi y_0}{b}\right),$$

while all other coefficients p_{ij} $(i = 1, 2,\ j = 3, 4, ..., 7)$ governed only by the shell parameters and the numbers of two interacting impact-induced modes are presented in Appendix.

III. Method of Solution

In order to solve a set of two nonlinear equations (15) and (16), we apply the method of multiple time scales [26] via the following expansions:

$$\xi_{ij}(t) = \varepsilon X^1_{ij}(T_0, T_1) + \varepsilon^2 X^2_{ij}(T_0, T_1), \quad (17)$$

where $ij = \alpha\beta$ or $\gamma\delta$, $T_n = \varepsilon^n t$ are new independent variables, among them: $T_0 = t$ is a fast scale characterizing motions with the natural frequencies, and $T_1 = \varepsilon t$ is a slow scale characterizing the modulation of the amplitudes and phases of the modes with nonlinearity.

Considering that

$$\frac{d^2}{dt^2}\, \xi_{ij} = \varepsilon\left(D_0^2 X^1_{ij}\right) + \varepsilon^2\left(D_0^2 X^2_{ij} + 2\varepsilon D_0 D_1 X^1_{ij}\right),$$

where $ij = \alpha\beta$ or $\gamma\delta$, and $D_i^n = \partial^n/\partial T_i^n$ $(n = 1, 2,\ i = 0, 1)$, and substituting the proposed solution (17) in (15) and (16), after equating the coefficients at like powers of ε to zero, we are led to a set of recurrence equations to various orders:

to order ε

$$p_{11}D_0^2X_1^1 + p_{12}D_0^2X_2^1 + \Omega_1^2X_1^1 = 0, \quad (18)$$

$$p_{21}D_0^2X_1^1 + p_{22}D_0^2X_2^1 + \Omega_2^2X_2^1 = 0; \quad (19)$$

to order ε^2

$$p_{11}D_0^2X_1^2 + p_{12}D_0^2X_2^2 + \Omega_1^2X_1^2 = -2p_{11}D_0D_1X_1^1 - 2p_{12}D_0D_1X_2^1 - p_{13}(X_1^1)^2 - p_{14}(X_2^1)^2 - p_{15}X_1^1X_2^1, \quad (20)$$

$$p_{21}D_0^2X_1^2 + p_{22}D_0^2X_2^2 + \Omega_2^2X_2^2 = -2p_{21}D_0D_1X_1^1 - 2p_{22}D_0D_1X_2^1 - p_{23}(X_1^1)^2 - p_{24}(X_2^1)^2 - p_{25}X_1^1X_2^1, \quad (21)$$

where for simplicity is it denoted $X_1^1 = X^1_{\alpha\beta}$, $X_2^1 = X^1_{\gamma\delta}$, $X_1^2 = X^2_{\alpha\beta}$, $X_2^2 = X^2_{\gamma\delta}$, $\Omega_1 = \Omega_{\alpha\beta}$, and $\Omega_2 = \Omega_{\gamma\delta}$.

A. Solution of Equations at Order of ε

Following Rossikhin et al. [27], we seek the solution of (18) and (19) in the form:

$$X_1^1 = A_1(T_1)\, e^{i\omega_1 T_0} + A_2(T_1)\, e^{i\omega_2 T_0} + \text{cc}, \quad (22)$$

$$X_2^1 = \alpha_1 A_1(T_1)\, e^{i\omega_1 T_0} + \alpha_2 A_2(T_1)\, e^{i\omega_2 T_0} + \text{cc}, \quad (23)$$

where $A_1(T_1)$ and $A_2(T_1)$ are unknown complex functions, cc is the complex conjugate part to the preceding terms, and $\bar{A}_1(T_1)$ and $\bar{A}_2(T_1)$ are their complex conjugates, ω_1 and ω_2 are unknown frequencies of the coupled process of impact interaction of the impactor and the target, and α_1 and α_2 are yet unknown coefficients.

Substituting (22) and (23) in (18) and (19) and gathering the terms with $e^{i\omega_1 T_0}$ and $e^{i\omega_2 T_0}$ yields

$$\left(-p_{11}\omega_1^2 - p_{12}\alpha_1\omega_1^2 + \Omega_1^2\right)A_1e^{i\omega_1T_0} + \left(-p_{11}\omega_2^2 - p_{12}\alpha_2\omega_2^2 + \Omega_1^2\right)A_2e^{i\omega_2T_0} + \text{cc} = 0, \quad (24)$$

$$\left(-p_{21}\omega_1^2 - p_{22}\alpha_1\omega_1^2 + \alpha_1\Omega_2^2\right)A_1e^{i\omega_1T_0} + \left(-p_{21}\omega_2^2 - p_{22}\alpha_2\omega_2^2 + \Omega_2^2\alpha_2\right)A_2e^{i\omega_2T_0} + \text{cc} = 0. \quad (25)$$

In order to satisfy equations (24) and (25), it is a need to vanish to zero each bracket in these equations. As a result, from four different brackets we have

$$\alpha_1 = -\frac{p_{11}\omega_1^2 - \Omega_1^2}{p_{12}\omega_1^2}, \quad (26)$$

$$\alpha_1 = -\frac{p_{21}\omega_1^2}{p_{22}\omega_1^2 - \Omega_2^2}, \quad (27)$$

$$\alpha_2 = -\frac{p_{11}\omega_2^2 - \Omega_1^2}{p_{12}\omega_2^2}, \quad (28)$$

$$\alpha_2 = -\frac{p_{21}\omega_2^2}{p_{22}\omega_2^2 - \Omega_2^2}. \quad (29)$$

Since the left-hand side parts of relationships (26) and (27), as well as (28) and (29) are equal, then their right-hand side parts should be equal as well. Now equating the corresponding right-hand side parts of (26), (27) and (28), (29) we are led to one and the same characteristic equation for determining the frequencies ω_1 and ω_2:

$$\left(\Omega_1^2 - p_{11}\omega^2\right)\left(\Omega_2^2 - p_{22}\omega^2\right) - p_{12}^2\omega^4 = 0, \quad (30)$$

whence it follows that

$$\omega^2_{1,2} = \frac{\left(p_{22}\Omega_1^2 + p_{11}\Omega_2^2\right) \pm \sqrt{\Delta}}{2\left(p_{11}p_{22} - p_{12}^2\right)}, \quad (31)$$

$$\Delta = \left(p_{22}\Omega_1^2 - p_{11}\Omega_2^2\right)^2 + 4\,\Omega_1^2\Omega_2^2 p_{12}^2.$$

Reference to relationships (31) shows that as the impactor mass $M \to 0$, the frequencies ω_1 and ω_2 tend to the natural frequencies of the shell vibrations Ω_1 and Ω_2, respectively. Coefficients s_1 and s_2 depend on the numbers of the natural modes involved in the process of impact interaction, $\alpha\beta$ and $\gamma\delta$, and on the coordinates of the contact force application $x_0,\ y_0$, resulting in the fact that their particular combinations could vanish coefficients s_1 and s_2 and, thus, coefficients $p_{12} = p_{21} = 0$.

