Utilizing Grillage of One-Dimensional Elements for Stability Problems of Rectangular Plates Resting on Elastic Foundations

Abdulhalim Karasin, Rehber Akdogan, Murat Arda Ugurlu

Abstract— In many cases solution for **s**tability of plate resting on elastic foundation problems have been available for limited regular geometries, but where irregular boundaries or partial contact are encountered difficulties arise because it will be necessary to describe the governing equation of motion in a general mathematical form. The intention of this study is to extend analytical solutions of the discrete one-dimensional beam elements resting on elastic foundation for solution of plate buckling problems. The solution can be stated as an extension of the so-called discrete parameter approach where the physical domain is broken down into discrete sub-domains. The derivations of the governing differential equations and geometric stiffness terms obtained to observe the influences of foundation parameters. Analytical solution of the discrete one-dimensional elements extended for solution of complex plate problems.

Keywords— Geometric stiffness matrices, elastic foundation, stability, grillage of beams, FEM.

I. Introduction

Plates on elastic foundations have received considerable attention due to their wide applicability in many engineering disciplines. Since the interaction between structural foundations and supporting soil has a great importance in many engineering applications, a considerable amount of research has been conducted on plates on elastic foundations. Many studies have been done to find a convenient representation of physical behaviour of a real structural component supported on a foundation[1-5]. The usual approach in formulating problems of beams, plates, and shells continuously supported by elastic media is based on the inclusion of the foundation reaction in the corresponding differential equation of the beam, plate, or shell.

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This simplest simulation of an elastic foundation is considered to provide vertical reaction by a composition of closely spaced independent vertical linearly elastic springs. Thus the relation between the pressure and deflection of the foundation can be written as;

$p(x,y)=k_1\, w(x,y)$

where p(x,y) is distributed reaction from the foundation due to applied load and k1 is Winkler parameter with the unit of force per unit area/per unit length (force/length3).

Then the governing differential equation or Lagrange's equation of a plate on Winkler foundation subjected to combined action of transverse load and biaxial in-plane loading can be derived as;

$$D\left(\frac{\partial^4 w}{\partial x^4}+2\frac{\partial^4 w}{\partial x^2\partial y^2}+\frac{\partial^4 w}{\partial y^4}\right)+k_1 w + N_x\frac{\partial^2 w}{\partial y^2}+N_y\frac{\partial^2 w}{\partial x^2}=q(x,y)$$

where, w is vertical deflection, D is flexural rigidity of the plate q(x,y) is the external load on the plate and Nx and Ny are in-plane loads in x and y directions respectively,

This equation is applicable to all types of plates resting on one-parameter elastic foundation problems. If there are no any loads other than the in-plane loads the equation will define an eigenvalue problem which lets us to find out the critical buckling loads. That is the governing differential equation of the plates under static buckling of plates is

$$D\left(\frac{\partial^4 w}{\partial x^4}+2\frac{\partial^4 w}{\partial x^2\partial y^2}+\frac{\partial^4 w}{\partial y^4}\right)+k_1 w + N_x\frac{\partial^2 w}{\partial y^2}+N_y\frac{\partial^2 w}{\partial x^2}=0$$

Currently, there exist approximate and numerical methods to solve the governing differential equations of plates resting on one-parameter elastic foundation. Many studies have been done related to such problems. Before embarking on a review of these results, it is useful to examine where we stand in relation to one-dimensional elements supported by such foundations. Wang, et al. [6] presented relationships between the buckling loads determined using classical Kirchhoff plate theory and shear deformable plate theories on Pasternak foundation. Using the relationships buckling solutions of the Reissner-Mindlin and Reddy plate theories can be readily obtained from known buckling solutions of the Kirchhoff plate theory without the Pastemak foundation. It is also noted that the relationships are also applicable for the Winkler foundation as a special case of the Pasternak foundation. Saha, et al. [7] studied the dynamic stability of a rectangular plate on non-homogeneous foundation, subjected to uniform compressive in-plane bi-axial dynamic loads and supported on completely elastically restrained boundaries. They derived the equation governing the small amplitude motion of the system by a variational method. They also studied the effects of stiffness and geometry of the foundation, boundary conditions, static load factor, in-plane load ratio and aspect ratio on the stability boundaries of the plate for first- and second-order simple and combination resonances.

