Numerically Stable Algorithms for Scattering by Impedance Cylinders

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***Abstract*—**Electric and Magnetic Field Integral Equations (EFIE&MFIE) on impedance cylinders with smoothly parametrized cross section contours under Transverse Magnetic and Electric (TM&TE) excitations are considered. It is possible to achieve super-algebraic convergence with accurate calculation of the kernels of integral equations. Unless the impedance value is too small or too big, these equations are Fredholm of the second kind and are subject to stable discretization procedures. Otherwise numerical stability requires Analytical Regularization. Numerical results to show evidence for these points are given.

***Keywords*—** Analytical regularization, electromagnetic scattering, impedance cylinders, super-algebraic convergence.

I. Introduction

THE surface integral equation formulation of the time harmonic electromagnetic scattering problems is indispensable for modern computational electromagnetics since it enables the investigation of the phenomenon throughout the space with data collected from a one-less dimensional space, owing it to equivalence principles modeled by surface integral transforms via Green's formulae [1]. Impedance type of boundary condition imposed on such surfaces can be regarded as the next level of physical equivalence as it treats the part of the space that is in or out this boundary impenetrable, summarizing the phenomenon in or out it with a relation on this boundary ([2]-[4]). On the other hand, cylindrical obstacles i.e. uniform along one Cartesian direction can be observed quite commonly in many modern engineering problems such as transmission lines, waveguides, metamaterials, nano-rods, photonics etc. involving multilayered multiple objects or lattice structures like gratings of one or two dimensions [5]. One can easily follow the new attempts for creating instruments to better the efficiency of such solvers (e.g.[6]).

In this study, we present the elements of a high order, super-algebraically convergent solution of the classical electric and magnetic field integral equations (EFIE&MFIE) for arbitrary but smoothly parametrized cross sectioned cylinders with impedance boundary conditions under transverse electric and magnetic (TE&TM) excitations rigorously with Galerkin method supported by features of a fast and scalable algorithm. For perfectly electrically conductive (PEC) boundaries which can be considered a subset of impedance boundary conditions, but open cross-sections, these elements were elaborated in [7]. Here we introduce the algorithm for a more general version of the boundary condition on a multitude of cylindrical surfaces, cross-sections of which are given by infinitely smooth closed contours.

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A. On Numerical Stability

Even on the simplest of such contours, i.e. circular impedance cylinders, the classical separation of variables based formulation has to be linked to the integral equation formulation of the corresponding boundary value problem followed with its regularization due to ill conditioning of the linear algebraic system [8]. The EFIE&MFIE here are second kind Fredholm integral equations when the impedance amplitude is neither too big nor too small, thus their numerical discretization is well-conditioned [9]. But in other variants of impedance values this beneficial feature diminishes and algorithm gets ill-conditioned. The establishment of the classical left and right regularization of such operators [10] are problem specific and not always valid. For the class of problems considered here they were elaborated in [11] and called "Analytical Regularization Method (ARM)" which basically involves finding a canonical problem with singularity equivalence to the one under investigation, which has an analytical solution. This was mostly applied on PEC boundaries [12], [13].

According to [14], having handled this aspect of the stability problem translates to the fact that *the necessity to establish an operator with bounded inverse regarding the integral equation formulations*, has been worked out, but *the necessity to work within a proper inner product space requiring "higher-order" choices of basis and testing functions which in turn results into a well-conditioned matrix operator for the problem* has also got to be expanded on still.

This actually points to usage of higher-order formulations to improve the accuracy control of the method next to its being scalable and fast [15]. An approach for splitting the kernels to accomplish the task has been outlined in [16]. Here/in [16], treatment involves evaluation of the integral kernels on same/different grids for integration and observation points analytically/numerically. Still benefiting from spectral accuracy, with grids getting denser, the instruments of the former/latter lead to solutions with controlled/degraded accuracy especially when the kernel is hyper-singular.

Rest of the text follows with formulation including posing of the boundary value problem, followed by details of the numerical calculation. Then the numerical results to demonstrate the above mentioned features will take place.