B. Solution of Equations at Order of ε^2 for the Case of Impact-Induced 2:1 Internal Resonance

Now substituting (22) and (23) in (20) and (21), we obtain

$$\begin{aligned} &p_{11}D_0^2X_1^2 + p_{12}D_0^2X_2^2 + \Omega_1^2X_1^2 \\ &\quad = -2i\omega_1(p_{11}+\alpha_1 p_{12})e^{i\omega_1T_0}D_1A_1 \\ &\quad -2i\omega_2(p_{11}+\alpha_2 p_{12})e^{i\omega_2T_0}D_1A_2 \\ &\quad -(p_{13}+\alpha_1^2p_{14}+\alpha_1p_{15})A_1\left[A_1e^{2i\omega_1T_0}+\bar{A}_1\right] \\ &\quad -(p_{13}+\alpha_2^2p_{14}+\alpha_2p_{15})A_2\left[A_2e^{2i\omega_2T_0}+\bar{A}_2\right] \\ &\quad -2\left[p_{13}+\alpha_1\alpha_2p_{14}+(\alpha_1+\alpha_2)p_{15}\right]A_1 \\ &\quad \times\left[A_2e^{i(\omega_1+\omega_2)T_0}+\bar{A}_2e^{i(\omega_1-\omega_2)T_0}\right]+\text{cc}, \end{aligned} \tag{32}$$

$$\begin{aligned} &p_{21}D_0^2X_1^2 + p_{22}D_0^2X_2^2 + \Omega_2^2X_2^2 \\ &\quad = -2i\omega_1(p_{21}+\alpha_1 p_{22})e^{i\omega_1T_0}D_1A_1 \\ &\quad -2i\omega_2(p_{21}+\alpha_2 p_{22})e^{i\omega_2T_0}D_1A_2 \\ &\quad -(p_{23}+\alpha_1^2p_{24}+\alpha_1p_{25})A_1\left[A_1e^{2i\omega_1T_0}+\bar{A}_1\right] \\ &\quad -(p_{23}+\alpha_2^2p_{24}+\alpha_2p_{25})A_2\left[A_2e^{2i\omega_2T_0}+\bar{A}_2\right] \\ &\quad -2\left[p_{23}+\alpha_1\alpha_2p_{24}+(\alpha_1+\alpha_2)p_{25}\right]A_1 \\ &\quad \times\left[A_2e^{i(\omega_1+\omega_2)T_0}+\bar{A}_2e^{i(\omega_1-\omega_2)T_0}\right]+\text{cc}. \end{aligned} \tag{33}$$

Reference to equations (32) and (33) shows that the following impact-induced two-to-one internal resonance could occur:

$$\omega_1 = 2\omega_2. \tag{34}$$

Condition (34) could be initiated by the impactor during the impact interaction with the target, i.e., the doubly curved shallow panel, since the frequencies ω_1 and ω_2 depend not only on the natural frequencies of two induced modes of the vibration of the target, but they depend also by the mass of the striker and the position of the impact, what it is evident from (31).

Thus, when the frequencies ω_1 and ω_2 are coupled by the impact-induced two-to-one internal resonance (34), equations (32) and (33) could be rewritten in the following form:

$$\begin{aligned} &p_{11}D_0^2X_1^2 + p_{12}D_0^2X_2^2 + \Omega_1^2X_1^2 \\ &\quad = B_1\exp(i\omega_1T_0) + B_2\exp(i\omega_2T_0) \\ &\quad +\text{Reg}+\text{cc}, \end{aligned} \tag{35}$$

$$\begin{aligned} &p_{21}D_0^2X_1^2 + p_{22}D_0^2X_2^2 + \Omega_2^2X_2^2 \\ &\quad = B_3\exp(i\omega_1T_0) + B_4\exp(i\omega_2T_0) \\ &\quad +\text{Reg}+\text{cc}, \end{aligned} \tag{36}$$

where all regular terms are designated by Reg, and

$$B_1 = -2i\Omega_1^2\omega_1^{-1}D_1A_1 - \left(p_{13}+\alpha_2^2p_{14}+\alpha_2p_{15}\right)A_2^2,$$

$$\begin{aligned} B_2 = -2i\Omega_1^2\omega_2^{-1}D_1A_2 - 2[&p_{13}+\alpha_1\alpha_2p_{14} \\ &+(\alpha_1+\alpha_2)\,p_{15}]A_1\bar{A}_2, \end{aligned}$$

$$B_3 = -2i\Omega_2^2\omega_1^{-1}\alpha_1D_1A_1 - \left(p_{23}+\alpha_2^2p_{24}+\alpha_2p_{25}\right)A_2^2,$$

$$\begin{aligned} B_4 = -2i\Omega_2^2\omega_2^{-1}\alpha_2D_1A_2 - 2[&p_{23}+\alpha_1\alpha_2p_{24} \\ &+(\alpha_1+\alpha_2)\,p_{25}]A_1\bar{A}_2. \end{aligned}$$

Let us show that the terms with the exponents $\exp(\pm i\omega_iT_0)$ $(i = 1, 2)$ produce circular terms. For this purpose we choose a particular solution in the form

$$\begin{aligned} X_{1\,p}^2 &= C_1\exp(i\omega_1T_0)+\text{cc}, \\ X_{2\,p}^2 &= C_2\exp(i\omega_1T_0)+\text{cc}, \end{aligned} \tag{37}$$

or

$$\begin{aligned} X_{1\,p}^2 &= C_1'\exp(i\omega_2T_0)+\text{cc}, \\ X_{2\,p}^2 &= C_2'\exp(i\omega_2T_0)+\text{cc}, \end{aligned} \tag{38}$$

where C_1, C_2 and C_1', C_2' are arbitrary constants.

Substituting the proposed solution in (35) and (36) or in (37) and (38), we are led to the following sets of equations, respectively:

$$\begin{cases} p_{12}\omega_1^2\left(\alpha_1C_1 - C_2\right) = B_1, \\ p_{21}\omega_1^2\left(-C_1 + \frac{1}{\alpha_1}\,C_2\right) = B_3, \end{cases} \tag{39}$$

or

$$\begin{cases} p_{12}\omega_2^2\left(\alpha_2C_1' - C_2'\right) = B_2, \\ p_{21}\omega_2^2\left(-C_1' + \frac{1}{\alpha_2}\,C_2'\right) = B_4. \end{cases} \tag{40}$$

From the sets of equations (39) and (40) it is evident that the determinants comprised from the coefficients standing at C_1, C_2 and C_1', C_2' are equal to zero, therefore, it is impossible to determine the arbitrary constants C_1, C_2 and C_1', C_2' of the particular solutions (37) and (38), what proves the above proposition concerning the circular terms.

In order to eliminate the circular terms, the terms proportional to $e^{i\omega_1T_0}$ and $e^{i\omega_2T_0}$ should be vanished to zero putting $B_i = 0\,(i = 1, 2, 3, 4)$. So we obtain four equations for defining two unknown amplitudes $A_1(t)$ and $A_2(t)$. However, it is possible to show that not all of these four equations are linear independent from each other.

For this purpose, let us first apply the operators $(p_{22}D_0^2 + \Omega_2^2)$ and $(-p_{12}D_0^2)$ to (35) and (36), respectively, and then add the resulting equations. This procedure will allow us to eliminate X_2^2. If we apply the operators $(-p_{12}D_0^2)$ and $(p_{11}D_0^2+\Omega_1^2)$ to (35) and (36), respectively, and then add the resulting equations. This procedure will allow us to eliminate X_1^2. Thus, we obtain

$$\begin{aligned}&\left[(p_{11}p_{22}-p_{12}^2)D_0^4+(p_{11}\Omega_2^2+p_{22}\Omega_1^2)D_0^2+\Omega_1^2\Omega_2^2\right]X_1^2\\&\quad=\left[(p_{22}D_0^2+\Omega_2^2)B_1-p_{12}D_0^2B_3\right]\exp(i\omega_1T_0)\\&\quad+\left[(p_{22}D_0^2+\Omega_2^2)B_2-p_{12}D_0^2B_4\right]\exp(i\omega_2T_0)\\&\quad+\mathrm{Reg}+\mathrm{cc},\end{aligned} \tag{41}$$

$$\begin{aligned}&\left[(p_{11}p_{22}-p_{12}^2)D_0^4+(p_{11}\Omega_2^2+p_{22}\Omega_1^2)D_0^2+\Omega_1^2\Omega_2^2\right]X_2^2\\&\quad=\left[-p_{12}D_0^2B_1+(p_{11}D_0^2+\Omega_1^2)B_3\right]\exp(i\omega_1T_0)\\&\quad+\left[-p_{12}D_0^2B_2+(p_{11}D_0^2+\Omega_1^2)B_4\right]\exp(i\omega_2T_0)\\&\quad+\mathrm{Reg}+\mathrm{cc}.\end{aligned} \tag{42}$$

To eliminate the circular terms from equations (41) and (42), it is necessary to vanish to zero the terms in each square bracket. As a result we obtain

$$\begin{cases}(\Omega_2^2-p_{22}\omega_1^2)B_1+p_{12}\omega_1^2B_3=0\\p_{12}\omega_1^2B_1+(\Omega_1^2-p_{11}\omega_1^2)B_3=0\end{cases} \tag{43}$$

and

$$\begin{cases}(\Omega_2^2-p_{22}\omega_2^2)B_2+p_{12}\omega_2^2B_4=0\\p_{12}\omega_2^2B_2+(\Omega_1^2-p_{11}\omega_2^2)B_4=0\end{cases} \tag{44}$$

