Spatial energy distribution in a harmonic oscillator and the golden section

B. Müller, E. Koyutürk, D. Göncü

***Abstract*—**The high school physics curriculum mostly targets the time dependent physical quantities of the harmonic oscillator e.g. velocity, acceleration and energies, because harmonic oscillations are defined as "periodic time dependent changes of physical quantities".

The focus of our interest in this paper is indeed the spatial dependence of the physical quantities, especially the spatial distribution of the three energies, which continuously change their amount in a harmonic oscillator system: kinetic energy (E_{kin}), potential energy (E_{pot}) and elastic potential energy (E_{epe}). The idea for this has been given by some students of the 11'th grade from the German high school of Istanbul (Özel İstanbul Alman Lisesi) in [1]May 2015.

We found out that the golden section plays an important role in the spatial energy distribution, especially between potential energy and elastic potential energy. Until today the role of the golden section has not been mentioned in such a simple system like the harmonic oscillator, but in more complicated systems. We cited the role of the golden section in the KAM- theorem and in Burgers-turbulence and found parallelisms to our results.

***Keywords*—**golden section, stability, spring pendulum, gravitational force

I. INTRODUCTION

The occurrence of the golden section is well known in many areas of nature. In this paper we investigated its role in the harmonic oscillator.

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We consider a simple system of a spring pendulum with the spring constant D, by which the mass can be neglected, a mass point with mass m, hanging at the end of the spring, where the spring is in a relaxed position at the beginning, meaning the elastic potential energy (E_{epe}) equals zero. (The mass is then released to oscillate vertically to the ground, so the mass only moves one dimensionally in y direction). In this case the oscillation length equals to $2\hat{y}$, where $\hat{y}$ is the amplitude with $\hat{y} = \frac{m \cdot g}{D}$ as the maximal distance from the equilibrium point. All other effects such as air resistance, spatial extension of the mass point etc. will not be taken into account.

In a second step we enhanced our considerations to an infinitely expanded harmonic oscillator where we found an interesting trace for a possible explanation in context to the investigations of the double pendulum.

The mathematical origin of the golden section can be shown by finding the following quadratic equation: If the ratio between the length x (the major part) and $1-x$ (the minor part), which give 1 when added up and the ratio between the major part and the whole can be equated, the solution of this equation with the quadratic formula is the length of the major part also called the golden section Φ. Therefore the minor part equals. 1-Φ.

$$\frac{x}{1-x} = \frac{1}{x} \Rightarrow x^2 + x - 1 = 0 \Rightarrow$$

$$x_{1/2} = -\frac{1}{2} \pm \sqrt{\frac{5}{4}} \Rightarrow x_1 = \frac{\sqrt{5}-1}{2} = \Phi = 0{,}618...$$

(major part) and the minor part equals

$$1 - x_1 = 1 - \Phi = 0{,}382...$$

Note that there are many other ways to get Φ, e.g. by the Fibonacci-sequence, chain fractions etc.

In the recent years the Binet-Fibonacci Formula for Fibonacci numbers is treated as a q-number (and q-operator) with Golden ratio bases $q=\varphi$ and $Q=-\frac{1}{\varphi}$ by PASHAEV and NALCI, where φ is the positive root of equation $x^2-x-1=0$. Quantum harmonic oscillator for this golden calculus is derived so that its spectrum is given just by Fibonacci numbers [11].

Indeed our considerations are restricted to the mechanical harmonic oscillator. In section II and III it will be derived and shown, that the place, where potential energy (E_{pot}) and elastic potential energy (E_{epe}) equal themselves, divides the total oscillation length exactly in the golden section ratio. In section IV and V the occurrence and the meaning of the golden section in other areas e.g. planetary systems, double pendulum and shockwaves [1]-[6] with the intention of finding parallelisms and explanations to our subject will be shown. In section VI and VII the oscillation length of the spring pendulum will be expensed from zero to infinity. Especially in the case of infinite oscillation length there is an interesting trace of explanation related to the work of RICHTER [1].