II. Formulation

A. Posing of the boundary value problem

Let Ω denote the region out the infinitely smooth contour Γ on x-y plane. Let also $\boldsymbol{S}$ symbolize ***S***ingle layer potential, and limits of four relevant boundary potentials to it on Γ, i.e. $\{\boldsymbol{S},\boldsymbol{R},\boldsymbol{V},\boldsymbol{D}\}$ be belonging to itself, its outer normal derivatives w.r.t integ***R***ation and obser***V***ation points, and ***D***ouble layer potential respectively [9]. They appear as elements when one expresses the scattering field which satisfies the homogenous Helmholtz equation via 3rd Green formula and then uses it to define an integral equation according to the boundary conditions on Γ for the solutions in Ω obeying Sommerfeld radiation condition [17].

Let us denote the block matrix operator $\boldsymbol{K}$ and block vectors $\boldsymbol{X}$ and $\boldsymbol{G}$ for unknown current densities and incident electromagnetic fields as follows [1] ($\boldsymbol{T}$: transverse, $\boldsymbol{L}$: longitudinal):

$$\boldsymbol{K}=\begin{bmatrix}\boldsymbol{R} & \boldsymbol{\alpha S}\\ \boldsymbol{\alpha^{-1} D} & -\boldsymbol{V}\end{bmatrix};\quad \boldsymbol{X}=\begin{bmatrix}\boldsymbol{T_\beta}\\ \boldsymbol{L_\gamma}\end{bmatrix};\ \boldsymbol{G}=\begin{bmatrix}\boldsymbol{T_\gamma^{inc}}\\ \boldsymbol{L_\beta^{inc}}\end{bmatrix} \tag{1}$$

Transverse/longitudinal indicate directions that are parallel two orthogonal basis on cylindrical surface $\Gamma\bigcup z\in(-\infty,\infty)$ respectively. The values for coefficients in (1) as well as integral equations on the boundary Γ with constant impedance $\boldsymbol{\eta}$, according to incidence polarizations, are specified in the table below (***m***:magnetic, ***e***:electric):

	Impedance Relation	α	β	γ	1st row of K	2nd row of K
TM	$L_\gamma=T_\beta/\eta$	$-j\omega\mu$	*m*	*e*	**EFIE**	**MFIE**
TE	$L_\gamma=-T_\beta\eta$	$j\omega\varepsilon$	*e*	*m*	**MFIE**	**EFIE**

Table 1. Information to select an equation under specific excitation

Let Γ now consist of a multitude of infinitely smooth contours, i.e. $\Gamma=\bigcup_{i=1}^{N}\Gamma_i$ defining interfaces between regions bounded by impedance boundaries. Then for both polarizations mentioned above, with corresponding values of constitutive parameters outer sides of Γ_i, the following block matrix representation of system of Fredholm integral equations of the second kind is valid where $\boldsymbol{I}$ is unit diagonal matrix operator:

$$\langle\left[\frac{1}{2}\boldsymbol{I}+\boldsymbol{H}\right]\boldsymbol{Y}=\boldsymbol{B}\rangle_\upsilon;\ \ \upsilon=\begin{cases}1, TM-EFIE\ or\ TE-MFIE\\ 2, TM-MFIE\ or\ TE-EFIE\end{cases}$$
$$\boldsymbol{H}=\left(\boldsymbol{K_{ij}}\right)_{\boldsymbol{N}\times\boldsymbol{N}};\boldsymbol{Y}=(\boldsymbol{X_i})_{\boldsymbol{N}};\ \boldsymbol{B}=\left(\boldsymbol{G_j}\right)_{\boldsymbol{N}} \tag{2}$$

Here, υ is row selector from (1) according to Table 1, e.g. TE-EFIE is formally obtained by neglecting 1st rows of the input blocks in $<>_\upsilon$ (υ=2) and selecting data from 2nd row of Table 1. The impedance relation in Table 1, leads to arrival at any of 4 intended equations with single unknown on Γ.