From equations (43) and (44) it is evident that the determinant of each set of equations is reduced to the characteristic equation (30), whence it follows that each pair of equations is linear dependent, therefore for further treatment we should take only one equation from each pair in order that these two chosen equations are to be linear independent. Thus, for example, taking the first equations from each pair and considering relationships (27) and (29), we have

$$B_1+\alpha_1B_3=0, \tag{45}$$

$$B_3+\alpha_2B_4=0. \tag{46}$$

Substituting values of B_1-B_4 in (45) and (46), we obtain the following solvability equations:

$$2i\omega_1D_1A_1+\frac{b_1}{k_1}A_2^2=0, \tag{47}$$

$$2i\omega_2D_1A_2+\frac{b_2}{k_2}A_1\bar{A}_2=0, \tag{48}$$

where

$$k_i=\frac{\Omega_1^2+\alpha_i^2\Omega_2^2}{\omega_i^2}\quad(i=1,2),$$

$$b_1=p_{13}+\alpha_2^2p_{14}+\alpha_2p_{15}+\alpha_1(p_{23}+\alpha_2^2p_{24}+\alpha_2p_{25}),$$

$$\begin{aligned}b_2=2\{&p_{13}+\alpha_1\alpha_2p_{14}+(\alpha_1+\alpha_2)p_{15}\\&+\alpha_2[p_{23}+\alpha_1\alpha_2p_{24}+(\alpha_1+\alpha_2)p_{25}]\}.\end{aligned}$$

Let us multiply equations (47) and (48) by $\bar{A}_1$ and $\bar{A}_2$, respectively, and find their complex conjugates. After adding every pair of the mutually adjoint equations with each other and subtracting one from another, as a result we obtain

$$2i\omega_1\left(\bar{A}_1D_1A_1-A_1D_1\bar{A}_1\right)+\frac{b_1}{k_1}\left(A_2^2\bar{A}_1+\bar{A}_2^2A_1\right)=0, \tag{49}$$

$$2i\omega_1\left(\bar{A}_1D_1A_1+A_1D_1\bar{A}_1\right)+\frac{b_1}{k_1}\left(A_2^2\bar{A}_1-\bar{A}_2^2A_1\right)=0, \tag{50}$$

$$2i\omega_2\left(\bar{A}_2D_1A_2-A_2D_1\bar{A}_2\right)+\frac{b_2}{k_2}\left(A_1\bar{A}_2^2+\bar{A}_1A_2^2\right)=0, \tag{51}$$

$$2i\omega_2\left(\bar{A}_2D_1A_2+A_2D_1\bar{A}_2\right)+\frac{b_2}{k_2}\left(A_1\bar{A}_2^2-\bar{A}_1A_2^2\right)=0. \tag{52}$$

Representing $A_1(T_1)$ and $A_2(T_1)$ in equations (49)–(52) in the polar form

$$A_i(T_1)=a_i(T_1)e^{i\varphi_i(T_1)}\quad(i=1,2), \tag{53}$$

we are led to the system of four nonlinear differential equations in $a_1(T_1)$, $a_2(T_1)$, $\varphi_1(T_1)$, and $\varphi_2(T_1)$

$$(a_1^2)^{\cdot}=-\frac{b_1}{k_1\omega_1}\,a_1a_2^2\sin\delta, \tag{54}$$

$$\dot{\varphi}_1-\frac{b_1}{2k_1\omega_1}\,a_1^{-1}a_2^2\cos\delta=0, \tag{55}$$

$$(a_2^2)^{\cdot}=\frac{b_2}{k_2\omega_2}\,a_1a_2^2\sin\delta, \tag{56}$$

$$\dot{\varphi}_2-\frac{b_2}{2k_2\omega_2}\,a_1\cos\delta=0, \tag{57}$$

where $\delta=2\varphi_2-\varphi_1$, and a dot denotes differentiation with respect to T_1.

From equations (54) and (56) we could find that

$$\frac{b_2}{k_2\omega_2}(a_1^2)^{\cdot}+\frac{b_1}{k_1\omega_1}(a_2^2)^{\cdot}=0 \tag{58}$$

Multiplying equation (58) by MV_0 and integrating over T_1, we obtain the first integral of the set of equations (54)–(57), which is the law of conservation of energy,

$$MV_0\left(\frac{b_2}{k_2\omega_2}\,a_1^2+\frac{b_1}{k_1\omega_1}\,a_2^2\right)=K_0, \tag{59}$$

where K_0 is the initial energy.

Considering that $K_0=\frac{1}{2}MV_0^2$, equation (59) is reduced to the following form:

$$\frac{b_2}{k_2\omega_2}\,a_1^2+\frac{b_1}{k_1\omega_1}\,a_2^2=\frac{V_0}{2}. \tag{60}$$

Let us introduce into consideration a new function $\xi(T_1)$ in the following form:

$$a_1^2=\frac{k_2\omega_2}{b_2}\,E_0\xi(T_1),\quad a_2^2=\frac{k_1\omega_1}{b_1}\,E_0\left[1-\xi(T_1)\right], \tag{61}$$

where $E_0=V_0/2$.

sumed that the shell is simply supported and partial differential equations have been obtained in terms of shell's transverse displacement and Airy's stress function. The local bearing of the shell and impactor's materials has been neglected with respect to the shell deflection in the contact region, and therefore, the contact force is found analytically without utilizing the Hertzian contact law. The equations of motion have been reduced to a set of infinite nonlinear ordinary differential equations of the second order in time and with cubic and quadratic nonlinearities in terms of the generalized displacements. Assuming that only two natural modes of vibrations dominate and interact during the process of impact and applying the method of multiple time scales, the set of equations has been obtained, which allows one to find the time dependence of the contact force and to determine the contact duration and the maximal contact force.

Further the approach proposed by Rossikhin et al. [21] has been generalized by the authors [22] for studying the influence of the impact-induced three-to-one internal resonances on the low velocity impact response of a nonlinear doubly curved shallow shell with rectangular platform.

Such an additional nonlinear phenomenon as the internal resonance could be examined only via analytical treatment, since any of existing numerical procedures could not catch this subtle phenomenon. Moreover, impact-induced internal resonance phenomena should be studied in detail as their initiation during impact interaction may lead to the fact that the impacted shell could occur under extreme loading conditions resulting in its invisible and/or visible damage and even failure.

Because of this and to investigate the mechanism of impact-induced internal resonance, in the present paper, a semi-analytical method proposed previously [21, 22] is applied to investigate doubly curved shallow panels with a rectangular boundary and simply supported boundary condition subjected to impact load, resulting in the impact-induced two-to-one internal resonance.

II. Problem Formulation and Governing Equations

Assume that an elastic or rigid sphere of mass M moves along the z-axis towards a thin-walled doubly curved shell with thickness h, curvilinear lengths a and b, principle curvatures k_x and k_y and rectangular base, as shown in Fig. 1. Impact occurs at the moment $t = 0$ with the low velocity εV_0 at the point N with Cartesian coordinates x_0, y_0, where ε is a small dimensionless parameter.

