New sufficient conditions for the Hadamard stability of a Mooney-Rivlin elastic solid in uniaxial deformation

Pilade Foti, Aguinaldo Fraddosio, Salvatore Marzano, Mario Daniele Piccioni

Abstract— A procedure for obtaining a lower bound estimate of the critical load for a homogeneously deformed Mooney-Rivlin incompressible cylinder is presented. By considering a lower bound estimate for the second variation of the total energy functional based on the Korn inequality, we establish sufficient conditions for the infinitesimal Hadamard stability of a distorted configuration. We then sketch the procedure for determining an optimal lower bound estimate of the critical load in a uniaxial compressive loading process and discuss its effectiveness for applications by comparing our results to other estimates proposed in the literature.

Keywords— Non-linear elasticity, bifurcation, stability, lower bound estimates.

I. Introduction

Nonlinear elasticity is a suitable theoretical framework for modeling a number of phenomena experienced by solid bodies undergoing large elastic deformations. New research issues (cf., e.g., [1-2]) concern the analysis of solid-solid phase transformations for materials characterized by non-convex strain energy density functions, whereas classical issues are the stability of large elastic deformations and the possibility of bifurcations related to the loss of stability.

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For a large class of mechanical problems, the instability phenomena may be related both to the material response and to the overall features of the particular boundary value problem.

In these cases, the nonlinear material response together with the geometry of the body and the boundary conditions contribute to the onset of instabilities, and consequently the stability problem becomes much more complex. Even by performing a local stability analysis, i.e., by studying the stability in a neighborhood of the given fundamental equilibrium deformation, explicit results are available only for very special cases.A relative simplification in a local stability analysis may be obtained by employing the classical method of adjacent equilibria, i.e., by checking if there exist other equilibrium solutions close to the given primary equilibrium state through the analysis of the linearized equilibrium equations from the fundamental deformation. This approach also shows the close correlation between bifurcation and stability issues. In this vein, the Hadamard criterion of infinitesimal stability (cf. [3], §68 bis) is a classical and widely used tool for testing the stability of an equilibrium configuration, since the violation of the so-called Hadamard stability condition plays the role of an indicator of possible bifurcations from the primary equilibrium. Indeed, in a monotonic loading process governed by a loading parameter, it is usually assumed that the body remains in the fundamental equilibrium state (no bifurcation allowed) until the Hadamard stability condition ceases to hold. Then, possible bifurcation modes may arise at the critical load λ_{cr}, defined as the value of the load parameter which first renders the Hadamard functional zero.

All the above considerations show that a crucial issue is the study of the sign of the Hadamard functional, which actually is not a simple matter. This motivates the development of procedures for determining lower bound estimates for the critical load λ_{cr}, i.e., a load λ_{LB} below which the infinitesimal stability criterion is definitely satisfied. These procedures are mainly based on the use of the Korn inequality like, for example, the lower bound estimates proposed in [4]-[7].

It is important to emphasize that the availability of a lower bound estimate for the critical load may be particularly useful in bifurcation problems. Indeed, during a loading process one usually identifies the load which corresponds to a special bifurcation mode with the load which first renders the Hadamard functional zero, but with reference to a particular subclass of incremental displacements. Then, one may wonder if such a special solution is actually the first bifurcation mode (among other bifurcations) which occurs during the prescribed loading process, and consequently if the corresponding bifurcation load is actually the "true" critical load and not an upper bound for the critical load. Since it is extremely difficult to check the sign of the Hadamard functional on the whole class of admissible incremental displacements, a possible answer may be obtained by checking whether the gap between the bifurcation load related to the special bifurcation mode and the lower bound estimate for the critical load is sufficiently small or not.

In this context, we proposed in [8] a new procedure for determining a lower bound estimate λ_{LB} for λ_{cr} in the case of hyperelastic solids; there, we obtained interesting improvements on the available estimates by enhancing both the quality of the estimate and the easiness of the application of the procedure. In [8], we also developed some considerations concerning the relation between upper and lower bound estimates for the critical load with reference to the analysis in [9] of a bifurcation from a *homogeneous* fundamental deformation. As a possible development of this research line, we are currently analyzing the possibility of extending this approach to a case of *inhomogeneous* deformation by considering the bifurcation problem in [10] as a representative example.

Here, by adopting an approach similar to that in [11] for incompressible bodies, we consider the homogeneous *uniaxial compression* of a Mooney-Rivlin incompressible, isotropic, homogeneous elastic cylinder. In particular, we provide some remarks concerning the effectiveness of the considered estimates by evaluating the gap between the lower bound estimate and the critical load. Moreover, we discuss the effectiveness of our procedure by comparing our estimates to those obtained in [7]. As a major result, we find that our lower bound estimate globally improves the estimate proposed in [7] and that our procedure is advantageous for applications, since it yields a very simple method for estimates which is easily implementable in numerical codes. More explicitly, our procedure only requires the determination of the incremental elasticity tensor, the principal Cauchy stresses, the Lagrange multiplier related to the incompressibility constraint and the Korn constant.

In the future, we intend to apply our method for the determination of lower bound estimates of the critical load by considering special (but representative) boundary-value problems inspired by the analyses in [12]-[14].