B. Canonical form of boundary potentials

The integral equation in (2) is subject to discretization via Galerkin method assuming a smooth parametric representation of $\Gamma_i(\theta)$ where $\theta\in(-\pi,\pi]$. These contours have to be isomorphic to the unit circle, evaluation of boundary potentials on which form the corresponding canonic local singular expansions in [11] and [18]. With $e^{j\omega t}$ time dependence, the Green's function of 2D free space with wave number k and its local singular expansion is ($c_{0,1,2}$ are some constants, see e.g. in [19])

$$\mathcal{G}_2(q,p)=\frac{1}{4j}H_0^{(2)}(k|q-p|)=$$
$$\frac{1}{2\pi}ln|q-p|\left(1+c_0\mathcal{O}(|q-p|^2)\right)+\mathcal{O}\left(c_1+c_2\mathcal{O}(|q-p|^2)\right) \tag{3}$$

On unit circle, the distance between two points is $|q-p|=\left|2sin\frac{\theta-\tau}{2}\right|$ with polar coordinates q=(1,θ), p=(1,τ). According to (3) natural logarithm of this distance is the canonical singularity and has the following Fourier series expansion analytically and relation with hyper-singularity via the parametric differentiation property as $\delta\to0$ (since x→0, 2sin(x/2)=$\mathcal{O}$(x) it also means (θ-τ)→0):

$$\delta=2sin\frac{\theta-\tau}{2};\qquad ln|\delta|=-\frac{1}{2}\sum_{\substack{n=-\infty\\ n\neq0}}^{\infty}a_n e^{in(\theta-\tau)};$$
$$a_n=|n|^{-1};\ \frac{1}{\delta^2}=-\frac{\partial^2}{\partial\theta^2}ln|\delta|=\frac{1}{2}\sum_{\substack{n=-\infty\\ n\neq0}}^{\infty}a_n^{-1}e^{in(\theta-\tau)} \tag{4}$$

Since the difference between evaluation of $\mathcal{G}_2(q,p)$ on unit circle and on any other $\Gamma_i(\theta)$ is a real-analytical function (a function converging to its Taylor series expansion at a point), the canonical integrals for the boundary potentials in Table 2 can be suggested describing the singular behavior of their kernels while $\delta\to0$.

Boundary potential	Canonical Parametric Integral Transform
$\boldsymbol{S}\left(\zeta(\theta)\right)$	$\int_{-\pi}^{\pi}\zeta(\tau)\left\{\frac{1}{2\pi}ln|\delta|+P_S(\theta,\tau)\right\}l(\tau)d\tau$
$\boldsymbol{R}\left(\xi(\theta)\right)$	$\frac{1}{2}\xi(\theta)+\int_{-\pi}^{\pi}\xi(\tau)\left\{\frac{1}{2\pi l(\tau)}P_R(\theta,\tau)\right\}l(\tau)d\tau$
$\boldsymbol{V}\left(\zeta(\theta)\right)$	$-\frac{1}{2}\zeta(\theta)+\frac{1}{2\pi l(\theta)}\int_{-\pi}^{\pi}\zeta(\tau)P_V(\theta,\tau)l(\tau)d\tau$

$\boldsymbol{D}\big(\xi(\theta)\big)$	$\frac{1}{2\pi l(\theta)l(\tau)}\left[\frac{\partial^2}{\partial\theta^2}\int_{-\pi}^{\pi}\xi(\tau)ln\lvert\delta\rvert l(\tau)d\tau + \int_{-\pi}^{\pi}\xi(\tau)P_D(\theta,\tau)l(\tau)d\tau\right]$

Table 2. The integral transforms via the boundary potentials

Here $l(\theta)=\sqrt{x'(\theta)^2+y'(\theta)^2}$ is the arc-length applies to each of the $\Gamma_i(\theta)$ parametrizations. The explicit form of the corresponding boundary potentials obeying the form in Table 2 can be given starting from the following matrix equation that summarizes the kernels of boundary potentials:

$$\begin{bmatrix}\boldsymbol{\mathcal{K}(S)}\\ \boldsymbol{\mathcal{K}(R)}\\ \boldsymbol{\mathcal{K}(V)}\\ \boldsymbol{\mathcal{K}(D)}\end{bmatrix}=\begin{bmatrix}1/jk & 0\\ 0 & 4a/jk\\ 0 & -4b/jk\\ kab & (2ab-c)/kR\end{bmatrix}\begin{bmatrix}jk\mathcal{G}_2\\ \frac{k}{4j}\mathcal{G}_2'\end{bmatrix};$$

$$\begin{cases}a=\hat{n}'\cdot\hat{R}\\ b=\hat{n}\cdot\hat{R}\\ c=\hat{n}'\cdot\hat{n}\\ R=\lvert q-p\rvert\end{cases};\quad \begin{matrix}\mathcal{G}_2(q,p)=\frac{1}{4j}H_0^{(2)}(kR)\\ \mathcal{G}'_2(q,p)=\frac{jk}{4}H_1^{(2)}(kR)\end{matrix} \qquad (5)$$