II. ENERGY BALANCE OF THE SYSTEM

For getting an appropriate comparison of the three energies E_{pot}, E_{epe} and kinetic energy (E_{kin}), it needs a coordinate-system, in which the deepest point of the oscillation has zero potential energy. If the mass point oscillates between $0 \le y \le 2\hat{y}$ we set at its highest position $y=0$, so in the equilibrium point y equals $\hat{y}$ and at the deepest point y equals $2\hat{y}$. Under this presumption the potential energy is written as $E_{pot}=m\cdot g\cdot 2\hat{y}-m\cdot g\cdot y$, we get $E_{pot}=0$ at the deepest point $y=2\hat{y}$ and its maximum at the highest point $y=0$. Indeed the elastic potential energy $E_{epe}=\frac{1}{2}D\cdot y^2$ has its minimum at $y=0$ (relaxed spring) and its maximum at $y=2\hat{y}$ (maximal extension of the spring). We can set $E_{kin}=\frac{1}{2}m\cdot v^2$ where v is the velocity of the mass point. The following equation shows the summation of the three energies.

$$m\cdot g\cdot 2\hat{y}-m\cdot g\cdot y+\frac{1}{2}D\cdot y^2+\frac{1}{2}m\cdot v^2=E_{total} \tag{1}$$

Note $E_{total}=m\cdot g\cdot 2\hat{y}=D\cdot 2\hat{y}^2$ (1a)

III. WHERE DO $E_{epe}=E_{pot}$ EQUAL THEMSELVES?

To find the place of the y, where the potential energy and the elastic potential energy have the same amount, we set

$$E_{pot}=E_{epe}\Rightarrow m\cdot g\cdot 2\hat{y}-m\cdot g\cdot y=\frac{1}{2}D\cdot y^2 \tag{1b}$$

(1b) is a quadratic equation with the variable y and its solution (with the quadratic formula) is:

$$y_{1/2}=-\frac{m\cdot g}{D}\pm\sqrt{\frac{m\cdot g}{D}\cdot\left(\frac{m\cdot g}{D}+4\hat{y}\right)} \tag{2}$$

In the equilibrium point there is a balance between gravitation force ($F_G=m\cdot g$) and the tension force ($F_{tension}=D\cdot\hat{y}$), so the following equation applies

$$\hat{y}=\frac{m\cdot g}{D} \tag{3}$$

(3) is inserted in (2):

$$y_{1/2}=-\hat{y}\pm\sqrt{5\hat{y}^2} \tag{2a}$$

The solution y_2 has no physical meaning, because it is outside of the oscillation interval $0\le y\le 2\hat{y}$. Indeed the other solution can be rewritten as $y_1=\hat{y}\cdot\left(\sqrt{5}-1\right)$. This is the place, where the amount of elastic potential energy and potential energy are equal. The result of the comparison between y_1 and the total oscillation length $2\hat{y}$ is the following equation:

$$\frac{y_1}{2\hat{y}}=\frac{\sqrt{5}-1}{2}. \tag{4}$$

This is exactly the division length $\Phi=\frac{\sqrt{5}-1}{2}=0{,}618...$ of the golden section.

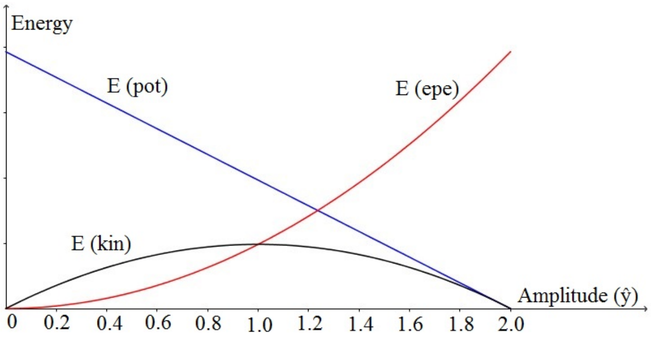

Fig. 1 The system with $\hat{y} = 2.0$. The relation of each energy with the total energy is squares or cubes of the golden section.