II. Bounds from below of the Hadamard functional for a class of boundary-value problems

In this Section, we first refer to a class of boundary-value problems in order to obtain a suitable format of the Hadamard functional for a Mooney-Rivlin incompressible cylinder. Then, we determine a lower bound estimate of the Hadamard functional based on the Korn inequality, which allows to establish sufficient conditions for the infinitesimal stability of a distorted configuration.

We consider a homogeneous, isotropic incompressible elastic circular cylinder $\mathcal{C}$ with radius R_0 and height H_0 and name $\rho_0 := H_0/R_0$ its referential slenderness ratio. Although the procedure we develop below is general, here we assume that the mechanical response of $\mathcal{C}$ is modeled by the Mooney-Rivlin strain energy density function

$$W(\mathbf{F})=\frac{C_1}{2}(I-3)+\frac{C_2}{2}(II-3), \tag{1}$$

where $\mathbf{F}$ denotes the deformation gradient, which must satisfy the incompressibility constraint

$$\det\mathbf{F}=1. \tag{2}$$

In (1), the quantities

$$I := \mathbf{F}\cdot\mathbf{F} = \mathrm{tr}(\mathbf{B}), \quad II := \mathbf{F}^{-T}\cdot\mathbf{F}^{-T} = \mathrm{tr}\left(\mathbf{B}^{-1}\right) \tag{3}$$

are a set of two principal invariants of the left Cauchy-Green strain tensor $\mathbf{B} := \mathbf{F}\,\mathbf{F}^{T}$ and C_1, C_2 are *positive* material constants with $\mu := 2(C_1+C_2)$ the referential shear modulus.

We further assume that the body may undergo uniquely *homogeneous*, isochoric deformations determined by the following form of the strain tensor:

$$\tilde{\mathbf{B}}_\lambda = \lambda^2\ \mathbf{e}\otimes\mathbf{e} + \alpha^2(\mathbf{I} - \mathbf{e}\otimes\mathbf{e}), \tag{4}$$

where the constant axial stretch $\lambda > 0$ acts as the load parameter, $\alpha > 0$ is the constant cross sectional stretch, and $\mathbf{e}$ coincides with the axis of $\mathcal{C}$. In particular, the incompressibility constraint (2) relates α to the parameter λ as follows:

$$\alpha = \lambda^{-1/2}; \tag{5}$$

thus, the strain tensor (4) at a given equilibrium configuration determined by the stretch $\lambda > 0$ may be written as

$$\tilde{\mathbf{B}}_\lambda = \lambda^2\ \mathbf{e}\otimes\mathbf{e} + \lambda^{-1}(\mathbf{I} - \mathbf{e}\otimes\mathbf{e}), \tag{6}$$

which shows that $\tilde{\mathbf{B}}_\lambda$ has two equal principal stretches in the cross section of the cylinder that are different from the stretch along the axis of the cylinder.

It is worth noting that the representations (4) or (6) for a homogeneous equilibrium strain tensor are shared by a number of significant mixed or all around dead load tractions boundary-value problems like, for example, the uniaxial compression and the uniaxial extension of a circular cylinder (see, e.g., [11]), the dead-load equibiaxial traction on the lateral surface of a parallelepiped accompanied by an orthogonal uniaxial compression of the same amount on the bases (see, e.g., [11], [12], and [14], and the uniform traction of a cube, namely the well-known *Rivlin's cube problem* (see, e.g., [15]-[16]).

In correspondence of an equilibrium strain tensor (6), the equilibrium total Cauchy stress $\mathbf{T}_\lambda$ for an isotropic incompressible Mooney-Rivlin elastic material has the form (see, e.g., [3]

$$\mathbf{T}_\lambda = \mathrm{W}_{\mathbf{F}}\left(\tilde{\mathbf{F}}_\lambda\right)\tilde{\mathbf{F}}_\lambda^{\mathrm{T}} - \mathrm{p}_\lambda \mathbf{I} = \left(\mathrm{C}_1\,\lambda^2 - \mathrm{C}_2\,\lambda^{-2}\right)\mathbf{e}\otimes\mathbf{e} + \left(\mathrm{C}_1\,\lambda^{-1} - \mathrm{C}_2\,\lambda\right)\left(\mathbf{I} - \mathbf{e}\otimes\mathbf{e}\right) - \mathrm{p}_\lambda \mathbf{I} \tag{7}$$

where $\mathrm{W}_{\mathbf{F}}(\cdot)$ denotes the derivative of W with respect to $\mathbf{F}$ and p_λ, the "pressure-like" field related to the incompressibility constraint, must be constant in order to satisfy the equilibrium field equations and, in particular, it is determined form the traction boundary conditions.