According to the Donnell-Mushtari nonlinear shallow shell theory [23], the equations of motion in terms of lateral

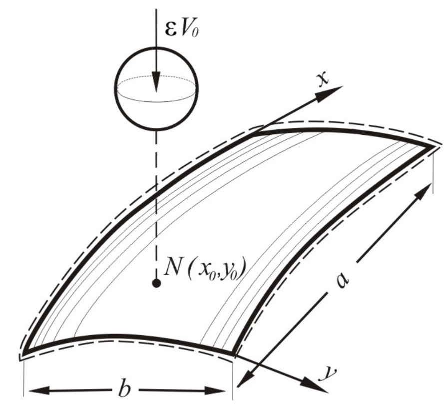

Figure 1: Geometry of the doubly curved shallow shell

deflection w and Airy's stress function ϕ have the form

$$\frac{D}{h}\left(\frac{\partial^4 w}{\partial x^4} + 2\,\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4}\right) = \frac{\partial^2 w}{\partial x^2}\,\frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 w}{\partial y^2}\,\frac{\partial^2 \phi}{\partial x^2} - 2\,\frac{\partial^2 w}{\partial x \partial y}\,\frac{\partial^2 \phi}{\partial x \partial y} + k_y\,\frac{\partial^2 \phi}{\partial x^2} + k_x\,\frac{\partial^2 \phi}{\partial y^2} + \frac{F}{h} - \rho \ddot{w}, \tag{1}$$

$$\frac{1}{E}\left(\frac{\partial^4 \phi}{\partial x^4} + 2\,\frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial y^4}\right) = -\frac{\partial^2 w}{\partial x^2}\,\frac{\partial^2 w}{\partial y^2} + \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 - k_y\,\frac{\partial^2 w}{\partial x^2} - k_x\,\frac{\partial^2 w}{\partial y^2}, \tag{2}$$

where $D = \frac{Eh^3}{12(1-\nu^2)}$ is the cylindrical rigidity, ρ is the density, E and ν are the elastic modulus and Poisson's ratio, respectively, t is time, $F = P(t)\delta(x - x_0)\delta(y - y_0)$ is the contact force, $P(t)$ is yet unknown function, δ is the Dirac delta function, x and y are Cartesian coordinates, overdots denote time-derivatives, $\phi(x, y)$ is the stress function which is the potential of the in-plane force resultants

$$N_x = h\,\frac{\partial^2 \phi}{\partial y^2}, \quad N_y = h\,\frac{\partial^2 \phi}{\partial x^2}, \quad N_{xy} = -h\,\frac{\partial^2 \phi}{\partial x \partial y}. \tag{3}$$

The equation of motion of the sphere is written as

$$M\ddot{z} = -P(t) \tag{4}$$

subjected to the initial conditions

$$z(0) = 0, \quad \dot{z}(0) = \varepsilon V_0, \tag{5}$$

where $z(t)$ is the displacement of the sphere, in so doing

$$z(t) = w(x_0, y_0, t). \tag{6}$$

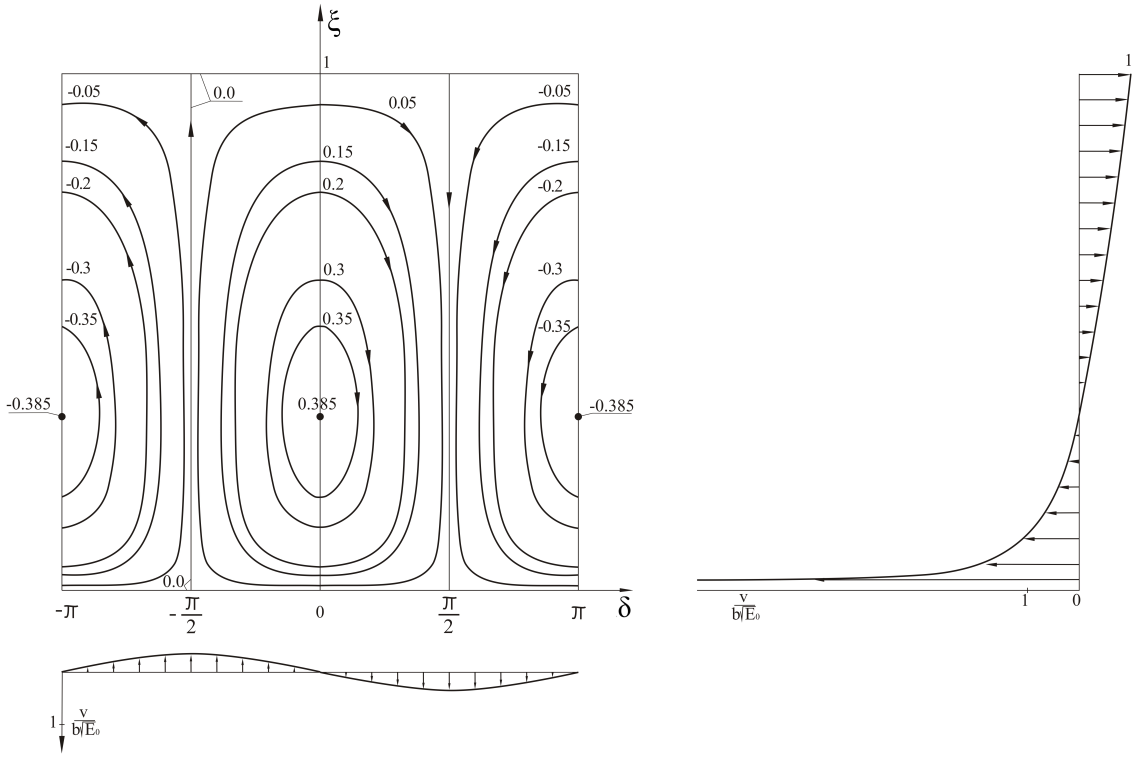

Figure 2: Phase portrait: $\Omega_1 = 2\Omega_2$

velocity along the vertical lines $\delta = \pm\pi n$ $(n = 0, 1, 2, ...)$ has the aperiodic character, while in the vicinity of the line $\xi = 1/3$ it possesses the periodic character.

IV. Initial Conditions

In order to construct the final solution of the problem under consideration, i.e. to solve the set of equations (54)–(57) involving the functions $a_1(T_1)$, $a_2(T_1)$, or $\xi(T_1)$, as well as $\varphi_1(T_1)$, and $\varphi_2(T_1)$, or $\delta(T_1)$, it is necessary to use the initial conditions

$$w(x, y, 0) = 0, \tag{72}$$

$$\dot{w}(x_0, y_0, 0) = \varepsilon V_0, \tag{73}$$

$$\frac{b_2}{k_2\omega_2}\, a_1^2(0) + \frac{b_1}{k_1\omega_1}\, a_2^2(0) = E_0. \tag{74}$$

The two-term relationship for the displacement w (9) within an accuracy of ε according to (17) has the form

$$w(x,y,t) = \epsilon\left[X^1_{\alpha\beta}(T_0,T_1)\sin\left(\tfrac{\alpha\pi x}{a}\right)\sin\left(\tfrac{\beta\pi y}{b}\right) + X^1_{\gamma\delta}(T_0,T_1)\sin\left(\tfrac{\gamma\pi x}{a}\right)\sin\left(\tfrac{\delta\pi y}{b}\right)\right] + O(\varepsilon^2). \tag{75}$$

Substituting (22) and (23) in (75) with due account for (53) yields

$$\begin{aligned} w(x,y,t) &= 2\epsilon\Big\{a_1(\varepsilon t)\cos\left[\omega_1 t + \varphi_1(\varepsilon t)\right] \\ &+a_2(\varepsilon t)\cos\left[\omega_2 t + \varphi_2(\varepsilon t)\right]\Big\}\sin\left(\tfrac{\alpha\pi x}{a}\right)\sin\left(\tfrac{\beta\pi y}{b}\right) \\ &+2\epsilon\Big\{\alpha_1 a_1(\varepsilon t)\cos\left[\omega_1 t + \varphi_1(\varepsilon t)\right] \\ &+\alpha_2 a_2(\varepsilon t)\cos[\omega_2 t+\varphi_2(\varepsilon t)]\Big\}\sin\left(\tfrac{\gamma\pi x}{a}\right)\sin\left(\tfrac{\delta\pi y}{b}\right)\Big] \\ &+O(\varepsilon^2). \end{aligned} \tag{76}$$

Differentiating (76) with respect to time t and limiting ourselves by the terms of the order of ε, we could find the velocity of the shell at the point of impact as follows

$$\begin{aligned} \dot{w}(x_0,y_0,t) &= -2\epsilon\Big\{\omega_1 a_1(\varepsilon t)\sin\left[\omega_1 t + \varphi_1(\varepsilon t)\right] \\ &+\omega_2 a_2(\varepsilon t)\sin\left[\omega_2 t + \varphi_2(\varepsilon t)\right]\Big\}s_1 \\ &-2\epsilon\Big\{\alpha_1\omega_1 a_1(\varepsilon t)\sin\left[\omega_1 t + \varphi_1(\varepsilon t)\right] \\ &+\alpha_2\omega_2 a_2(\varepsilon t)\sin\left[\omega_2 t + \varphi_2(\varepsilon t)\right]\Big\}s_2 + O(\varepsilon^2). \end{aligned} \tag{77}$$