In the following section the relation of the three energies with the total energy at this special point y_1 are compared. According to (1) and (1a) the total energy equals $m \cdot g \cdot 2\hat{y}$. If y_1 is inserted into the equation of E_{pot} the relation of it to the sum of all three energies E_{total} turns out as the following equation.

$$\frac{E_{pot}}{E_{total}} = \frac{m \cdot g \cdot 2\hat{y} - m \cdot g \cdot \hat{y} \cdot \left(\sqrt{5}-1\right)}{m \cdot g \cdot 2\hat{y}} = 1 - \frac{\sqrt{5}-1}{2} = \Phi^2 = 0{,}382...$$
(5)

Due to the presumption of potential energy being equal with the elastic potential energy at y_1 the relation of the elastic potential energy with the total energy gives the same result:

$$\frac{E_{epe}}{E_{total}} = \Phi^2 \qquad (6)$$

Consequently the relation of the kinetic energy to the total energy amounts $\frac{E_{kin}}{E_{total}} = 1 - 2\Phi^2 = \Phi^3$, (7) suggesting that the relation of each energy with the total energy is squares or cubes of the golden section. These results are shown in Fig. 1.

IV. THE GOLDEN SECTION IN COUPLED OSCILLATIONS AND THE KAM-THEOREM

PETER H. RICHTER performs an elaborated consideration about planetary movements and coupled oscillations. A computer simulation containing an artificial planetary system, in which the sun is the biggest mass, Jupiter's mass is only 1/1000 of the sun's and a test planet with a very small mass are located, can demonstrate that such systems are not stable, in a case of casually chosen distances between the planets. If the test planet is too close to Jupiter and if such approaches are repeated in a regulation the test planet will be ejected from the system. This incident brings up the question whether our planetary system is stable. The answer is that there is self-evidently stability, in the opposite case the mankind would not stay alive. The distance relation between two planers is roughly the golden section. If this relation was to be expressed through an equation, the distance between the n-th planet and the sun should have been $d(n)$, which would cause the distance between the sun and the subsequent planet to be $d(n+1)$, so that the equation would be the following ([8] LANDSCHEIDT, 1995):

$$\frac{d(n)}{d(n+1)} \approx \Phi$$

It should be noted that this is an arithmetical consideration. Φ is the most irrational number and causing the approaches and disturbances between the planets to be extremely irregular. ([7] SCHOLZ, 1987) Therefore the resonance effects cannot occur followed by stability as a result.

A scientific and detailed explanation about the stability-problems has been given 1962 by KOLMOGOROV und ARNOLD [2] at Moscow and independently by MOSER ([3] MOSER, 1962) at Göttingen (KAM-theorem).In recent time it is explained e.g. by KÖNIG, K. and RÜSSELER, K. [10]. In our paper these problems are demonstrated experimentally by a planar coupled oscillator, the so called double pendulum. The configuration of the double pendulum is described by two angles φ_1 und φ_2.

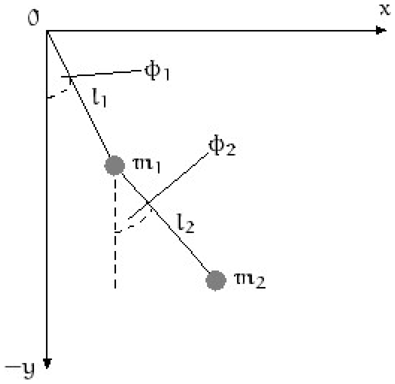

Fig. 2 The configuration of the double pendulum is described by two angles φ_1 und φ_2

To simplify the system the masses of the points are considered equal and the air resistance is neglected. The movement of the double pendulum can only be described when the angles, velocity and torque values of both pendulums are known. The dynamics take place in a four dimensional space. It is considered that every movement of the double pendulum is a trace in the phase space, which is in fluent movement. When the properties of this "fluid" are observed, that regular and chaotic behavior occur together, which is similar to the partial laminar and partial turbulent stream in an incompressible ideal fluid. If the double pendulum was to be pushed, eight types of movement could be observed. Nevertheless it is important to mention that in a case, where the gravitational force is involved, meaning that the double pendulum is not horizontally positioned, transitions between all the types of movements from one to eight shall be found. Sometimes one of the pendulum finds itself in an instable position, where it has to "decide" whether to oscillate back or jump over, which consequently leads to a chaotic movement. The gravitational force gives a torque to the system and the total angular momentum is no more a conservation quantity.