We now particularize to the present situation of homogeneous deformations a general method developed in [8] and [11] (also valid for *inhomogeneous* deformations) for the determination of a lower bound estimate of the Hadamard functional. We assume that an equilibrium configuration determined by the parameter $\lambda > 0$ is known and recall that the Hadamard criterion of infinitesimal stability for incompressible bodies (see [17]) sets that an equilibrium deformation $\tilde{\mathbf{f}}_\lambda(\mathbf{X})$ with gradient $\tilde{\mathbf{F}}_\lambda(\mathbf{X})$ is stable if the second variation of the total energy functional is non-negative for each divergence-free admissible incremental displacement $\mathbf{u} := \mathbf{u}(\tilde{\mathbf{x}})$ (from the deformed configuration) which vanishes on the constrained boundary. Under the current assumptions, this condition on the second variation of the total energy may be written as follows:

$$\varepsilon(\mathbf{u}) := \int_{\tilde{\Omega}} \tilde{\mathbb{C}}_\lambda[\mathrm{grad}\mathbf{u}]\cdot\mathrm{grad}\mathbf{u} + \int_{\tilde{\Omega}} \mathrm{p}_\lambda\left(\mathrm{grad}\mathbf{u}\right)^{\mathrm{T}}\cdot\left(\mathrm{grad}\mathbf{u}\right) \geq 0, \tag{8}$$

where $\tilde{\mathbb{C}}_\lambda$ is the fourth-order *instantaneous elasticity tensor* defined as follows:

$$\tilde{\mathbb{C}}_\lambda[\mathbf{N}] = \mathrm{W}_{\mathbf{FF}}\left(\tilde{\mathbf{F}}_\lambda\right)\left[\mathbf{N}\,\tilde{\mathbf{F}}_\lambda\right]\tilde{\mathbf{F}}_\lambda^{\mathrm{T}} \tag{9}$$

for each second order tensor $\mathbf{N}$. Notice that in the present analysis $\tilde{\mathbb{C}}_\lambda$ is constant because we are dealing with homogeneous deformations; in particular, $\tilde{\mathbb{C}}_\lambda$ is determined by the material parameters C_1, C_2, which are implicitly assumed as prescribed, and by the constant stretch λ which is known from the equilibrium problem. The functional (8) is commonly called the *Hadamard functional*.

We now sketch the main steps for determining first a suitable format of the integrand function in (8) and then a bound from below for the whole functional (8). We first introduce the decomposition

$$\mathrm{grad}\mathbf{u} =: \mathbf{H} = \mathbf{E} + \mathbf{W}, \tag{10}$$

where $\mathbf{E}$ is symmetric and traceless, because of the incompressibility constraint, and $\mathbf{W}$ is skew-symmetric. Using (9) for the class of Mooney-Rivlin materials (1), with the aid of some calculations based on the general procedure in [11, section 2] we get

$$\tilde{\mathbb{C}}_\lambda[\mathbf{H}]\cdot\mathbf{H} + \mathrm{p}_\lambda\,\mathbf{H}^{\mathrm{T}}\cdot\mathbf{H} = \left(\mathrm{C}_1\,\tilde{\mathbf{B}}_\lambda + 3\mathrm{C}_2\,\tilde{\mathbf{B}}_\lambda^{-1} + \mathrm{p}_\lambda\mathbf{I}\right)\cdot\mathbf{E}^2 + 2\left(\mathbf{W}\mathbf{T}_\lambda\right)\cdot\mathbf{E} + \left(\mathbf{W}\mathbf{T}_\lambda\right)\cdot\mathbf{W}, \tag{11}$$

where $\tilde{\mathbf{B}}_\lambda$ and $\mathbf{T}_\lambda$, respectively given by (4) and (7), are clearly *coaxial* tensors. It is then convenient to introduce a orthonormal basis $\{\mathbf{i}_1, \mathbf{i}_2, \mathbf{e}\}$ built on the principal directions of $\mathbf{T}_\lambda$, where $\mathbf{i}_1, \mathbf{i}_2$ are two orthogonal arbitrary directions in the cross section of the cylinder since two principal stretches of $\mathbf{T}_\lambda$ are equal. In such a basis, we define the components of $\mathbf{E}$ and $\mathbf{W}$ as follows:

$$\begin{gathered}\{\gamma_1,\ \gamma_2,\ \gamma_3\} := \{\mathrm{E}_{12}, \mathrm{E}_{23}, \mathrm{E}_{31}\},\\ \{\varepsilon_1,\ \varepsilon_2,\ \varepsilon_3\} := \{\mathrm{E}_{11}, \mathrm{E}_{22}, \mathrm{E}_{33}\},\\ \{\omega_1,\ \omega_2,\ \omega_3\} := \{\mathrm{W}_{12}, \mathrm{W}_{23}, \mathrm{W}_{31}\},\end{gathered} \tag{12}$$

so that we readily have

$$\begin{gathered}\mathbf{W}\cdot\mathbf{W} = 2\left(\omega_1^2 + \omega_2^2 + \omega_3^2\right),\\ \mathbf{E}\cdot\mathbf{E} = \mathbf{I}\cdot\mathbf{E}^2 = \varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2 + 2\left(\gamma_1^2 + \gamma_2^2 + \gamma_3^2\right).\end{gathered} \tag{13}$$