Substituting (76) in the first initial condition (72) yields

$$a_1(0)\cos\varphi_1(0) + a_2(0)\cos\varphi_2(0) = 0, \tag{78}$$

$$\alpha_1 a_1(0)\cos\varphi_1(0) + \alpha_2 a_2(0)\cos\varphi_2(0) = 0. \tag{79}$$

From equations (78) and (79) we find that

$$\cos\varphi_1(0) = 0, \quad \cos\varphi_2(0) = 0, \tag{80}$$

whence it follows that

$$\varphi_1(0) = \pm\frac{\pi}{2}, \quad \varphi_2(0) = \pm\frac{\pi}{2}, \tag{81}$$

and

$$\cos\delta_0 = \cos\left[2\varphi_2(0) - \varphi_1(0)\right] = 0, \tag{82}$$

i.e.,

$$\delta_0 = \pm\frac{\pi}{2} \pm 2\pi n. \tag{83}$$

The signs in (81) should be chosen considering the fact that the initial amplitudes are positive values, i.e. $a_1(0) > 0$ and $a_2(0) > 0$. Assume for definiteness that

$$\varphi_1(0) = -\frac{\pi}{2}, \quad \varphi_2(0) = \frac{\pi}{2}. \tag{84}$$

Substituting now (77) in the second initial condition (73) with due account for (84), we obtain

$$\omega_1(s_1 + \alpha_1 s_2)a_1(0) - \omega_2(s_1 + \alpha_2 s_2)a_2(0) = E_0. \tag{85}$$

From equations (74) and (85) we could determine the initial amplitudes

$$a_2(0) = \frac{\omega_1(s_1 + \alpha_1 s_2)}{\omega_2(s_1 + \alpha_2 s_2)}\, a_1(0) - \frac{E_0}{\omega_2(s_1 + \alpha_2 s_2)}, \tag{86}$$

$$c_1 a_1^2(0) + c_2 a_1(0) + c_3 = 0, \tag{87}$$

where

$$c_1 = 1 + \frac{b_1 k_2 \omega_1 (s_1 + \alpha_1 s_2)^2}{b_2 k_1 \omega_2 (s_1 + \alpha_2 s_2)^2},$$

$$c_2 = -\frac{b_1 k_2 (s_1 + \alpha_1 s_2) 2E_0}{b_2 k_1 \omega_2 (s_1 + \alpha_2 s_2)^2},$$

$$c_3 = \frac{b_1 k_2 E_0^2}{b_2 k_1 \omega_1 \omega_2 (s_1 + \alpha_2 s_2)^2} - \frac{k_2 \omega_2 E_0}{b_2}.$$

From equations (86) and (87) it is evident that the initial magnitudes depend on the mass and the initial velocity of the impactor, on the coordinates of the point of impact, as well as on the numbers of the two modes induced by the impact.

Considering (82), from (66) we find the value of constant G_0

$$G_0 = 0 \tag{88}$$

Reference to (67) shows that G_0 could be zero in three cases: at $\xi_0 = 0$, $\xi_0 = 1$, or when $\cos\delta_0 = 0$. The above analysis of the phase portrait has revealed that the case $\xi_0 = 0$ is not realized. As for the case $\xi_0 = 1$, then the solution for the phase modulated motion takes the form of (71). However, for the found magnitudes of the initial phase difference δ_0 (83), the value of $\tan\left(\frac{\delta_0}{2} + \frac{\pi}{4}\right)$ in (71) is either equal to zero or to infinity, what means that this case could not be realized as well.

That is why in further treatment we will analyze only the third case, resulting in the amplitude modulated motion (70) with

$$\delta(T_1) = \delta_0 = \text{const}. \tag{89}$$

Thus, we have determined all necessary constants from the initial conditions, therefore we could proceed to the construction of the solution for the contact force.

V. Contact Force

Substituting relationship (77) differentiated one time with respect to time t in (4), we could obtain the contact force $P(t)$

$$\begin{aligned} P(t) &= 2\epsilon M\Big\{\omega_1^2 a_1(\varepsilon t)\cos\left[\omega_1 t + \varphi_1(\varepsilon t)\right] \\ &+\omega_2^2 a_2(\varepsilon t)\cos\left[\omega_2 t + \varphi_2(\varepsilon t)\right]\Big\}s_1 \\ &+2\epsilon M\Big\{\alpha_1\omega_1^2 a_1(\varepsilon t)\cos\left[\omega_1 t + \varphi_1(\varepsilon t)\right] \\ &+\alpha_2\omega_2^2 a_2(\varepsilon t)\cos\left[\omega_2 t + \varphi_2(\varepsilon t)\right]\Big\}s_2 + O(\varepsilon^2). \end{aligned} \tag{90}$$

From equations (55) and (57) with due account for (89) it follows that

$$\begin{aligned} \varphi_1(T_1) &= \text{const} = \varphi_1(0), \\ \varphi_2(T_1) &= \text{const} = \varphi_2(0). \end{aligned} \tag{91}$$

Considering (91) and (84), equation (90) is reduced to

$$\begin{aligned} P(t) &= 2\epsilon M\omega_2^2\Big\{8(s_1 + \alpha_1 s_2)a_1(\varepsilon t)\cos\omega_2 t \\ &-(s_1 + \alpha_2 s_2)a_2(\varepsilon t)\Big\}\sin\omega_2 t. \end{aligned} \tag{92}$$

Substituting (61) in (92), we finally obtained

$$\begin{aligned} P(t) &= 2\epsilon M\omega_2^2\sqrt{E_0}\Big\{8(s_1 + \alpha_1 s_2) \\ &\times\sqrt{\tfrac{k_2\omega_2}{b_2}}\sqrt{\xi(\varepsilon t)}\cos\omega_2 t \\ &-(s_1 + \alpha_2 s_2)\sqrt{\tfrac{k_1\omega_1}{b_1}}\sqrt{1 - \xi(\varepsilon t)}\Big\}\sin\omega_2 t, \end{aligned} \tag{93}$$

where the function $\xi(\varepsilon t)$ is defined by (70).

Since the duration of contact is a small value, then $P(t)$ could be calculated via an approximate formula, which is obtained from (92) at $\varepsilon t \approx 0$

$$\begin{aligned} P(t) &\approx 16\epsilon M\omega_2^2\left(\cos\omega_2 t - \tfrac{1}{8}\,\text{æ}\right) \\ &\times(s_1 + \alpha_1 s_2)a_1(0)\sin\omega_2 t + O(\varepsilon^2), \end{aligned} \tag{94}$$

where the dimensionless parameter æ

$$\text{æ} = \frac{(s_1 + \alpha_2 s_2)}{(s_1 + \alpha_1 s_2)}\,\frac{a_2(0)}{a_1(0)} \tag{95}$$

is defined by the parameters of two impact-induced modes coupled by the two-to-one internal resonance (34), as well as by the coordinates of the point of impact and the initial velocity of impact.

The dimensionless time $\tau = \omega_2 t$ dependence of the dimensionless contact force P^*

$$P^*(\tau) \approx \left(\cos\tau - \frac{1}{8}\text{æ}\right)\sin\tau, \tag{96}$$

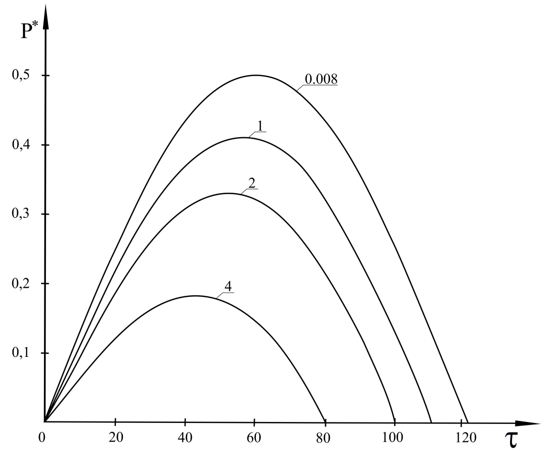

Figure 3: Dimensionless time dependence of the dimensionless contact force

where

$$P^*(\tau) = \frac{P(t)}{16\varepsilon M\omega_2^2(s_1+\alpha_1 s_2)a_1(0)},$$

is shown in Figure 3 for the different magnitudes of the parameter æ: 0.008, 1, 2, and 4.