Another observation on RICHTER's work is the fact, that the double pendulum system is stable if the energies are very high, then the influence of the gravitational force can be neglected. So in a system with low energy, in which the gravitational force is playing a role, a transition from stability into chaos can be observed.

A resonance situation could occur if

$k_1 \cdot \omega_1 + k_2 \cdot \omega_2 = 0$ with $k_1, k_2 \in Z$ and the ratio of frequencies $\frac{\omega_1}{\omega_2}$ rational number.

Indeed the greatest stability that can be found is the case of the so called „KAM-Torus". ([4] ARNOLD, 1978) This applies $\frac{\omega_1}{\omega_2} = \Phi$. This is a typical example showing the stabilizing function of Φ. The role of the golden section has also been mentioned in other types of oscillating system, e.g. in the turning points of the spherical pendulum ([9] ESSÉN, H., APAZIDIS, N., 2009)

V. THE GOLDEN SECTION IN SHOCK WAVES AS A RESULT OF THE BURGERS-EQUATION

The previous researches on turbulence have shown that the quality of diverse turbulence-models can be tested easier: not with the full governing equations of fluid mechanics (Navier-Stokes equations), but only the one dimensional analogy, in which the Burgers-equation is also involved.

$$\frac{\partial v}{\partial t} + v \cdot \frac{\partial v}{\partial y} = \nu \cdot \frac{\partial^2 v}{\partial y^2} \qquad (8)$$

ν stands for „kinematic viscosity" and v the velocity of the fluid in y-direction.

This equation derives from a mathematical model, which BURGERS ([5] BURGERS, 1948) developed for the illustration of the turbulence theory. (8) has to be interpreted in this sense, that the velocity v is a deviation of the fluid particle in y - direction, whereas the fundamental streaming is two dimensional, but has no movement in x - direction.

The numeric solution of (11) gives a time dependent "shock wave", whilst the amplitudes are decreasing. If the amplitudes should not decrease a periodic force-term is added e.g. $A \cdot \sin(2\pi \cdot y)$ with $A \in R$. If the elongations at a certain point in the area $0 \le y \le 1$ were to be fixed („stop" the travelling wave), a Galilei-transformation could be inserted, this results into the following equation:

$$\frac{\partial v}{\partial t} = \frac{1}{R} \cdot \frac{\partial^2 v}{\partial y^2} - \frac{\partial v}{\partial y} \cdot (v - 1) - A \cdot \sin(2\pi \cdot y) \qquad (9)$$

R = Reynolds-Number

It shows that in the quasi stationary solution $\left(\frac{\partial v}{\partial t} = 0 \right)$ one of the two zero points ($v = 0$ is exactly the shock-region (the midpoint of the swirl)) and is located at $y = \frac{\Phi + 1}{2}$. The other zero point is found at $y = \Phi^2$. ([6] MÜLLER, B., 2005).

It should be noted that the "elongations" of the shock-waves are the y-velocities of the fluid particles, so the zero point means a change from "up" to "down" in y - direction and in addition at this point there is the maximal spatial gradient of the velocity.

$\left(\frac{\partial v}{\partial y} = \max. \right)$ This makes the center of the swirl, the only stable point in a turbulent system. In this case also the stabilizing function of Φ is shown. Another context of the Burgers equation to the Golden ratio is given by PASHAEV, where the nonlinear complex q-Burgers equation has been solved and Fibonacci numbers has been described as a special type of q-numbers with matrix Binet formula [12].