In view of the analysis we develop in the next section for the case of uniaxial compression, we make the further hypotheses that the principal stretch $\lambda \in]0, 1[$, so that the two *equal* principal Cauchy stresses $\mathrm{t}_1, \mathrm{t}_2$ in the cross section of the cylinder are always greater than the principal t_3 stress along the axis. Indeed, by (7) we easily see that

$$\mathrm{t}_1 = \mathrm{t}_2 =: \mathrm{t} = \mathrm{C}_1\,\lambda^{-1} - \mathrm{C}_2\,\lambda - \mathrm{p}_\lambda > \mathrm{t}_3 := \mathrm{C}_1\lambda^2 - \mathrm{C}_2\lambda^{-2} - \mathrm{p}_\lambda, \tag{14}$$

provided that $\lambda \in \left]0, 1\right[$ and that the material moduli are positive, as it occurs under the current assumptions. Then, by (7), (14) and (12) we immediately have

$$\begin{aligned} &2\left(\mathbf{W}\mathbf{T}_\lambda\right)\cdot\mathbf{E} + \left(\mathbf{W}\mathbf{T}_\lambda\right)\cdot\mathbf{W} = \\ &2\,t\,\omega_1^2 + \left(t+t_3\right)\left(\omega_2^2+\omega_3^2\right) - \left(t-t_3\right)\left(2\omega_2\gamma_2 + 2\omega_3\gamma_3\right) \end{aligned} \tag{15}$$

We now denote $\tilde{\eta}_\lambda$ as any real number such that

$$\tilde{\eta}_\lambda + t + t_3 > 0 \tag{16}$$

and consider the following inequalities (see (14) and (16)):

$$\begin{aligned} &\left(\tilde{\eta}_\lambda + 2t\right)\omega_1^2 \geq 0 \\ &\left(\left(t-t_3\right)\gamma_2 - \left(\tilde{\eta}_\lambda + t + t_3\right)\omega_2\right)^2 \geq 0 \\ &\left(\left(t-t_3\right)\gamma_3 - \left(\tilde{\eta}_\lambda + t + t_3\right)\omega_3\right)^2 \geq 0. \end{aligned} \tag{17}$$

Then, by adding together the terms in (17) we obtain

$$\begin{aligned} &2\,t\,\omega_1^2 + \left(t+t_3\right)\left(\omega_2^2+\omega_3^2\right) - \left(t-t_3\right)\left(2\omega_2\gamma_2 + 2\omega_3\gamma_3\right) \geq \\ &-\frac{\left(t-t_3\right)^2}{\tilde{\eta}_\lambda + t + t_3}\left(\gamma_2^2+\gamma_3^2\right) - \tilde{\eta}_\lambda\left(\omega_1^2+\omega_2^2+\omega_3^2\right). \end{aligned} \tag{18}$$

Thus, in view of (15), it follows that

$$\begin{aligned} &\int_{\tilde{C}} 2\left(\mathbf{W}\mathbf{T}_\lambda\right)\cdot\mathbf{E} + \left(\mathbf{W}\mathbf{T}_\lambda\right)\cdot\mathbf{W} \geq \\ &-\frac{\left(t-t_3\right)^2}{\tilde{\eta}_\lambda + t + t_3}\int_{\tilde{C}}\left(\gamma_2^2+\gamma_3^2\right) - \tilde{\eta}_\lambda\int_{\tilde{C}}\left(\omega_1^2+\omega_2^2+\omega_3^2\right). \end{aligned} \tag{19}$$

We now determine a bound from below for the left-hand side term in (19) by using the well-known *Korn inequality* (see [18])

$$\int_{\tilde{C}} \mathbf{W}\cdot\mathbf{W} \;\leq\; \left(\tilde{\kappa}_\lambda - 1\right)\int_{\tilde{C}} \mathbf{E}\cdot\mathbf{E}, \tag{20}$$

where the Korn constant $\tilde{\kappa}_\lambda \geq 1$ depends either on the geometry of the cylinder or on the constrained boundary of C, where the displacement are prescribed. By $(13)_1$ we see that

$$-\frac{1}{2}\tilde{\eta}_\lambda \int_{\tilde{C}} \mathbf{W}\cdot\mathbf{W} = -\tilde{\eta}_\lambda\int_{\tilde{C}} \omega_1^2+\omega_2^2+\omega_3^2; \tag{21}$$

thus, we may distinguish two cases determined by inequality (20).

Case a) For $\tilde{\eta}_\lambda \leq 0$ the terms in (21) are positive or equal to zero and consequently from (19) we have the bound from below

$$\int_{\tilde{C}} 2\left(\mathbf{W}\mathbf{T}_\lambda\right)\cdot\mathbf{E} + \left(\mathbf{W}\mathbf{T}_\lambda\right)\cdot\mathbf{W} \geq -\frac{\left(t-t_3\right)^2}{\tilde{\eta}_\lambda + t + t_3}\int_{\tilde{C}}\left(\gamma_2^2+\gamma_3^2\right), \tag{22}$$

which does not involve the Korn inequality.