Reference to Figure 3 shows that the decrease in the parameter æ results in the increase of both the maximal contact force and the duration of contact. In other words, from Figure 3 it is evident that the peak contact force and the duration of contact depend essentially upon the parameters of two impact-induced modes coupled by the two-to-one internal resonance (34).

VI. Particular Case

It has been emphasized in the above reasoning that the initial amplitudes and phases, as well as the contact force depend essentially on the location of the point of impact, i.e., on the coordinates of this point.

Let us consider below a particular case when one of the coefficients s_1 or s_2 vanishes to zero due to a particular combination the position of the impact point and the numbers of impact-induced modes of vibration. We will examine below the case $s_1 \neq 0$ and $s_2 = 0$, since another one, when $s_1 = 0$ and $s_2 \neq 0$, could be treated in a similar way.

Thus for this particular case the coefficients $p_{22} = 1$, $p_{12} = p_{21} = 0$, and nonlinear differential equations (15) and (16) are reduced to the following set of equations:

$$p_{11}\ddot{\xi}_{\alpha\beta} + \Omega^2_{\alpha\beta}\xi_{\alpha\beta} + p_{13}\xi^2_{\alpha\beta} + p_{14}\xi^2_{\gamma\delta} + p_{15}\xi_{\alpha\beta}\xi_{\gamma\delta} + p_{16}\xi^3_{\alpha\beta} + p_{17}\xi_{\alpha\beta}\xi^2_{\gamma\delta} = 0, \quad (97)$$

$$\ddot{\xi}_{\gamma\delta} + \Omega^2_{\gamma\delta}\xi_{\gamma\delta} + p_{23}\xi^2_{\gamma\delta} + p_{24}\xi^2_{\alpha\beta} + p_{25}\xi_{\alpha\beta}\xi_{\gamma\delta} + p_{26}\xi^3_{\gamma\delta} + p_{27}\xi^2_{\alpha\beta}\xi_{\gamma\delta} = 0, \quad (98)$$

where only one coefficient p_{11} depend on impactor's mass, the coordinates of the impact point, as well as on the parameters of the target and the numbers of the induced modes.

Reference to (97) and (98) shows that, despite from (15) and (16), the linear parts of (97) and (98) are uncoupled. Therefore using expansions (17) and the method of multiple time scales, we arrive at the following set of recurrence equations to various orders:

to order ε

$$p_{11}D_0^2X_1^1 + \Omega_1^2X_1^1 = 0, \quad (99)$$

$$D_0^2X_2^1 + \Omega_2^2X_2^1 = 0; \quad (100)$$

to order ε^2

$$p_{11}D_0^2X_1^2 + \Omega_1^2X_1^2 = -2p_{11}D_0D_1X_1^1 - p_{13}(X_1^1)^2 - p_{14}(X_2^1)^2 - p_{15}X_1^1X_2^1, \quad (101)$$

$$D_0^2X_2^2 + \Omega_2^2X_2^2 = -2D_0D_1X_2^1 - p_{23}(X_1^1)^2 - p_{24}(X_2^1)^2 - p_{25}X_1^1X_2^1, \quad (102)$$

where for simplicity once again is it denoted $X_1^1 = X_{\alpha\beta}^1$, $X_2^1 = X_{\gamma\delta}^1$, $X_1^2 = X_{\alpha\beta}^2$, $X_2^2 = X_{\gamma\delta}^2$, $\Omega_1 = \Omega_{\alpha\beta}$, and $\Omega_2 = \Omega_{\gamma\delta}$.

The solution of (99) and (100) has the form

$$X_1^1 = A_1(T_1)\,e^{i\omega_1T_0} + \bar{A}_1(T_1)e^{-i\omega_1T_0}, \quad (103)$$

$$X_2^1 = A_2(T_1)\,e^{i\Omega_2T_0} + \bar{A}_2(T_1)e^{-i\Omega_2T_0}, \quad (104)$$

where $\omega_1^2 = \Omega_1^2/p_{11}$.

Substituting (103) and (104) in (101) and (102) yields

$$\begin{aligned} p_{11}D_0^2X_1^2 + \Omega_1^2X_1^2 &= -2i\omega_1p_{11}D_1A_1e^{i\omega_1T_0} \\ &-p_{13}A_1^2e^{2i\omega_1T_0} - p_{13}A_1\bar{A}_1 - p_{14}A_2^2e^{2i\Omega_2T_0} \\ &-p_{14}A_2\bar{A}_2 - p_{15}A_1A_2e^{i(\omega_1+\Omega_2)T_0} \\ &+A_1\bar{A}_2e^{i(\omega_1-\Omega_2)T_0} + \text{cc}, \end{aligned} \quad (105)$$

$$\begin{aligned} D_0^2X_2^2 + \Omega_2^2X_2^2 &= -2i\Omega_2D_1A_2e^{i\omega_2T_0} \\ &-p_{23}A_2^2e^{2i\omega_2T_0} - p_{23}A_2\bar{A}_2 - p_{24}A_1^2e^{2i\omega_1T_0} \\ &-p_{24}A_1\bar{A}_1 - p_{25}A_1A_2e^{i(\omega_1+\omega_2)T_0} \\ &-p_{25}A_1\bar{A}_2e^{i(\omega_1-\omega_2)T_0} + \text{cc}. \end{aligned} \quad (106)$$

Reference to equations (105) and (106) shows that the following impact-induced two-to-one internal resonances could occur:

$$\omega_1 = 2\Omega_2, \quad (107)$$

or

$$\Omega_2 = 2\omega_1. \quad (108)$$

A. Internal Resonance $\omega_1 = 2\Omega_2$

In this case we obtain the following solvability equations:

$$2i\omega_1D_1A_1 + \frac{p_{14}}{p_{11}}A_2^2 = 0, \quad (109)$$

$$2i\omega_2 D_1 A_2 + p_{25} A_1 \bar{A}_2 = 0. \tag{110}$$

Comparison of solvability equations (109) and (110) with those obtained above for the general case, i.e., with (47) and (48), shows that they are similar in structure within an accuracy of the coefficients. That is why the solution of (47) and (48) presented in Sec. III *B* is valid for equations (109) and (110) as well under the proper substitution of coefficients b_1/k_1 and b_2/k_2 with p_{14}/p_{11} and p_{25}, respectively.

In the case under consideration, the two-term relationship for the displacement w (9) within an accuracy of ε according to (17) has the form

$$\begin{aligned} w(x,y,t) = 2\epsilon\Big\{ & a_1(\varepsilon t)\cos\left[\omega_1 t + \varphi_1(\varepsilon t)\right] \\ & \times \sin\left(\tfrac{\alpha\pi x}{a}\right)\sin\left(\tfrac{\beta\pi y}{b}\right) \\ & + a_2(\varepsilon t)\cos[\Omega_2 t + \varphi_2(\varepsilon t)]\sin\left(\tfrac{\gamma\pi x}{a}\right)\sin\left(\tfrac{\delta\pi y}{b}\right)\Big\} \\ & + O(\varepsilon^2). \end{aligned} \tag{111}$$

Differentiating (111) with respect to time t and limiting ourselves by the terms of the order of ε, we could find the velocity of the shell at the point of impact as follows

$$\dot{w}(x_0, y_0, t) = -2\epsilon\omega_1 s_1 a_1(\varepsilon t)\sin\left[\omega_1 t + \varphi_1(\varepsilon t)\right] + O(\varepsilon^2). \tag{112}$$

Now substituting (111) and (112) in the initial conditions (72)-(74), we could find the initial amplitudes and phases:

$$\varphi_1(0) = -\frac{\pi}{2}, \quad \varphi_2(0) = \frac{\pi}{2}. \tag{113}$$

$$a_1(0) = \frac{V_0}{2\omega_1 s_1}, \quad a_2(0) = \sqrt{\frac{V_0 p_{25}}{2\Omega_2}\left(1 - \frac{V_0 p_{11}}{2\omega_1 s_1^2 p_{14}}\right)}. \tag{114}$$

As a result the contact force could be written as

$$\begin{aligned} P(t) &\approx 2\varepsilon M\omega_1^2 a_1(0)\sin\omega_1 t = \varepsilon M V_0 \omega_1 s_1^{-1}\sin\omega_1 t, \\ &= 2\varepsilon M V_0 \Omega_2 s_1^{-1}\sin 2\Omega_2 t. \end{aligned} \tag{115}$$

Reference to (115) shows that the contact force depends on the mass of the striker, its initial velocity, the coordinates of the point wherein the impact occur ed, as well as on the frequencies and umbers of two modes coupled by the impact-induced two-to-one internal resonance (107).