A broad range of the engineering problems has been solved by computer-based methods some numerical and approximate methods, such as finite element, finite difference, boundary element and framework methods in applied mechanics [8-12] have been developed to overcome such problems..Owing to its convenience in solution of plate problems as a numerical method the finite strip method have attracted much attention from many authors as [13-16]. It is suggested a procedure incorporating the finite strip method together with spring systems proposed for treating plates on elastic supports. In the studies the spring systems simulates different elastic supports, such as elastic foundation, line and point elastic supports for mixed boundary conditions.

In order to simplify the problem it is possible to use a grid of beam elements to model plates. For the elastic foundation soil model underneath plate problems the solution will be too much complex and there is no analytical solution other than for elementary cases.

There are many researches concerning analysis of beam element resting on elastic foundation Among the references, Eisenberger and Clastornik [17] developed the formulations based on interpolation (shape) functions for of solution beams by finite element method with the exact stiffness matrices. This derivations extended to an analytical solution for the shape functions of a beam segment on a generalized elastic foundation given in [18-19], leading to element-level matrices. Conceptually, the matrices obtained in a similar way of the finite element method, except that each discrete element utilized is equipped with an exact solution. In this study it is proposed to investigate an improved finite grid solution for stability problems of plates on a one-parameter elastic foundation. The solution can be stated as an extension of the so-called discrete parameter approach where the physical continuous domain is broken down into discrete sub-domains, each endowed with a response suitable for the purpose of mimicking problem at hand..

II. Theory and Problem Definition

In engineering practice, beside static case often one or both of the stability and dynamic effects have to be taken into consideration to the plate design problems. It will be necessary to describe the governing equation of motion of plates in a general mathematical form for such cases. This can be achieved by inserting both of the inertia force due to the lateral translation and in-plane loading simultaneously, in an appropriate way, into the governing differential equation for static case.

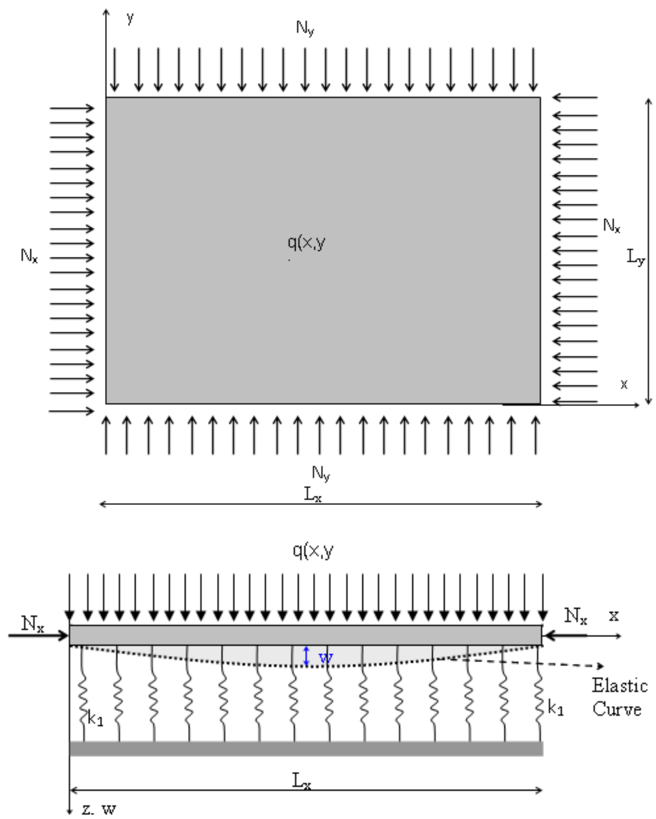

Fig. 1. The Representation of a Modal Plates Resting on Winkler Foundation Under the Combined Action of Transverse Load and Biaxial In-Plane Loads.

Bazant [20] stated that buckling of plates is analogous to buckling of columns and frames. The similarities are bifurcation type of buckling with similar near the critical load and the possibility of solving the critical loads from linear eigenvalue problem. Dynamic problems of the plates with arbitrary contours and arbitrary boundary condition are very difficult or often impossible to solve by the classical methods based on the plate governing equations. In some respects dynamic behaviour of plates resembles to that of beams. Therefore the plates can be modeled as an assemblage of individual beam elements interconnected at their neighboring joints as represented in Fig. (1). By representing the plate with assemblage of individual beam elements interconnected at their neighboring joints, the system cannot truly be equal to the continuous structure, however sufficient accuracy can be obtained similar to the static case shown in Fig. (2).