Case b) For $\tilde{\eta}_\lambda > 0$, by (20)-(21) it follows that

$$\begin{aligned} &-\frac{1}{2}\tilde{\eta}_\lambda \int_{\tilde{C}} \mathbf{W}\cdot\mathbf{W} = -\tilde{\eta}_\lambda\int_{\tilde{C}} \omega_1^2+\omega_2^2+\omega_3^2 \;\geq \\ &-\frac{1}{2}\left(\tilde{\kappa}_\lambda - 1\right)\tilde{\eta}_\lambda \int_{\tilde{C}} \mathbf{E}\cdot\mathbf{E}, \end{aligned} \tag{23}$$

and consequently, in view of $(13)_2$, a lower bound for the left-hand side term of (19) is now given by

$$\begin{aligned} &\int_{\tilde{C}} 2\left(\mathbf{W}\mathbf{T}_\lambda\right)\cdot\mathbf{E} + \left(\mathbf{W}\mathbf{T}_\lambda\right)\cdot\mathbf{W} \geq -\frac{\left(t-t_3\right)^2}{\tilde{\eta}_\lambda + t + t_3}\int_{\tilde{C}}\left(\gamma_2^2+\gamma_3^2\right) \\ &-\frac{1}{2}\left(\tilde{\kappa}_\lambda - 1\right)\tilde{\eta}_\lambda\int_{\tilde{C}} \varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2 + 2\left(\gamma_1^2+\gamma_2^2+\gamma_3^2\right). \end{aligned} \tag{24}$$

We now complete the estimate from below of the functional (8) by analyzing the first term in the right-hand side of (11). To this aim, we let $\tilde{\beta}_\lambda$ be any real number such that

$$\tilde{\beta}_\lambda := \min_{\mathbf{E}}\left\{\frac{\left(C_1\,\tilde{\mathbf{B}}_\lambda + 3C_2\,\tilde{\mathbf{B}}_\lambda^{-1} + p_\lambda\mathbf{I}\right)\cdot\mathbf{E}^2}{\mathbf{E}\cdot\mathbf{E}}\right\} \tag{25}$$

for each symmetric, traceless tensor $\mathbf{E}$, so that it results (see $(13)_2$)

$$\begin{aligned} &\int_{\tilde{C}}\left(C_1\,\tilde{\mathbf{B}}_\lambda + 3C_2\,\tilde{\mathbf{B}}_\lambda^{-1} + p_\lambda\mathbf{I}\right)\cdot\mathbf{E}^2 \geq \tilde{\beta}_\lambda\int_{\tilde{C}}\left(\mathbf{E}\cdot\mathbf{E}\right) = \\ &\tilde{\beta}_\lambda\int_{\tilde{C}} \varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2 + 2\left(\gamma_1^2+\gamma_2^2+\gamma_3^2\right). \end{aligned} \tag{26}$$

Consequently, by (8), (11), (22) and (26) we have the following possibility, which is related to the *Case a)* above discussed.

Case a) $\tilde{\eta}_\lambda \leq 0$

$$\varepsilon(\mathbf{u}) \geq \tilde{\beta}_\lambda \int_{\mathcal{C}} \varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2 + 2\left(\gamma_1^2 + \gamma_2^2 + \gamma_3^2\right) - \frac{(t-t_3)^2}{\tilde{\eta}_\lambda + t + t_3} \int_{\mathcal{C}} \left(\gamma_2^2 + \gamma_3^2\right); \tag{27}$$

thus, since by (16) it results $-\frac{(t-t_3)^2}{\tilde{\eta}_\lambda + t + t_3} \leq 0$, we readily have

$$\varepsilon(\mathbf{u}) \geq \aleph(\mathbf{u}) := \left[\frac{1}{2}\left(2\tilde{\beta}_\lambda - \frac{(t-t_3)^2}{\tilde{\eta}_\lambda + t + t_3}\right)\right] \left[\int_{\mathcal{C}} \varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2 + 2\left(\gamma_1^2 + \gamma_2^2 + \gamma_3^2\right)\right], \tag{28}$$

which clearly shows that the lower bound $\aleph(\mathbf{u})$ for the Hadamard functional $\varepsilon(\mathbf{u})$ is non-negative for any admissible incremental displacement $\mathbf{u}$ if there exists a real number $\tilde{\beta}_\lambda$ satisfying (25) and at least one value of $\tilde{\eta}_\lambda$ satisfying (16) such that

$$\tilde{\beta}_\lambda \geq f(\tilde{\eta}_\lambda) := \frac{(t-t_3)^2}{2(\tilde{\eta}_\lambda + t + t_3)}. \tag{29}$$

Case b) $\tilde{\eta}_\lambda > 0$

In view of (8), (11), (24) and (26), we now have

$$\varepsilon(\mathbf{u}) \geq \left(\tilde{\beta}_\lambda - \frac{1}{2}(\tilde{\kappa}_\lambda - 1)\,\tilde{\eta}_\lambda\right) \int_{\mathcal{C}} \varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2 + 2\left(\gamma_1^2 + \gamma_2^2 + \gamma_3^2\right) - \frac{(t-t_3)^2}{\tilde{\eta}_\lambda + t + t_3} \int_{\mathcal{C}} \left(\gamma_2^2 + \gamma_3^2\right); \tag{30}$$

therefore, we obtain the bound from below

$$\varepsilon(\mathbf{u}) \geq \Im(\mathbf{u}) := \left[\frac{1}{2}\left(2\tilde{\beta}_\lambda - (\tilde{\kappa}_\lambda - 1)\,\tilde{\eta}_\lambda - \frac{(t-t_3)^2}{\tilde{\eta}_\lambda + t + t_3}\right)\right] \left[\int_{\mathcal{C}} \varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2 + 2\left(\gamma_1^2 + \gamma_2^2 + \gamma_3^2\right)\right], \tag{31}$$