Here $\hat{n}'$ and $\hat{n}$ are outward unit normals at integration and observation points respectively and $\hat{R}$ is the unit vector from integration point to observation point where k is the wave number of the observation domain. Let us add the following local singular expansion to (3) for $\mathcal{G}_2'(q,p)$ implicitly to cast the most dominant singularity in (5) ($c_{3,4,5,6}$ are some constants, see e.g. [19]);

$$\frac{\mathcal{G}_2'(q,p)}{\lvert q-p\rvert}=-\frac{1}{2\pi\lvert q-p\rvert^2}-\frac{k}{2\pi}ln\lvert q-p\rvert\times$$
$$\times\big(c_3+c_4\mathcal{O}(\lvert q-p\rvert^2)\big)+k\mathcal{O}\big(c_5+c_6\mathcal{O}(\lvert q-p\rvert^2)\big) \qquad (6)$$

Notice that when $q=p$, $a=b=0$ and $c=1$ in (5). Therefore on Γ, $\mathcal{K}(\boldsymbol{S})$ is weakly singular, $\mathcal{K}(\boldsymbol{R})$ and $\mathcal{K}(\boldsymbol{V})$ involve principle values equal to ½ with a smooth tail while $\mathcal{K}(\boldsymbol{D})$ has finite part in Hadamard sense, i.e. is hyper-singular with a weakly singular tail, also qualifying the kernels in Table 2 [1]. Rather than these direct values, it is important to show their limits while observation made approaches to Γ, exist [9] since those values are subject to boundary conditions. This was proven in [18] specifically for closed as well as open smoothly parametrized boundary contours, quite intricately for $\boldsymbol{D}$ as well as $\boldsymbol{S}$, $\boldsymbol{R}$, $\boldsymbol{V}$ where the task is much straightforward to accomplish (see Table 3). Looking at (3), (4) and (6), one can easily relate (5) to kernels in Table 2 as $\delta\to 0$, when Γ is unit circle since $l(\theta)=l(\tau)=1$. For arbitrary $\Gamma_i(\theta)$, one can also show that the local singular expansion of R^2 performing it once for integration and once for observation points on R of (5) will lead to;

$$R^2=\lvert q(\theta)-p(\tau)\rvert^2=\lvert q(\tau+\delta)-p(\tau)\rvert\lvert q(\theta)-p(\theta+\delta)\rvert$$
$$=\Big(l(\tau)\delta+\mathcal{O}\big(\delta^{3/2}\big)\Big)\Big(l(\theta)\delta+\mathcal{O}\big(\delta^{3/2}\big)\Big)$$
$$=l(\theta)l(\tau)\delta^2+\mathcal{O}(\delta^3),\quad \delta\to 0 \qquad (7)$$

This clarifies the transition from (5) to kernels in Table 2 for arbitrary smooth Γ in the light shed from (3) to (7).

C. Values of boundary potentials on Γ

The high-order discretization of (2) for the problems of interest, requires the integration of the kernels in (5) accurately. Achieving spectral accuracy leads to super-algebraic convergence. The factorized representation of the kernels, consist of canonically singular parts multiplied by infinitely smooth parts ($\mathcal{K}_0$) plus solely infinitely smooth parts ($\mathcal{K}_1$), i.e. $\mathcal{K}=\mathcal{K}_0+\mathcal{K}_1$. Canonical singularities appearing in Table 2, have the analytical expressions of their spectrum given in (4). In Table 3 $\mathcal{K}_0$ is given as defined above, in terms of these canonical singularities. Then the determination of the spectrum of $\mathcal{K}_0$, can be performed via Fast Fourier Transform (FFT) for the corresponding infinitely smooth part accurately and convolving them with spectrum of the singularities factorizing them either when $\delta\neq 0$ or $\delta\to 0$. Formally one can define $\mathcal{K}_1=\mathcal{K}-\mathcal{K}_0$ and calculate the difference functions occurring by extraction of kernel $\mathcal{K}_0$ when $\delta\neq 0$. Their limits when $\delta\to 0$ are given also in Table 3. Then corresponding spectrum of it can be determined accurately by the well-known fast and scalable procedure i.e. FFT. Therefore, the samples for Fourier transform are involved analytically for the singularities and efficiently numerically for infinitely smooth parts as well as the finite limits in the kernels as $\delta\to 0$ avoiding the big numbers that may occur during calculation. To specify, let us define the following short-hand notations for recurring expressions ($\epsilon=0.577215\ldots$, Euler-Mascheroni constant):