Relationship for the time-dependence of the contact force (115) obtained above analytically verifies the assumption for the contact force proposed in [1] a known sine-dependent input to the analysis.

Note that another particular case of the impact-induced two-to-one internal resonance (108) could be treated in a similar way.

As for the particular case when $s_1 = s_2 = 0$, then it could not be realized in this problem, since this condition results in the violation of the second initial condition (73).

VII. Conclusion

In the present paper, a new approach proposed recently by the authors for the analysis of the impact interactions of nonlinear doubly curved shallow shells with rectangular base under the low-velocity impact by an elastic sphere [21, 22] has been generalized for investigating the impact-induced two-to-one internal resonance.

Such an approach differs from the Galerkin method, wherein resonance phenomena are not involved [1]. Since is it assumed that shell's displacements are finite, then the local bearing of the shell and impactor's materials is neglected with respect to the shell deflection in the contact region. In other words, the Hertz's theory, which is traditionally in hand for solving impact problems, was not used in the present study; instead, the method of multiple time scales has been adopted, which is used with much success for investigating vibrations of nonlinear systems subjected to the conditions of the internal resonance, as well as to find the time dependence of the contact force.

It has been shown that the time dependence of the contact force depends essentially on the position of the point of impact and the parameters of two impact-induced modes coupled by the internal resonance. Besides, the contact force depends essentially on the magnitude of the initial energy of the impactor. This value governs the place on the phase plane, where a mechanical system locates at the moment of impact, and the phase trajectory, along which it moves during the process of impact.

It is shown that the intricate $P(t)$ dependence at impact-induced internal resonance (90) give way to rather simple sine time-dependence, what is an accordance with a priori assumption of some researchers about a sine-like character of the contact force with time [1], [28]–[31].

The procedure suggested in the present paper could be generalized for the analysis of impact response of plates and shells when their motions are described by three or five nonlinear differential equations.

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APPENDIX

$$p_{13} = \frac{8\pi^2 E}{a^3 b^3 \rho}\, B_{\alpha\beta\alpha\beta\alpha\beta}\, \frac{1}{2}\, K_{\alpha\beta} = -\frac{4E}{6\rho\alpha\beta}\Bigg\{ k_y\left(\frac{\beta}{b}\right)^2 + k_x\left(\frac{\alpha}{a}\right)^2 + 16\left(\frac{\alpha\beta}{ab}\right)^2\left[k_y\left(\frac{\alpha}{a}\right)^2 + k_x\left(\frac{\beta}{b}\right)^2\right]\left[\left(\frac{\alpha}{a}\right)^2 + \left(\frac{\beta}{b}\right)^2\right]^{-2}\Bigg\},$$

$$p_{23} = \frac{8\pi^2 E}{a^3 b^3 \rho}\, B_{\gamma\delta\gamma\delta\gamma\delta}\, \frac{1}{2}\, K_{\gamma\delta} = -\frac{4E}{6\rho\gamma\delta}\Bigg\{ k_y\left(\frac{\delta}{b}\right)^2 + k_x\left(\frac{\gamma}{a}\right)^2 + 16\left(\frac{\gamma\delta}{ab}\right)^2\left[k_y\left(\frac{\gamma}{a}\right)^2 + k_x\left(\frac{\delta}{b}\right)^2\right]\left[\left(\frac{\gamma}{a}\right)^2 + \left(\frac{\delta}{b}\right)^2\right]^{-2}\Bigg\},$$

$$\begin{aligned} p_{14} &= \frac{8\pi^2 E}{a^3 b^3 \rho}\, B_{\gamma\delta\gamma\delta\alpha\beta}\left(K_{\gamma\delta} - \frac{1}{2}\, K_{\alpha\beta}\right) \\ &= -\frac{4E}{\rho ab}\Bigg\{ 8\frac{a\delta^2}{b\beta\alpha}\left(\frac{\gamma}{a}\right)^2\left[k_y\left(\frac{\gamma}{a}\right)^2 + k_x\left(\frac{\delta}{b}\right)^2\right] \\ &\times\left[4 - \left(\frac{\alpha^2}{\gamma^2} - 2\right)\left(\frac{\beta^2}{\delta^2} - 2\right)\right]\left[\left(\frac{\gamma}{a}\right)^2 + \left(\frac{\delta}{b}\right)^2\right]^{-2} \\ &\times\left(\frac{\alpha^2}{\gamma^2} - 4\right)^{-1}\left(\frac{\beta^2}{\delta^2} - 4\right)^{-1} \\ &+ \frac{1}{2}\left[k_y\frac{a\delta^2\alpha}{b\gamma^2\beta}\left(\frac{\alpha^2}{\gamma^2} - 4\right)^{-1} + k_x\frac{b\gamma^2}{a\delta^2}\frac{\beta}{\alpha}\left(\frac{\beta^2}{\delta^2} - 4\right)^{-1}\right]\Bigg\}, \end{aligned}$$

$$\begin{aligned} p_{24} &= \frac{8\pi^2 E}{a^3 b^3 \rho}\, B_{\alpha\beta\alpha\beta\gamma\delta}\left(K_{\alpha\beta} - \frac{1}{2}\, K_{\gamma\delta}\right) \\ &= -\frac{4E}{\rho ab}\Bigg\{ 8\frac{a\beta^2}{b\gamma\delta}\left(\frac{\alpha}{a}\right)^2\left[k_y\left(\frac{\alpha}{a}\right)^2 + k_x\left(\frac{\beta}{b}\right)^2\right] \\ &\times\left[4 - \left(\frac{\gamma^2}{\alpha^2} - 2\right)\left(\frac{\delta^2}{\beta^2} - 2\right)\right]\left[\left(\frac{\alpha}{a}\right)^2 + \left(\frac{\beta}{b}\right)^2\right]^{-2} \\ &\times\left(\frac{\gamma^2}{\alpha^2} - 4\right)^{-1}\left(\frac{\delta^2}{\beta^2} - 4\right)^{-1} \\ &+ \frac{1}{2}\left[k_y\frac{a\beta^2\gamma}{b\alpha^2\delta}\left(\frac{\gamma^2}{\alpha^2} - 4\right)^{-1} + k_x\frac{b\alpha^2}{a\beta^2}\frac{\delta}{\gamma}\left(\frac{\delta^2}{\beta^2} - 4\right)^{-1}\right]\Bigg\}, \end{aligned}$$