VI. THE OSCILLATION OF THE OBJECT FROM A POINT y' $(0 \le y' \le 2\hat{y})$ INSIDE THE ORIGINAL OSCILLATION LENGTH

If the object is released to oscillate from a deeper point than $y = 0$, here y' in the same oscillating system applies $\left[0 \le y \le 2\hat{y}\right]$ the following energy balance:

$$m \cdot g \cdot (2\hat{y} - y') - m \cdot g \cdot (y - y') + \frac{1}{2} D \cdot y^2 + \frac{1}{2} m \cdot v^2$$
$$= E_{total}$$
$$= m \cdot g \cdot (2\hat{y} - y') + \frac{1}{2} D \cdot y'^2$$

(10)

And it follows $y \ge y'$.

(10) displays that the place $y_1 = \hat{y} \cdot (\sqrt{5} - 1)$, where $E_{pot} = E_{epe}$ does not change dependent from the place y', from which the object is released to oscillate, because in the component E_{pot} the terms $m \cdot g \cdot y'$ eliminate themselves.

VII. THE OBJECT OSCILLATES FROM A POINT $2\hat{y} + \varepsilon$ OUTSIDE OF THE ORIGINAL OSCILLATION LENGTH

In this case the oscillating object would push the spring about ε at the place $y = 0$, where $E_{epe} = 0$. Then the spring has an elastic potential energy at its "new" highest point. For the investigation of this important case we create an extended coordinate-system and we set $y = 0$ as the "new" highest place of the object, so in contrast to the origin oscillation length the "new" total oscillation length, where the object moves, is in the interval $\left[0 \le y \le 2\hat{y} + 2\varepsilon\right]$.

In the following we will call it "extended oscillation length"

From now on the following energy balance is acquired:

$$m \cdot g \cdot 2(\hat{y} + \varepsilon) - m \cdot g \cdot y + \frac{1}{2} D \cdot (y - \varepsilon)^2 + \frac{1}{2} m \cdot v^2$$
$$= E_{total}$$
$$= \frac{1}{2} D \cdot (2\hat{y} + \varepsilon)^2$$

(11)

In this case, the "new" place y_1, where $E_{pot} = E_{epe}$ is

$$y_1 = \varepsilon - \hat{y} + \sqrt{5\hat{y}^2 + 2\hat{y} \cdot \varepsilon} \qquad (12)$$

Obviously this place is *not* in the golden section relation to the extended oscillation length, but we have in general for $\varepsilon > 0$:

$$\frac{y_1}{2(\hat{y} + \varepsilon)} \ne \Phi \qquad (13)$$

Also considering the relation to the original oscillation length applies $\frac{y_1}{2\hat{y}} \ne \Phi$ (13a). The fundamental difference to the original length and the inner original length is the change in y_1, which depends on ε. Now it shall be investigated, if at least one $\varepsilon > 0$ exists, where (13a) can be set equal to Φ. If (13a) will be plotted as a function of ε, we get

$$f(\varepsilon) = \frac{y_1}{2(\hat{y} + \varepsilon)} = \frac{\varepsilon - \hat{y} + \sqrt{5\hat{y}^2 + 2\hat{y} \cdot \varepsilon}}{2(\hat{y} + \varepsilon)} \qquad (14)$$

$f(\varepsilon)$ equals Φ again, if $\varepsilon = 8 \cdot \left(\Phi + \frac{3}{2}\right) \cdot \hat{y}$

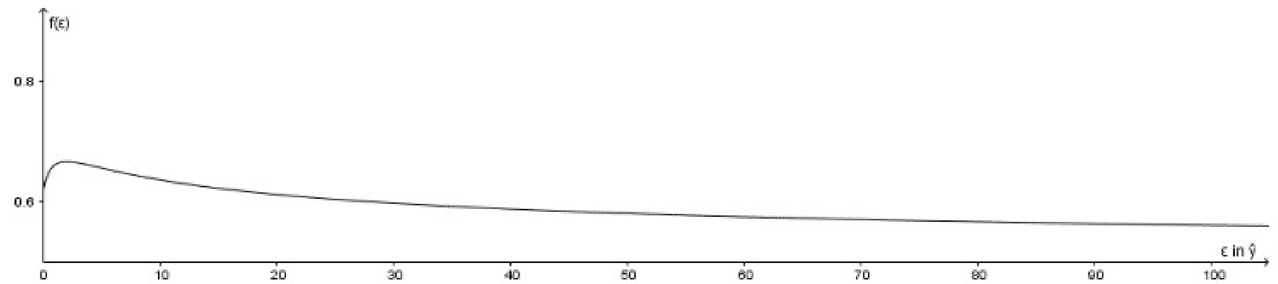

Fig. 3 The fundamental difference to the original length and the inner original length is the change in y_1, which depends on ε.