$$\bar{J}_0(x)=J_1(x)/x;\qquad \bar{\bar{J}}_0(x)=J_0(x)-2\bar{J}_0(x);$$
$$\begin{bmatrix}P(\theta)\\ M(\theta)\end{bmatrix}^{(i,j)}=\begin{bmatrix}x^{(i)}(\theta)x^{(j)}(\theta)+y^{(i)}(\theta)y^{(j)}(\theta)\\ x^{(i)}(\theta)y^{(j)}(\theta)-y^{(i)}(\theta)x^{(j)}(\theta)\end{bmatrix} \qquad (8)$$

Note that $J_0(x)=\mathcal{O}(1)$, $\bar{J}_0(x)=\mathcal{O}(1/2)$ and $\bar{\bar{J}}_0(x)=\mathcal{O}(x)$. Indices ($i$, j) denote the order of the derivative of the Cartesian coordinate of the parametrization.

$\boldsymbol{\mathcal{K}}$	$\lim_{\delta\to 0}\boldsymbol{\mathcal{K}_1}$	$\boldsymbol{\mathcal{K}_0}$
$\boldsymbol{\mathcal{K}(S)}$	$\frac{1}{4j}-\frac{\epsilon+ln\lvert kl/2\rvert}{2\pi}$	$-\frac{1}{2\pi}ln\lvert\delta\rvert J_0(kR)$
$\boldsymbol{\mathcal{K}(R)}$	$\frac{1}{2\pi l(\tau)}\left[\frac{kM^{(1,2)}}{2l^2}\right]$	$\frac{ka}{2\pi}ln\lvert\delta\rvert J_1(kR)$
$\boldsymbol{\mathcal{K}(V)}$	$\frac{1}{2\pi l(\theta)}\left[\frac{kM^{(1,2)}}{2l^2}\right]$	$-\frac{kb}{2\pi}ln\lvert\delta\rvert J_1(kR)$
$\boldsymbol{\mathcal{K}(D)}$	$\frac{1}{2\pi l(\theta)l(\tau)}\left[\pi(kl)^2\left(\frac{1}{4j}-\frac{\epsilon+1/2+ln\lvert kl/2\rvert}{2\pi}\right)+\frac{1}{12}-\left(\frac{P^{(1,3)}}{6l^2}-\frac{P^{(2,2)}}{4l^2}-\frac{\left(M^{(1,2)}\right)^2}{l^4}\right)\right]$	$\frac{-k^2ln\lvert\delta\rvert}{2\pi}\times\left(ab\bar{\bar{J}}_0(kR)+c\bar{J}_0(kR)\right)+\frac{1}{2\pi l(\theta)l(\tau)}\frac{\partial^2}{\partial\theta^2}ln\lvert\delta\rvert$

Table 3. Infinitely smooth parts of kernels of boundary potential as $\delta\to 0$ ($\mathcal{K}_1$) and canonic singularity times infinitely smooth parts of kernels of boundary potential ($\mathcal{K}_0$).

In the case of multiple interfaces, during forming a block row of (2), (θ_i, τ_j) parameters only require use of Table 3 when $i=j$. Otherwise when $i \neq j$, using (5) without any singularity extraction will lead to infinitely smooth parts in the kernel subject to FFT procedure as all infinitely smooth parts.

D. Discretization of boundary potentials

Assuming a non-zero right hand side as well as the unknown currents in (2), all of these functions as well as the functions in Table 3, to form the transforms in Table 2, are 2π-periodic to their parameters on Γ and have to be represented by corresponding single or double Fourier series. The transforms in Table 2 discretize the integrals owing to the orthogonality of complex exponentials in the given domain as described in [11] and [18]. Below in Table 4, one can formally find discrete version in other words Fourier spectrum truncated to integer M of Table 2, which is elaborated according to Table 3 ($*$ represents convolution operation). The single/double index Fourier coefficients are for unknown current densities (z)/infinitely smooth functions (g,j) in kernels ($\mathcal{K}_1$,$\mathcal{K}_0$) of Table 3 respectively. I as well as a_n and $1/a_n$ are diagonal matrices values of which are unity, Fourier coefficients of $ln|\delta|$ and $\partial^2 ln|\delta|/\partial\theta^2$ given in (4) ($[\ldots]^{<0>}$ notation indicates entry is zero to location of zero indices). Notice that the last one is achieved evidently by differentiating after the integral transform as indicated in Table 2.