which shows that the lower bound $\Im(\mathbf{u})$ for $\varepsilon(\mathbf{u})$ is non-negative for any admissible incremental displacement $\mathbf{u}$ if there exists a real number $\tilde{\beta}_\lambda$ satisfying (25) and at least one value of $\tilde{\eta}_\lambda$ satisfying (16) such that

$$\tilde{\beta}_\lambda \geq g(\tilde{\eta}_\lambda) := \frac{1}{2}\left((\tilde{\kappa}_\lambda - 1)\,\tilde{\eta}_\lambda + \frac{(t-t_3)^2}{(\tilde{\eta}_\lambda + t + t_3)}\right). \tag{32}$$

It is worth noting that inequalities (29) and (32) play the role of *sufficient* conditions for the Hadamard stability (8), since they represent sufficient conditions for the non-negativity of the lower bound estimates $\aleph(\mathbf{u})$ and $\Im(\mathbf{u})$ of the Hadamard functional $\varepsilon(\mathbf{u})$, respectively. Furthermore, *Case a)* and *Case b)* may discussed altogether by introducing the function (see (29) and (32))

$$h(\tilde{\eta}_\lambda) := \begin{cases} \dfrac{(t-t_3)^2}{2(\tilde{\eta}_\lambda + t + t_3)} & \text{if } \tilde{\eta}_\lambda \leq 0 \\[2ex] \dfrac{1}{2}\left((\tilde{\kappa}_\lambda - 1)\,\tilde{\eta}_\lambda + \dfrac{(t-t_3)^2}{(\tilde{\eta}_\lambda + t + t_3)}\right) & \text{if } \tilde{\eta}_\lambda > 0; \end{cases} \tag{33}$$

thus, a sufficient conditions for the Hadamard stability (8) requires the determination of a real number $\tilde{\beta}_\lambda$ satisfying (25) and of a value of $\tilde{\eta}_\lambda$ satisfying (16) such that

$$\tilde{\beta}_\lambda \geq h(\tilde{\eta}_\lambda). \tag{34}$$

Clearly, the smaller is $h(\tilde{\eta}_\lambda)$ the higher is the possibility of successfully fulfilling inequality (34). Having this in mind, the "optimal" choice of $\tilde{\eta}_\lambda$ satisfying (16) is here obtained by choosing the value of $\tilde{\eta}_\lambda$ in correspondence of which $h(\tilde{\eta}_\lambda)$ attains its *infimum*. To this aim, we now need to briefly discuss the main features of the function $h(\tilde{\eta}_\lambda)$.

We note from (16) and (14) that $h(\tilde{\eta}_\lambda)$ is strictly positive and such that

$$\lim_{\tilde{\eta}_\lambda \to -(t+t_3)} h(\tilde{\eta}_\lambda) = +\infty, \quad \lim_{\tilde{\eta}_\lambda \to +\infty} h(\tilde{\eta}_\lambda) = +\infty, \tag{35}$$

$$h'(\tilde{\eta}_\lambda) = \begin{cases} -\dfrac{(t-t_3)^2}{2(t + t_3 + \tilde{\eta}_\lambda)^2} < 0 & \text{if } \tilde{\eta}_\lambda \leq 0 \\[2ex] \dfrac{1}{2}\left[\tilde{\kappa}_\lambda - 1 - \dfrac{(t-t_3)^2}{(t + t_3 + \tilde{\eta}_\lambda)^2}\right] & \text{if } \tilde{\eta}_\lambda > 0, \end{cases}, \tag{36}$$

$$h''(\tilde{\eta}_\lambda) = \frac{(t-t_3)^2}{(t + t_3 + \tilde{\eta}_\lambda)^3} > 0 \qquad \forall\ \tilde{\eta}_\lambda > -(t + t_3). \tag{37}$$

Therefore, the function $h(\tilde{\eta}_\lambda)$ is *positive* and *strictly convex* for $\tilde{\eta}_\lambda > -(t+t_3)$, and it is *strictly decreasing* for $\tilde{\eta}_\lambda < 0$. If we then consider the *unique* real number $\tilde{\eta}_1$ that makes the first derivative $h'(\tilde{\eta}_\lambda)$ in $(36)_2$ zero, that is

$$\tilde{\eta}_1 = (\tilde{\kappa}_\lambda - 1)^{-1/2}(t-t_3) - (t+t_3) > -(t+t_3), \tag{38}$$

we conclude that, for $\tilde{\eta}_\lambda > -(t+t_3)$,

$$\inf h(\tilde{\eta}_\lambda) = \begin{cases} h(\tilde{\eta}_1) \text{ if } t+t_3 \le 0 \text{ or } t+t_3 > 0 \text{ and } \tilde{\eta}_1 > 0 \\ \\ h(0) \text{ if } t+t_3 > 0 \text{ and } \tilde{\eta}_1 \le 0 \end{cases} \tag{39}$$

so that (34) is replaced by the optimal sufficient inequality for the Hadamard stability

$$\tilde{\beta}_\lambda \ge \inf h(\tilde{\eta}_\lambda), \quad \text{with} \quad \tilde{\eta}_\lambda > -(t+t_3). \tag{40}$$