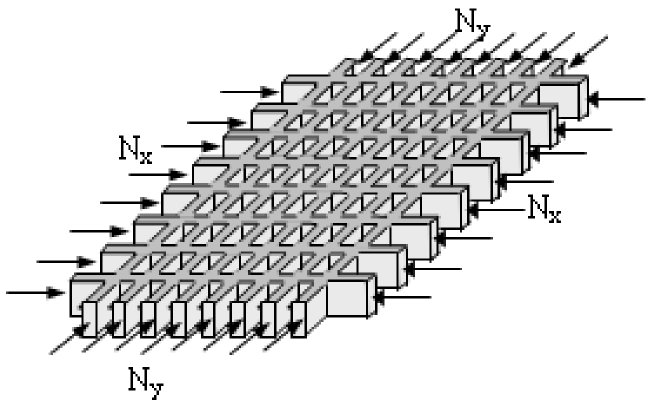

Fig. 2. The Representation of a Modal Plates by Parallel sets of one-dimensional elements replaced by the continuous surface under the action of biaxial in-plane loads.

By representing the plate shown in Fig. (2) with individual beam elements the problem can be reduced to one-dimensional. Then the governing equation of plate resting on Winkler foundation can be reduced in the form one dimensional beam element as;

$$D\frac{d^4w}{dx^4}+k_1w+N\frac{d^2w}{dy^2}=0$$

The main advantage of the reduction is that the exact geometric stiffness matrix can be determined for the beam elements. These matrices will be used as a basis of assembling the plate problems in a proper way [19]. Then dynamic problems of the plates resting on Winkler foundation with arbitrary loading and boundary conditions could be solved approximately..

III. Derivations and Formulations

With most elements developed to date, there exists no rigorous solution for plates except in the form of infinite Fourier series for a Levy-type solution. The series solutions are valid for very limited cases such as when the second parameter has been eliminated, and simple loading and boundary conditions exist. Networks of beam elements that have no such limitations can represent the plates. The properties of beam elements resting on elastic foundations will be a very useful tool to solve such complicate problems.

It is possible to calculate geometric stiffness matrix coefficients of a structural element with the procedures similar to that obtaining the element stiffness matrix [21]. The degrees of freedom of the element are the torsional rotation, vertical translation and bending rotation at each end. In the interest of consistency it can be assumed that the displacements within the span are defined again by the same interpolation functions as those already derived. This is not strictly correct because displacement interpolation functions under nodal displacements do not apply when inertia or axial forces are involved. Using the principle of virtual displacements the element geometric stiffness terms associated with a constant axial force can be evaluated, by evaluating the following integrals:

$$k_{Gij}=N\int_0^L\frac{d\psi_i(x)}{dx}\frac{d\psi_j(x)}{dx}dx \qquad (4)$$

where N is axial force and ψi and ψj are the shape functions associated with bending. Recalling the corresponding shape functions for both cases and substituting them into Eqn. (4) enables us to evaluate the geometric stiffness matrices [19].

Geometric stiffness matrix

For one-parameter foundation case as a compressive axial force applied to a beam element, it is obvious that its stiffness will reduce. The axial force influences can be included to the problem by the consistent geometric stiffness terms. It is possible to evaluate the terms without introducing any terms due to axial force into the governing differential equation.

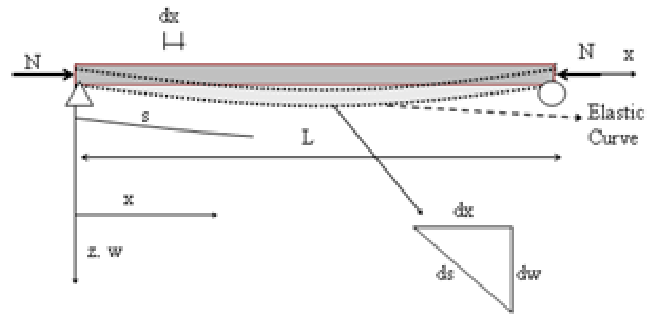

Fig. 3. The Deformed Shape of a Simply Supported Axially Loaded Beam Element.