It can be seen that the curve of Fig. 3 has a maximum at $\varepsilon = 2\hat{y}$ with $f(2\hat{y}) = \frac{2}{3}$, then it approaches Φ at a higher ε, and the most interesting fact is that for very high ε there is a convergence with $\lim_{\varepsilon\to\infty} f(\varepsilon) = 0{,}5$. We are aware, that this situation cannot be applied in an experiment, but nevertheless this result is very important for our theoretical investigations. When these two extreme values ($\frac{1}{2}, \frac{2}{3}$) are considered, it should be noted that these numbers are first "members" of the reciprocal Fibonacci-sequence are considered 1,1,2,3,5,8,13, 21… with

$$\lim_{n\to\infty} \frac{f(n)}{f(n+1)} = \Phi$$

According to this there are three locations in this function:

Table 1 f(ε) as a function of ε

ε	0	$2\hat{y}$	∞
$f(\varepsilon)$	Φ	$\frac{2}{3}$	$\frac{1}{2}$

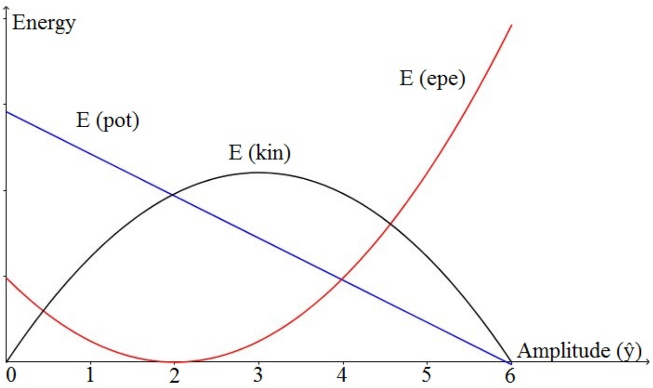

Fig. 4 the system at the maximum value of Fig. 3 with $\varepsilon = 2\hat{y}$.

If the three energies in the case $\varepsilon \to \infty$ with the energies at the origin oscillation length are to be compared, it can be seen that E_{kin} and E_{epe} predominate E_{pot} for high ε. In this case we have a system with very high energies *and* the role of the gravitation force decreases to zero. Now we remember to the results of RICHTER ([1] RICHTER, 1998), where it is shown, that in the case of high energies the double pendulum is stable and only if the gravitation force is involved *and* the total energy of the system is lower, the system gets chaotic, except the frequencies of the two pendulum have the relation $\frac{\omega_1}{\omega_2} = \Phi$. So we can conclude: If we have an origin or slower oscillating length, the gravitation force plays an important role with a great E_{pot}-component, then the golden section stabilizes the system with $\frac{E_{kin}}{E_{pot}} = \frac{E_{kin}}{E_{epe}} = \Phi$. By extended oscillation length the system will be stabilized by high energies and the stabilization by Φ seems "not necessary".