Notice that for the evaluation of the Korn constant in (38) it is convenient to use the classical estimate of Bernstein and Toupin [19]:

$$\begin{cases} \tilde{\kappa}_\lambda = 2 + \dfrac{12}{3+\rho^2(\lambda)} & \text{if } \rho(\lambda) \le \sqrt{3} \\ \\ \tilde{\kappa}_\lambda = 2 + \dfrac{2}{3}\rho^2(\lambda) & \text{if } \rho(\lambda) \ge \sqrt{3}, \end{cases} \tag{41}$$

where

$$\rho(\lambda) = \frac{H(\lambda)}{R(\lambda)} = \frac{\lambda\, H_0}{\lambda^{-1/2}\, \rho_0} = \rho_0\, \lambda^{3/2} \tag{42}$$

is the slenderness ratio of the deformed circular cylinder and $\rho_0 = H_0/R_0$ is the undistorted slenderness ratio.

III. Lower Bound Estimate of the Critical Load in a Uniaxial Compressive Loading Process

The procedure developed in the previous section is closely related to *stability* issues, since it yields conditions under which a deformed configuration is Hadamard stable. However, our method may be also employed for analyzing *bifurcation* problems through the classical approach of adjacent equilibria, wherein one usually considers a monotonic loading process ruled out by a parameter λ and checks if there is a "critical" value λ_{cr} of the load in correspondence of which the primary equilibrium deformations ceases to be Hadamard stable and a possible bifurcation mode may arise.

In such a context, the critical load λ_{cr} may be defined as the value of the load parameter which first renders the Hadamard functional zero, whereas the lower bound estimates λ_{LB} for the critical load may be considered as a load below which the infinitesimal Hadamard stability criterion is definitely satisfied. In particular, according to the discussion of the previous section, a bound from below λ_{LB} of the critical load λ_{cr} may be defined as the *smallest value* of the load λ in correspondence of which the inequality (40) is violated for the first time.

It is important to emphasize that the availability of a lower bound estimate for the critical load may not represent by itself a satisfactory information in bifurcation problems. Indeed, during a loading process one usually identifies the load corresponding to a *special* bifurcation mode with the load which first renders the Hadamard functional zero, but with reference to a particular subclass of incremental displacements. Then, one may wonder if such a special solution is actually the first bifurcation mode (among other bifurcations) which occurs during the prescribed loading process, and consequently if the corresponding bifurcation load is actually the "true" critical load and not an upper bound estimate for the critical load. Since it is extremely difficult to check the sign of the Hadamard functional on the whole class of admissible incremental displacements, a possible answer may be obtained by checking whether the gap between the bifurcation load related to the special bifurcation mode and the lower bound estimate for the critical load is sufficiently small or not.

In this Section, by following arguments partly developed in we sketch a procedure for determining an optimal lower bound estimate of the critical load in a *uniaxial compressive* monotonic loading process for the incompressible Mooney-Rivlin cylinder $\mathcal{C}$.

We then assume that the body forces are zero and that the cylinder is subject to the mixed dead-load boundary-value problem defined by zero traction on the lateral surface, zero tangential traction on the bases, and prescribed normal displacements at the bases. These prescriptions clearly correspond to a homogeneous uniaxial deformation. Since by the above the homogeneous fundamental deformation and the total Cauchy stress are determined by (6) and (7), respectively, then the boundary condition

$$\mathbf{T}_\lambda\, \mathbf{n} = \mathbf{0}, \qquad \mathbf{n}\cdot\mathbf{e}_3 = 0 \tag{43}$$

on the lateral surface of the cylinder immediately determines the "pressure-like" field:

$$p_\lambda = C_1\, \lambda^{-1} - C_2\, \lambda\,. \tag{44}$$

Thus, by substituting (44) into (7), we obtain the following expression for the total Cauchy stress:

$$\mathbf{T}_\lambda = \sigma(\lambda)\ \mathbf{e}\otimes\mathbf{e}, \tag{45}$$

where

$$\sigma(\lambda) := (C_1\ \lambda + C_2)\ (\lambda^3 - 1)\lambda^{-2} < 0 \quad \text{for} \quad \lambda \in]0,\ 1[\tag{46}$$

shows that $\mathbf{T}_\lambda$ corresponds to a compressive axial stress.

We now consider a monotonically decreasing loading process starting from the undistorted configuration, in order to evaluate the lower bound estimate λ_{LB} of the "critical" compressive stretch λ_{cr}, at which the Hadamard stability criterion is violated for the first time. Notice that at λ_{cr}, possible bifurcation modes may occur.