Consider a simply supported beam subjected to compressive axial load as shown in Fig. 3. Due to the load the element will deformed, the change in length of the element can be obtained by the difference of the arc length and the horizontal length. From the Figure the arc length is;

$$ds=\sqrt{dx^2+dw^2}=\sqrt{1+\left(\frac{dw}{dx}\right)^2}$$

The series solution is;

$$ds=dx\left(1+\frac{1}{2}\left(\frac{dw}{dx}\right)^2-\frac{1}{4}\left(\frac{dw}{dx}\right)^4+................\right)$$

Neglecting small terms

$$ds=dx\left(1+\frac{1}{2}\left(\frac{dw}{dx}\right)^2\right)$$

$$\Delta=ds-dx=\frac{1}{2}\left(\frac{dw}{dx}\right)^2dx$$

Work done by the axial force N, the strain energy stored in the system, is

$$N\Delta=\int_0^L N\frac{1}{2}\left(\frac{dw}{dx}\right)^2dx$$

in this equation w can be defined as

$$w(x)=\{N\}^T\{w\}$$

where $\{w\}$ is the joint displacement vector and $\{N\}$ is the shape functions have already been obtained matrix of the beam element resting on one or two-parameter elastic foundation. For constant axial load, it can be rewritten as;

$$\int_0^L N\frac{1}{2}\left(\frac{dw}{dx}\right)^2 dx = \frac{N}{2}\int_0^L \frac{dw(x)}{dx}\frac{dw(x)}{dx}dx$$
$$= \frac{1}{2}\{w\}\int_0^L \{N'\}^T N \ \{N'\}dx \ \{w\}$$
$$= \frac{1}{2}\{w\}[k_G] \ \{w\}$$

where $[k_G] = \int_0^L \{N'\}^T N \ \{N'\}dx$

represents consistent geometric stiffness matrix of the beam element. Using this equation, each terms of the matrix in general form can be evaluated by;

$$k_{G_{ij}} = N\int_0^L \psi_i' \ \psi_j' dx = N\int_0^L \frac{d\psi_i}{dx} \ \frac{d\psi_j}{dx}dx$$

By the equation the consistent geometric stiffness using the proper shape functions terms for conventional beam or beam element resting on one or two parameter elastic foundations can be evaluated. After evaluating the necessary integrations, the terms will be obtained as;

$$k_{G11} = \frac{N}{30L} \quad and \quad k_{G14} = -\frac{N}{30L}$$

$$k_{G22} = N\left(\frac{8p(1-\cos(2p)(1-\cosh(2p)+4\sin(2p)- 6\cosh(2p)\sin(2p)+\sin(4p)+4\sinh(2p)- 6\cos(2p)\sinh(2p)+\sinh(4p)}{8p(\cos(2p)+\cosh(2p)-2)^2}\right)$$

$$k_{G23} = N\left(\frac{4\sin(p)\sinh(p)(p\cosh(p)\sin(p)+ p\cos(p)\sinh(p) - 2\sin(p)\sinh(p)}{(\cos(2p)+\cosh(2p)-2)^2}\right)$$

$$k_{G25} = N\left(\frac{3\sin(p)\cosh(3p) - \cosh(p)\sin(3p)+ 3\cos(3p)\sinh(p)+2p\sin(3p)\sinh(p) - \cos(p)\sinh(3p) - 2p\sin(p)\sinh(3p)}{4p(\cos(2p)+\cosh(2p)-2)^2}\right)$$

$$k_{G26} = N\left(\frac{-4p\sin(p)\sinh(p)\cosh(p)\sin(p)+ p\cos(p)\sinh(p) - 2\sin(p)\sinh(p)}{(\cos(2p)+\cosh(2p)-2)^2}\right)$$

$$k_{G33} = N\left(\frac{p(8p(\cos(2p)-\cosh(2p))+6\cosh(2p)\sin(2p) +4\sinh(2p)-6\cos(2p)\sinh(2p)+\sinh(4p) -8p\sin(2p)\sinh(2p)-4\sin(2p)-\sin(4p))}{4(\cos(2p)+\cosh(2p)-2)^2}\right)$$

$$k_{G36} = N\left(\frac{p(2p(\cos(p)\cosh(3p) - \cosh(p)\cos(3p)) - 3\cosh(3p)\sin(p)+\sin(3p)\cosh(p) - \cos(p)\sinh(3p) +3\cos(3p)\sinh(p)+16p\sin(p)\sinh(p))}{2(\cos(2p)+\cosh(2p)-2)^2}\right)$$

where $p = \lambda L = \sqrt[4]{\frac{k_1}{4EI}}L$ and N is constant axial compressive force

It is noted that as foundation parameter k_1 tends to zero (or $p\to 0$), the terms in the geometric stiffness equation reduce to the conventional beam consistent mass terms obtained by Hermitian functions. The correctness of the terms is verified that the terms reduce to the following conventional terms in matrix form.

$$\lim_{p\to 0}[k_G] = \frac{N}{30L}\begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 4L^2 & -3L & 0 & -L^2 & 3L \\ 0 & -3L & 36 & 0 & -3L & -36 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -L^2 & -3L & 0 & 4L^2 & 3L \\ 0 & 3L & -36 & 0 & 3L & 36 \end{bmatrix}$$

The normalized terms represent the influence of the foundation parameter k_1 on the geometric stiffness terms given of the geometric stiffness to the corresponding terms of the conventional terms of the matrix are portrayed in Figs. 4 to 9.