$$\begin{aligned} p_{15} &= \frac{8\pi^2 E}{a^3 b^3 \rho}\left[B_{\gamma\delta\alpha\beta\alpha\beta}\, \frac{1}{2}\, K_{\alpha\beta} + B_{\alpha\beta\gamma\delta\alpha\beta}\left(K_{\gamma\delta} - \frac{1}{2}\, K_{\alpha\beta}\right)\right] \\ &= -\frac{32E}{\rho}\frac{\delta}{\gamma\, b^2}\Bigg\{\left(2 + 2\frac{\gamma^2\beta^2}{\delta^2\alpha^2} - \frac{\gamma^2}{\alpha^2}\right)\left(\frac{\alpha}{a}\right)^2 \\ &\times\left[k_y\left(\frac{\alpha}{a}\right)^2 + k_x\left(\frac{\beta}{b}\right)^2\right]\left[\left(\frac{\alpha}{a}\right)^2 + \left(\frac{\beta}{b}\right)^2\right]^{-2} \\ &+ \left(2\frac{\beta^2}{\delta^2} + 2\frac{\alpha^2}{\gamma^2} - 1\right)\left(\frac{\gamma}{a}\right)^2 \\ &\times\left[k_y\left(\frac{\gamma}{a}\right)^2 + k_x\left(\frac{\delta}{b}\right)^2\right]\left[\left(\frac{\gamma}{a}\right)^2 + \left(\frac{\delta}{b}\right)^2\right]^{-2}\Bigg\} \\ &\times\left(\frac{\gamma^2}{\alpha^2} - 4\right)^{-1}\left(\frac{\delta^2}{\beta^2} - 4\right)^{-1} \\ &- \frac{4E}{\rho}\frac{\beta^2}{\gamma\,\delta\, b^2}\sum_{p=1}^{2}\sum_{s=1}^{2}(-1)^{p+s} \\ &\times\left[k_y + k_x\frac{\beta^2 a^2}{\alpha^2 b^2}\left(1 + (-1)^p\,\frac{\delta}{\beta}\right)^2\left(1 + (-1)^s\,\frac{\gamma}{\alpha}\right)^{-2}\right] \\ &\times\left(\frac{\gamma}{\alpha} - (-1)^{p+s}\,\frac{\delta}{\beta}\right)^2\left(1 + (-1)^s\,\frac{\gamma}{\alpha}\right)^{-2} \\ &\times\left(\frac{\gamma}{\alpha} + 2\,(-1)^s\right)^{-1}\left(\frac{\delta}{\beta} + 2\,(-1)^p\right)^{-1} \\ &\times\left[1 + \frac{\beta^2 a^2}{\alpha^2 b^2}\left(1 + (-1)^p\,\frac{\delta}{\beta}\right)^2\left(1 + (-1)^s\,\frac{\gamma}{\alpha}\right)^{-2}\right]^{-2}, \end{aligned}$$

$$\begin{aligned} p_{25} &= \frac{8\pi^2 E}{a^3 b^3 \rho}\left[B_{\alpha\beta\gamma\delta\gamma\delta}\, \frac{1}{2}\, K_{\gamma\delta} + B_{\gamma\delta\alpha\beta\gamma\delta}\left(K_{\alpha\beta} - \frac{1}{2}\, K_{\gamma\delta}\right)\right] \\ &= -\frac{32E}{\rho}\frac{\beta}{\alpha b^2}\Bigg\{\left(2 + 2\frac{\alpha^2\delta^2}{\beta^2\gamma^2} - \frac{\alpha^2}{\gamma^2}\right)\left(\frac{\gamma}{a}\right)^2 \\ &\times\left[k_y\left(\frac{\gamma}{a}\right)^2 + k_x\left(\frac{\delta}{b}\right)^2\right]\left[\left(\frac{\gamma}{a}\right)^2 + \left(\frac{\delta}{b}\right)^2\right]^{-2} \\ &+ \left(2\frac{\delta^2}{\beta^2} + 2\frac{\gamma^2}{\alpha^2} - 1\right)\left(\frac{\alpha}{a}\right)^2 \\ &\times\left[k_y\left(\frac{\alpha}{a}\right)^2 + k_x\left(\frac{\beta}{b}\right)^2\right]\left[\left(\frac{\alpha}{a}\right)^2 + \left(\frac{\beta}{b}\right)^2\right]^{-2}\Bigg\} \\ &\times\left(\frac{\alpha^2}{\gamma^2} - 4\right)^{-1}\left(\frac{\beta^2}{\delta^2} - 4\right)^{-1} \\ &- \frac{4E}{\rho ab}\frac{\beta}{\alpha\, b^2}\sum_{p=1}^{2}\sum_{s=1}^{2}(-1)^{p+s} \\ &\times\left[k_y + k_x\frac{\beta^2 a^2}{\alpha^2 b^2}\left(1 + (-1)^p\,\frac{\delta}{\beta}\right)^2\left(1 + (-1)^s\,\frac{\gamma}{\alpha}\right)^{-2}\right] \\ &\times\left(\frac{\gamma}{\alpha} - (-1)^{p+s}\,\frac{\delta}{\beta}\right)^2\left(1 + (-1)^s\,\frac{\gamma}{\alpha}\right)^{-2} \\ &\times\left(\frac{\alpha}{\gamma} + 2\,(-1)^s\right)^{-1}\left(\frac{\beta}{\delta} + 2\,(-1)^p\right)^{-1} \\ &\times\left[1 + \frac{\beta^2 a^2}{\alpha^2 b^2}\left(1 + (-1)^p\,\frac{\delta}{\beta}\right)^2\left(1 + (-1)^s\,\frac{\gamma}{\alpha}\right)^{-2}\right]^{-2} \end{aligned}$$

$$p_{16} = \frac{32\pi^2 E}{a^3 b^3 \rho}\sum_i\sum_j B_{\alpha\beta ij\alpha\beta}B_{\alpha\beta\alpha\beta ij} = \frac{E}{16\rho}\frac{\alpha^4\pi^4}{a^4}\left(1 + \frac{\beta^4 a^4}{\alpha^4 b^4}\right),$$

$$p_{26} = \frac{32\pi^2 E}{a^3 b^3 \rho}\sum_i\sum_j B_{\gamma\delta ij\gamma\delta}B_{\gamma\delta\gamma\delta ij} = \frac{E}{16\rho}\frac{\gamma^4\pi^4}{a^4}\left(1 + \frac{\delta^4 a^4}{\gamma^4 b^4}\right),$$

$$p_{17} = \frac{32\pi^2 E}{a^3 b^3 \rho} \sum_i \sum_j \left(B_{\alpha\beta ij\alpha\beta} B_{\gamma\delta\gamma\delta ij}\right.$$
$$\left. + B_{\gamma\delta ij\alpha\beta} B_{\alpha\beta\gamma\delta ij} + B_{\gamma\delta ij\alpha\beta} B_{\gamma\delta\alpha\beta ij}\right)$$
$$= \frac{E}{4\rho} \frac{\delta^2 \beta^2 \pi^4}{b^4} \sum_{p=1}^{2} \sum_{s=1}^{2} \left[\frac{\gamma}{\alpha} - (-1)^{p+s} \frac{\delta}{\beta}\right]^2$$
$$\times \left[1 - \left(1 + (-1)^p \frac{\beta}{\delta}\right) \left(1 + (-1)^s \frac{\alpha}{\gamma}\right)^{-1}\right]^2$$
$$\times \left[1 + \frac{\beta^2 a^2}{\alpha^2 b^2} \left(1 + (-1)^p \frac{\delta}{\beta}\right)^2 \left(1 + (-1)^s \frac{\gamma}{\alpha}\right)^{-2}\right]^{-2}$$
$$\times \left[1 + (-1)^s \frac{\gamma}{\alpha}\right]^{-2},$$

$$p_{27} = \frac{32\pi^2 E}{a^3 b^3 \rho} \sum_i \sum_j \left(B_{\alpha\beta ij\gamma\delta} B_{\gamma\delta\gamma\delta ij}\right.$$
$$\left. + B_{\gamma\delta ij\gamma\delta} B_{\alpha\beta\gamma\delta ij} + B_{\gamma\delta ij\gamma\delta} B_{\gamma\delta\alpha\beta ij}\right)$$
$$= \frac{E}{4\rho} \frac{\beta^4 \pi^4}{b^4} \sum_{p=1}^{2} \sum_{s=1}^{2} \left(\frac{\gamma}{\alpha} - (-1)^{p+s} \frac{\delta}{\beta}\right)^2$$
$$\times \left[1 - \left(1 + (-1)^p \frac{\delta}{\beta}\right) \left(1 + (-1)^s \frac{\gamma}{\alpha}\right)^{-1}\right]^2$$
$$\times \left[1 + \frac{\beta^2 a^2}{\alpha^2 b^2} \left(1 + (-1)^p \frac{\delta}{\beta}\right)^2 \left(1 + (-1)^s \frac{\gamma}{\alpha}\right)^{-2}\right]^{-2}$$
$$\times \left[1 + (-1)^s \frac{\gamma}{\alpha}\right]^{-2}.$$