We emphasize that our approach for calculating λ_{LB} is somewhat easy. Indeed, given the geometry of the body by assigning the undistorted slenderness ratio $\rho_0 = H_0/R_0$ and the material parameters C_1, C_2, we consider a decreasing sequence of values for λ starting from $\lambda = 1$, and for each λ we calculate the principal stress $\sigma(\lambda)$ using (46), the Korn constant by means of (41)-(42), $\tilde{\mathbf{B}}_\lambda$ by (6) and $\tilde{\beta}_\lambda$ in (25) by employing, for example, a numerical minimization tool. Furthermore, since by (45)-(46) we have for the principal Cauchy stresses $t_1 = t_2 = t = 0$ and $t_3 = \sigma(\lambda) < 0$, then $t + t_3 = \sigma(\lambda) < 0$ and consequently (see (39) and (38)) $\inf h(\tilde{\eta}_\lambda) = h(\tilde{\eta}_1)$ with

$$\tilde{\eta}_1 = -\sigma(\lambda)\left[(\tilde{\kappa}_\lambda - 1)^{-1/2} + 1\right] > -\sigma(\lambda) > 0. \tag{47}$$

Therefore, we finally have

$$\inf h(\tilde{\eta}_\lambda) = h(\tilde{\eta}_1) = -\sigma(\lambda)\left[(\tilde{\kappa}_\lambda - 1)^{1/2} + \frac{1}{2}(\tilde{\kappa}_\lambda - 1)\right], \tag{48}$$

and consequently we conclude that λ_{LB} is the value of λ which first makes $\tilde{\beta}_\lambda - \inf h(\tilde{\eta}_\lambda)$ in (40) equal to zero.

In Fig. 1 we compare our estimates to that obtained in [7] for incompressible bodies whose strain energy density depends only on the second invariant of the left Cauchy-green stain tensor **B**. The expressions contained [7], but revisited and corrected according on how we understand their meaning, may be found in [11]. In particular, our estimate is better for thick cylinders (approximately 9%), whereas for slender cylinders the two estimates practically coincide.

Although in the above case of uniaxial compression our estimate improves the one in [7], the effectiveness of an estimate from below arises if one may show that its "distance" from the actual critical load is small enough. Here, we have compared λ_{LB} to the upper bound estimate reported in [7, Section 9]. This shows that for slender bodies, where the loss of stability is mainly due to geometrical effects and the onset of instability is accompanied by relatively small deformations, λ_{LB} accurately approximates the critical load. On the other hand, when an instability is strongly related to the non-linearity of the constitutive equation (small slenderness), we have found that our lower bound estimate may not represent a good approximation of the critical load. Thus, there remains the open problem of enhancing lower bound estimates when stable large deformations are anticipated. This represents an issue for possible future researches.

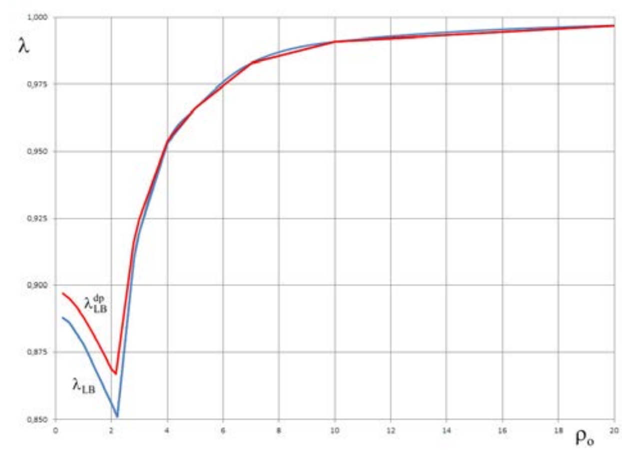

Fig. 1 Uniaxial compression for $C_1 = 0$, $C_2 > 0$: blue line our estimate, red line-the estimate in [5].

IV. Conclusions

On the basis of the approach sketched in [20], we have developed a new method for determining a lower bound estimate of the Hadamard functional based on the Korn inequality for Mooney-Rivlin incompressible, isotropic elastic solids, with the aim of obtaining an optimal bound from below of the critical load for a homogeneous uniaxial compressive loading process. The comparison to other recent proposals shows that our procedure is effective and, most of all, it has as further advantage the ease for applications.

We emphasize that our method may be extended to general cases of inhomogeneous deformations for different and more complex boundary value problems. What still deserves an improvement for the current problem of uniaxial compression is the estimate of the critical load for thick cylinders, for which the instability is basically due to large deformations rather than geometrical effects.

Finally, we underline that our procedure yields a strategy for checking if a primary deformation is on the stable "safe side" and also a method for establishing which is the first possible bifurcation, but this method does not allow to assert if a bifurcation mode actually occurs or, in other words, if there are actually local branches of post-bifurcating solutions. One may refer to [21] in order to find conditions for the existence of local branches of bifurcating solutions within the elliptic range, namely the classical strong ellipticity condition and the boundary complementing conditions for the fourth-order incremental elasticity tensor field which rules the equilibrium problem linearized around the bifurcation point.

Notice that the fulfilment of such conditions gives raise to the so-called *diffuse* bifurcations within the elliptic range, which are substantially related both to the geometry of the body and to the boundary conditions. Notice that these kinds of instabilities basically differ from other bifurcation phenomena, like for example the localization due to material softening, basically related to the violation of the strong ellipticity, which usually may be ascribed to the non-linearity of the material response and not to the influence of the boundary conditions.

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