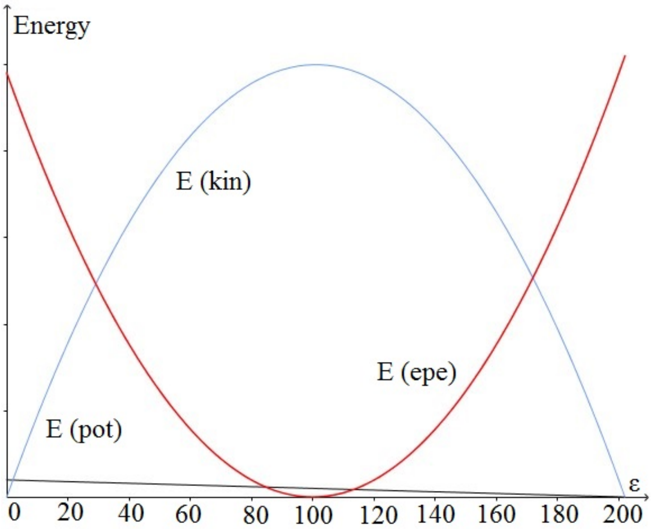

Fig. 5 the system with $\varepsilon = 200\hat{y}$.

In contrast to the case of original oscillation length from now on will be the focus of interest the place, where $E_{kin} = E_{epe}$. It is known from the original oscillation with $\varepsilon = 0$, that we find equality between kinetic and elastic potential energy at $y = 0$ and $y = \hat{y}$. But if $\varepsilon > 0$, the graph of the function $E_{epe}(y) = \frac{1}{2} D \cdot (y - \varepsilon)^2$ is a parabola, which is moved to the right hand side with minimum at $y = \varepsilon$.

Therefore it is expected, that the graph of the function

$$E_{kin}(y) = \frac{m \cdot g}{2} \cdot \left(\frac{(2\hat{y}+\varepsilon)^2}{\hat{y}} - \frac{(y-\varepsilon)^2}{\hat{y}} + 2y - 4\cdot(\hat{y}+\varepsilon) \right)$$

overlaps the graph of the elastic potential energy at two places, it results

$$y_{1/2} = \varepsilon + \frac{\hat{y}}{2} \pm \sqrt{\frac{1}{2}\varepsilon^2 + \varepsilon \cdot \hat{y} + \frac{\hat{y}^2}{4}} \qquad (15)$$

Is there a connection with Φ to be found? It is seen directly that $\frac{y_{1/2}}{2(\hat{y}+\varepsilon)} \neq \Phi$ for most of the ε. In analogy to the case $E_{epe} = E_{pot}$ a function $f(\varepsilon)$ can be plotted with (14), the following equation is the result:

$$f(\varepsilon)=\frac{y_{1/2}}{2(\hat{y}+\varepsilon)}=\frac{\varepsilon+\frac{\hat{y}}{2}\pm\sqrt{\frac{1}{2}\varepsilon^{2}+\varepsilon\cdot\hat{y}+\frac{\hat{y}^{2}}{4}}}{2(\hat{y}+\varepsilon)} \quad (16)$$

pendulum we cited the work of RICHTER, where he shows that in the cases of high energies or/and ignorance of gravitational forces the system is stable, but if the system has lower energies and the gravitation has been taken into account, the system gets chaotic *except* the relation of the