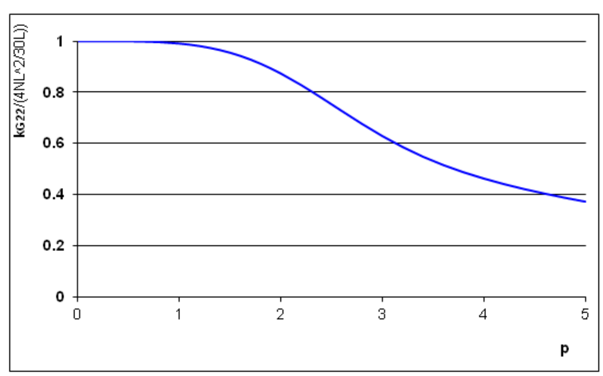

Fig. 4. The normalized consistent geometric stiffness term k_G22

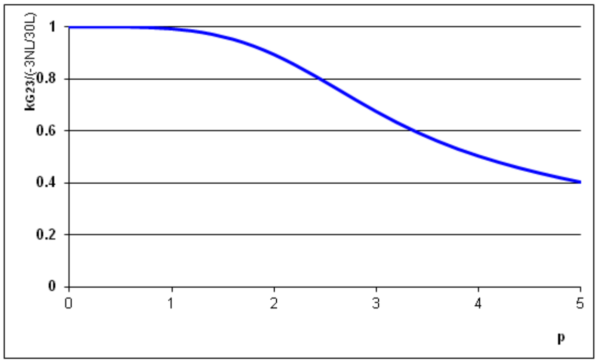

Fig. 5. The normalized consistent geometric stiffness term k_G23

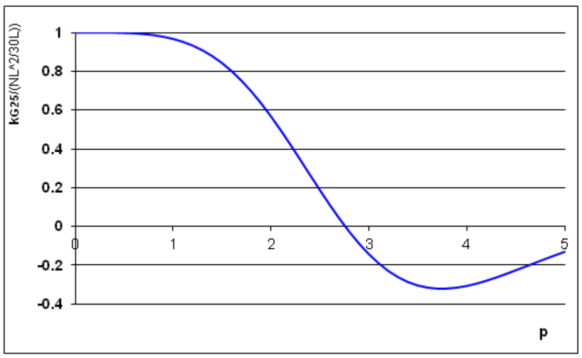

Fig. 6. The normalized consistent geometric stiffness term k_G25.