Boundary Potential	***Fourier Spectrum of Boundary Potential on Γ truncated to integer M***
S	$\left[-\left(\left[g_{s,-n}^{(S)}\right]_{M\times M}+\frac{2}{\pi}[a_n]_{M\times M}^{\langle 0\rangle}*\left[j_{s,-n}^{(S)}\right]_{M\times M}\right)\right]\left[z_n^{(S)}\right]_{M\times 1}$
R	$\left[\frac{1}{2}I_{M\times M}-\frac{i\pi}{2}\left(\left[g_{s,-n}^{(R)}\right]_{M\times M}+\frac{2}{\pi}[a_n]_{M\times M}^{\langle 0\rangle}*\left[j_{s,-n}^{(R)}\right]_{M\times M}\right)\right]\left[z_n^{(R)}\right]_{M\times 1}$
V	$\left[\frac{1}{2}I_{M\times M}+\frac{i\pi}{2}\left(\left[g_{s,-n}^{(V)}\right]_{M\times M}+\frac{2}{\pi}[a_n]_{M\times M}^{\langle 0\rangle}*\left[j_{s,-n}^{(V)}\right]_{M\times M}\right)\right]\left[z_n^{(V)}\right]_{M\times 1}$
D	$\left[\left[\frac{1}{a_n}\right]_{M\times M}^{\langle 0\rangle}-\left(\left[g_{s,-n}^{(D)}\right]_{M\times M}+\frac{2}{\pi}[a_n]_{M\times M}^{\langle 0\rangle}*\left[j_{s,-n}^{(D)}\right]_{M\times M}\right)\right]\left[z_n^{(D)}\right]_{M\times 1}$

Table 4. The spectrum of boundary potentials being used during the formation of the right hand side of the matrix equations

The specific composition of the matrix-vector multiplications in Table 4 are equated to corresponding Fourier coefficients of the excitation given in (1) on $\Gamma_i(\theta)$ which forms the right hand side in (2) after having selected the excitation polarization type in Table 1 as well as the equation to solve for, and composed the kernels according to (1).

III. NUMERICAL RESULTS

We will start with validations of the new established algorithms. This will be made by the method in [8] constructed for circular boundaries and involve the comparison of the Fourier coefficients of the unknowns on the boundaries. Then a combination of different smooth contours from [20] will be chosen and the solution performance in terms of convergence and numerical stability will be demonstrated on them. The numerical instability when the impedance values are either too big or too small will be witnessed numerically and the improvement via application of ARM will be presented where basically explained in the appendix.

APPENDIX: ANALYTICAL REGULARIZATION METHOD

The history and theoretical background of the ARM is thoroughly given in [11]. Basically it can be thought as an analytical preconditioning for the ill-posed systems and make them well-posed by finding a set of correctness that for Ax=b, x,b∈H, it is possible to find pair of spaces H_1,H_2: $H_1 \supset H \supset H_2$ that bounded inverse of A, A^{-1} exists [21]. Since the most suitable space representing the algebra in computers is l_2, we have no choice but H_1, H, H_2 all have to be l_2 the space of square summable sequences. Therefore the linear algebraic system of the first kind (LAES1: Ax=b, x,b∈l_2) which is ill-posed has to be converted to a linear algebraic equation of the second kind (LAES2: (I+H)y=g, y,g∈l_2) where I is identity and H and A are compact operators in l_2. The passage depends on finding the invertible left and right regularizing operators [10] that leads to LAR=I+H, by the definition of new unknown $y=R^{-1}x$ and new right hand side g=Lb. LAES2 with increasing truncation number of the infinite system, have uniformly bounded condition numbers thus leading to a numerically stable inversion procedure leading to guaranteed convergence unlike LAES1 where none of such features are under guarantee.

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