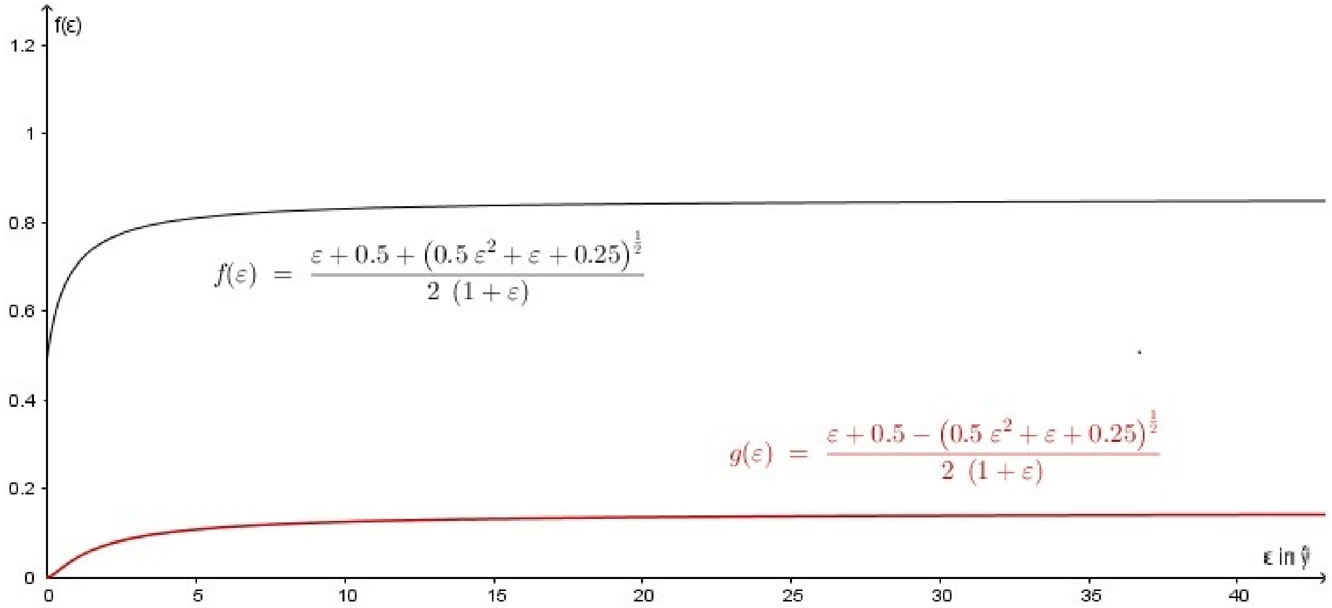

Fig. 6

In contrast to (14), which is shown in Fig. 3, in Fig. 6 there is no local maximum but there are two convergences.

In the case $\varepsilon\to\infty$ the limes $\lim\limits_{\varepsilon\to\infty} f(\varepsilon)=0{,}5\left(1\pm\frac{1}{\sqrt{2}}\right)$

(17)

To conclude the three convergences that have been found, may play the stabilizing role in the case of high energies.

VIII. SUMMARY

We investigated the spatial distribution of energies and other physical quantities by a harmonic oscillator with a spring pendulum.

At first we detected that the place where $E_{pot}=E_{epe}$ divides the oscillation length $2\hat{y}$ of the system exactly in the golden section relation Φ and therefore at this place the following relation between all three energies applies: $\frac{E_{kin}}{E_{pot}}=\frac{E_{kin}}{E_{epe}}=\Phi$.

It is known that the golden section occurs in all areas of natural science and art. We presented three examples of the physics area, where it plays an important role (double pendulum, KAM theorem and shock-waves). By the double

frequencies of both pendulum is $\frac{\omega_1}{\omega_2}=\Phi$. So the golden section can be interpreted as a stabilizing factor.

Indeed in our investigation the system is stable and we cannot choose between two frequencies, but the golden section is integrated in the energy relations.

Then we changed the oscillation length. At first we considered a system where the mass point oscillates inside the origin oscillation length $2\hat{y}$ with $\hat{y}=\frac{m\cdot g}{D}$ and no change of the place, where $E_{epe}=E_{pot}$, has been found. But if the oscillation length is extended $[0\le y\le 2(\hat{y}+\varepsilon)]$ where ε is the extending-factor, this place changes dependent of ε and there is not the relation Φ to the total oscillation length $2(\hat{y}+\varepsilon)$ anymore.

This phenomenon is expressed as a function $f(\varepsilon)$ and we found out that $f(\varepsilon)$ has a local maximum at $\varepsilon=2\hat{y}$ with $f(2\hat{y})=\frac{2}{3}$, and after the maximum this function converges $\lim\limits_{\varepsilon\to\infty} f(\varepsilon)=0{,}5$

The extreme values of this function show an interesting similarity to the Fibonacci-sequence.

We interpret this by finding a parallel to the work of RICHTER:

If the system has low energies and the gravitational force is dominant ($\varepsilon = 0$ or the mass point oscillates inside $[0 \le y \le 2\hat{y}]$, the occurrence of Φ in the energy relations gives the stability.

If the system has high energies ($\varepsilon \to \infty$) the gravitational force and E_{pot} in the energy balance can be neglected, then Φ disappears in the energy relations but a high energy system is stable by itself like in the case of the double pendulum.

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