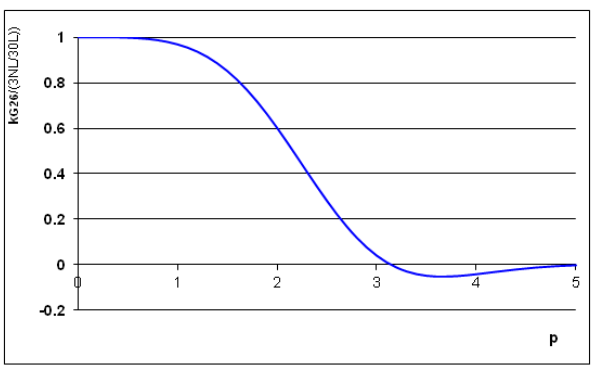

Fig. 7. The normalized consistent geometric stiffness term k_G26

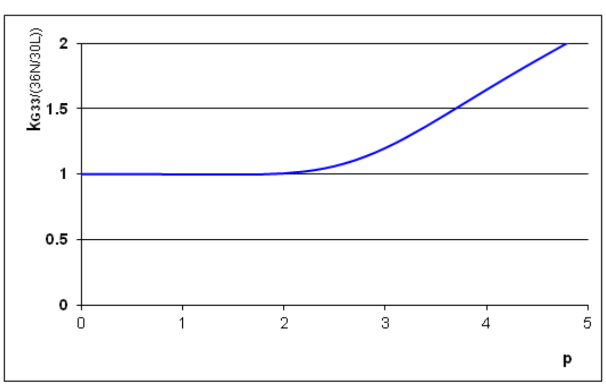

Fig. 8. The normalized consistent geometric stiffness term k_G33

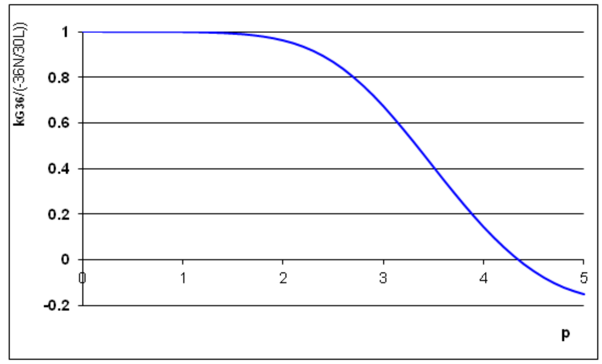

Fig. 9. The normalized consistent geometric stiffness term k_G36

From the figures the shape functions and stiffness terms related to beams on one- parameter elastic foundations are very sensitive to variation of foundation parameters after some limits. The solution obtained by inserting Hermitian polynomials into strain energy functional that has been derived [19]. In order to converge to the exact solution, the beam needs to be divided into smaller segments. The solution method is acceptable from two points of view. One is the usage of the same strain energy function and the second point, dividing the beam into smaller elements. On the other hand shape functions converge towards Hermitian polynomials when the parameter λL becomes smaller as portrayed graphically in [21].

IV. Validity of the Method

In this method plates through the lattice analogy at which the discrete elements are connected at finite nodal points is represented by one dimensional beam elements. Because of plane rigid intersection, the elements can resist torsion as well as bending moment and shear. Then finite element based matrix methods is used to determine stiffness and geometric stiffness matrices of one-dimensional beam elements resting on elastic foundations by exact shape functions. These individual element matrices are used to form the system load and stiffness matrices for plates. The matrix displacement method based on stiffness-matrix approach is suitable to solve gridworks with arbitrary load and boundary conditions. By using any convenient numbering scheme to collect all displacements for each nodal point in a convenient sequence the stiffness and geometric stiffness matrices of the system for any type of grids can be generated. As Hrennikoff [22] indicated the system cannot truly be equal to the continuous structure but solutions adequate for engineering purposes can be found with greater ease.

For checking the validity of the finite grid method (FGM), a comparison study was carried out for plates resting on Winkler foundations. For this reason a parametric study for uniformly loaded SSSS (all edges of the plate are simple supported), square plates on one-parameter foundation denoted to be accepted as benchmark results [23] refered to check the convergence, validity and accuracy of the solution method as a numerical solutions. For convenience and generality the following parameter have been introduced;

$$k = \frac{k_1 a^4}{D}$$

where a is the length of the square plate, k_1 is the Winkler foundation parameter and D is the flexural rigidity of the plate. For the square plate the buckling load parameters (Nc) have been compared with the benchmark [23] values. Tables 1 represents the non-dimensional buckling load parameters due to uniaxial and biaxial inplane loads for SSSS square plates on one-parameter elastic foundations

a=1,h=.01.D=1 E=10920000		ssss (Nc/π^2)					
		Nx=1,Ny=0			Nx=1,Ny=1		
Case	k	FG	Ref.	Error %	FG	Ref.	Error %
1	0	3.855	4	3.63	1.924	2	3.8
2	100	4.87	5.027	3.12	2.436	2.513	3.06

Table 1. Comparison of the finite grid solution with the benchmark results[23] for buckling load cases of the ssss square plate in foundation parameters

From the table it can be seen that the maximum relative error for both uniaxial and biaxial inplane loads on the square foundationless plate (k=0) and plate resting on Winkler foundation cases obtained as less than 4 %. This reflects a high degree of accuracy with respect to the reference benchmark values. The finite grid solution based on the matrices of one-dimensional beam properties for plate buckling problems verified with a high degree of accuracy.

V. Conclusion

the solution of buckling problems is usually much too complex and there is apparently no analytical solution other than simple cases for plates supported by Winkler foundations. The solution method of this technique as grid work analogy is based on a treatment of view of use the strain energy functions to obtain shape functions and stiffness matrices used to develop a more general simplified numerical approach for such complicated problems. It is noted that the shape functions and geometric stiffness terms related to beams on one-parameter elastic foundations are very sensitive to variation of foundation parameters after some limits. It can be concluded that the finite grid solution as a combination of finite element method, lattice analogy and matrix displacement analysis of grid works is a useful tool to improve the solution of plate stability problems.

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